Regularization of linear time-invariant differential-algebraic systems

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Abstract

We derive equivalent criteria for the existence of a feedback ensuring that a given linear and time-invariant differential-algebraic control system is regular or autonomous, respectively. Algebraic and geometric criteria are stated in terms of the involved matrices and the augmented Wong sequences. For systems which are not regularizable by feedback, we show that an additional behavioral equivalence transformation and a reorganization of input and state variables leads to a regular system, the index of which is at most one. This procedure is known, however our approach allows for a detailed characterization of the resulting regular system.

Keywords: differential-algebraic systems; descriptor systems; regularization; behavioral approach; feedback; Wong sequences.

1. Introduction

We study linear time-invariant systems given by differential-algebraic equations (DAEs) of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t) \tag{1}$$

where $E,A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$. Systems of that type are also called *descriptor systems*. The set of systems (1) is denoted by $\Sigma_{l,n,m}$ and we write $[E,A,B] \in \Sigma_{l,n,m}$. DAE systems of the form (1) naturally occur when modeling dynamical systems subject to algebraic constraints; for a further motivation we refer to [1, 7, 18, 20, 26] and the references therein. The system $[E,A,B]$ is called *regular*, if the matrix pencil $sE-A$ is regular, that is, $l = n$ and $\det(sE-A) \in \mathbb{R}[s]\setminus\{0\}$.

The function $u : \mathbb{R} \to \mathbb{R}^m$ is usually called *input* of the system, although one should keep in mind, that in the nonregular case u might be constrained and some of the state variables can play the role of an input, see Section 5. The tuple $(x,u) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ is said to be a *solution* of (1) if, and only if, it belongs to the *behavior* of (1):

$$\mathfrak{B}_{[E,A,B]} := \{(x,u) \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m) \mid Ex \in \mathscr{AC}(\mathbb{R};\mathbb{R}^l),\ (x,u) \text{ satisfies (1) for almost all } t\in\mathbb{R}\},$$

where $\mathscr{L}^1_{\mathrm{loc}}$ and $\mathscr{AC}$ denote the space of locally (Lebesgue) integrable and absolutely continuous functions, resp. DAE control systems based on the above behavior have been studied in detail e.g. in [1].

In the present paper we investigate different notions of regularization of DAE systems. In [23] a system is called regularizable, if there exists a proportional state feedback $u(t) = Fx(t)$ such that the closed-loop system $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = (A+BF)x(t)$ is regular. This is not always possible (obviously, $l = n$ is necessary); we derive different equivalent characterizations in Section 3. Another approach is to additionally allow for transformations which do not alter the behavior of the system $[E,A,B]$, i.e., behavioral equivalence as discussed in Section 4, and reinterpretation of variables. The latter means that certain state variables are reinterpreted as input variables and vice versa. This regularization procedure has been developed in [13] for general time-varying and nonlinear systems.

One purpose of the present paper is to give a complete picture of this regularization procedure for the case of linear time-invariant DAE systems. Particular emphasis is placed on the *feedback form* from [21] (see Section 2) which generalizes the famous *Brunovský form* [8] to DAEs and is the crucial tool to derive the regularized system in Section 5. As a result, the indices appearing in the feedback form allow to explicitly state

- the number of redundant equations,
- the number of free state variables,
- the number of constraint input variables.

Furthermore, we show that these numbers as well as the complete regularized system (without the transformation leading to it) can be computed from the augmented Wong sequences, which are simple subspace iterations.

We like to stress that the above approaches are different from the notion of regularizability used in [9, 10, 11, 12, 14, 15], which aim to find a feedback such that the closed-loop system is regular and additionally has index at most one (see Section 5 for the definition of the index). It is shown in [10, 11] that for $[E,A,B] \in \Sigma_{n,n,m}$ there exists some $F \in \mathbb{R}^{m\times n}$ such that $sE-(A+BF)$ is regular and has index at most 1 if, and only if, $[E,A,B]$ is impulse controllable, that is $\operatorname{im} E + A\ker E + \operatorname{im} B = \mathbb{R}^n$. This has been extended to the case of derivative and proportional output feedback in [11, 14, 15]; numerical methods have been investigated in [9, 12].

The paper is organized as follows: In Section 2 we recall the feedback form and how the entries in it can be calculated by the augmented Wong sequences; these results stem from [21]

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in great parts. The feedback form is exploited to study regularizability of DAE systems by state feedback in Section 3. First, we derive algebraic and geometric characterizations for the more general concept of autonomizability and then apply these to the case of regularizability. We also remark on the case of derivative state feedback. In Section 4 we introduce the concepts of behavioral equivalence and minimality of DAE systems. Again, minimality is characterized by algebraic and geometric conditions, where the feedback form is the main tool for the proof. Finally, in Section 5 we revisit the regularization procedure from [13]. It is shown that, given any DAE system $[E,A,B] \in \Sigma_{l,n,m}$, by a combination of behavioral equivalence transformation, proportional state feedback and reorganization of variables a new system $[\hat{E},\hat{A},\hat{B}]$ can be obtained where $s\hat{E}-\hat{A}$ is regular and of index at most one. In contrast to [13], in Theorem 5.1 the transformations to the regular index-1 system are stated explicitly and not by means of a numerical procedure.

2. Feedback

In this section we recall the feedback canonical form from [21], see also the survey [3]. The feedback form is described by a set of multi-indices and one matrix. We show that the entries of the multi-indices and the Jordan canonical form of the matrix are completely determined by the augmented Wong sequences.

For the definition of the feedback form some notation is warranted. For $k \in \mathbb{N}$ we define the matrices

$$N_k := \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{k\times k},$$

$$K_k := \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix}, \quad L_k := \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(k-1)\times k}.$$

For any multi-index $\alpha = (\alpha_1,\ldots,\alpha_k) \in \mathbb{N}^k$ we use the notation $\ell(\alpha) = k$, $|\alpha| = \sum_{i=1}^k \alpha_i$ and introduce

$$\begin{aligned} N_\alpha &:= \operatorname{diag}(N_{\alpha_1},\ldots,N_{\alpha_k}) \in \mathbb{R}^{|\alpha|\times|\alpha|},\\ K_\alpha &:= \operatorname{diag}(K_{\alpha_1},\ldots,K_{\alpha_k}) \in \mathbb{R}^{(|\alpha|-k)\times|\alpha|},\\ L_\alpha &:= \operatorname{diag}(L_{\alpha_1},\ldots,L_{\alpha_k}) \in \mathbb{R}^{(|\alpha|-k)\times|\alpha|},\\ E_\alpha &:= \operatorname{diag}(e_{\alpha_1}^{[\alpha_1]},\ldots,e_{\alpha_k}^{[\alpha_k]}) \in \mathbb{R}^{|\alpha|\times k}, \end{aligned}$$

where $e_i^{[n]}$ is the ith unit vector in $\mathbb{R}^n$. We also denote the set of all invertible $n\times n$-matrices over a ring $\mathscr{R}$ by $\mathbf{Gl}_n(\mathscr{R})$.

Theorem 2.1 (Feedback form [21])**.** *For any system $[E,A,B] \in \Sigma_{l,n,m}$ there exist $S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), V \in \mathbf{Gl}_m(\mathbb{R}), F \in \mathbb{R}^{m\times n}$ such that*

$$[SET, SAT+SBF, SBV] = \left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0\\ 0 & L_\beta & 0 & 0 & 0 & 0\\ 0 & 0 & K_\gamma^\top & 0 & 0 & 0\\ 0 & 0 & 0 & L_\delta^\top & 0 & 0\\ 0 & 0 & 0 & 0 & N_\kappa & 0\\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{c}}} \end{bmatrix}, \begin{bmatrix} N_\alpha^\top & 0 & 0 & 0 & 0 & 0\\ 0 & K_\beta & 0 & 0 & 0 & 0\\ 0 & 0 & L_\gamma^\top & 0 & 0 & 0\\ 0 & 0 & 0 & K_\delta^\top & 0 & 0\\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0\\ 0 & 0 & 0 & 0 & 0 & A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} E_\alpha & 0 & 0\\ 0 & 0 & 0\\ 0 & E_\gamma & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0 \end{bmatrix}\right], \tag{2}$$

where $\alpha \in \mathbb{N}^{n_\alpha}, \beta \in \mathbb{N}^{n_\beta}, \gamma \in \mathbb{N}^{n_\gamma}, \delta \in \mathbb{N}^{n_\delta}, \kappa \in \mathbb{N}^{n_\kappa}$ are multi-indices which are unique up to a permutation of their entries and $A_{\overline{c}} \in \mathbb{R}^{n_{\overline{c}}\times n_{\overline{c}}}$ is unique up to similarity.

As stated in Theorem 2.1 the entries of the multi-indices are uniquely determined by $[E,A,B]$. These entries can even be calculated with the help of three simple subspace iterations, the latter of which are defined as follows:

$$\begin{aligned} \mathscr{V}_0 &:= \mathbb{R}^n, & \mathscr{V}_{i+1} &:= A^{-1}(E\mathscr{V}_i + \operatorname{im} B),\ i \geq 0,\\ \mathscr{V}^* &:= \bigcap_{i\geq 0} \mathscr{V}_i, & &\\ \mathscr{W}_0 &:= \{0\}, & \mathscr{W}_{i+1} &:= E^{-1}(A\mathscr{W}_i + \operatorname{im} B),\ i \geq 0,\\ \mathscr{W}^* &:= \bigcup_{i\geq 0} \mathscr{W}_i, & &\\ \mathscr{X}_0 &:= \ker E, & \mathscr{X}_{i+1} &:= E^{-1}(A\mathscr{X}_i + \operatorname{im} B),\ i \geq 0,\\ \mathscr{X}^* &:= \bigcup_{i\geq 0} \mathscr{X}_i. & & \end{aligned}$$

Recall that, for some matrix $M \in \mathbb{R}^{l\times n}$, $M\mathscr{S} = \{\, x \in \mathbb{R}^l \mid x \in \mathscr{S} \,\}$ denotes the image of $\mathscr{S} \subseteq \mathbb{R}^n$ under M and $M^{-1}\mathscr{S} = \{\, x \in \mathbb{R}^n \mid Mx \in \mathscr{S} \,\}$ denotes the preimage of $\mathscr{S} \subseteq \mathbb{R}^l$ under M. Note that the subspaces $\mathscr{V}_i$, $\mathscr{W}_i$, $\mathscr{X}_i$ can be computed numerically, cf. [25].

The sequences $(\mathscr{V}_i)$ and $(\mathscr{W}_i)$ are called *augmented Wong sequences*, see [3, 6] and the references therein. The reason is that they are based on the Wong sequences ($B=0$) used in [2, 4, 5] and which have their origin in WONG [28] who was the first using both sequences (with $B=0$) for the analysis of matrix pencils. The sequence $(\mathscr{X}_i)$ is just a simple modification of $(\mathscr{W}_i)$ with a different initial value; in fact, $\mathscr{X}^* = \mathscr{W}^*$.

For any multi-index $\eta \in \mathbb{N}^p$ we introduce the notation

$$\overline{\eta}_i := \left|\{\, j \in \mathbb{N} \mid \eta_j \geq i \,\}\right|, \quad i = 1,\ldots,\max\eta,$$

where $\max\eta$ is the largest entry in η. Then the entries of the multi-indices appearing in the feedback form (2) are fully determined by the augmented Wong sequences in the following sense.

Theorem 2.2 (Indices in the feedback form)**.** *Let $[E,A,B] \in \Sigma_{l,n,m}$ and use the notation from Theorem 2.1 so that* (2) *holds. Then*

$$\begin{aligned} \overline{\alpha}_i &= \dim(\mathscr{V}^* \cap \mathscr{W}_i) - \dim(\mathscr{V}^* \cap \mathscr{X}_{i-1}), \quad i \geq 1,\\ \overline{\beta}_i &= \dim(\mathscr{V}^* \cap \mathscr{X}_{i-1}) - \dim(\mathscr{V}^* \cap \mathscr{W}_{i-1}), \quad i \geq 1,\\ \overline{\kappa}_i &= \dim \mathscr{X}_{i-1} - \dim \mathscr{W}_{i-1} - \overline{\beta}_i, \quad i \geq 1, \end{aligned}$$

and, for all $i \geq 2$,

$$\overline{\gamma}_1 = \operatorname{rk} B - \overline{\alpha}_1, \quad \overline{\gamma}_i = \dim \mathscr{W}_{i-1} - \dim \mathscr{X}_{i-2} - \overline{\alpha}_{i-1},$$

$$\overline{\delta}_1 = l - n + \overline{\beta}_1 - \overline{\gamma}_1,$$
$$\overline{\delta}_i = \dim(\mathscr{W}^* + \mathscr{V}_{i-2}) - \dim(\mathscr{W}^* + \mathscr{V}_{i-1}).$$

Proof. The formulas for $\overline{\alpha}_i, \ldots, \overline{\gamma}_i$ are shown in [21, Prop. 3.2]. The expression for $\overline{\delta}_i$ is a consequence of eqs. [3, (6.1) & (6.2)]. □

For a full determination of the feedback form by the augmented Wong sequences it remains to show that the ODE part $[I_{n_{\overline{c}}}, A_{\overline{c}}, 0]$ is determined by (a version of) the augmented Wong sequences as well. To this end, similar to [5, Prop. 2.6], consider the sequence of complex subspaces

$$\mathscr{W}_0^\lambda := \{0\},\ \mathscr{W}_{i+1}^\lambda := (A - \lambda E)^{-1}(E\mathscr{W}_i^\lambda + \mathrm{im}_{\mathbb{C}} B) \subseteq \mathbb{C}^n,\ i \geq 0,$$

where $\lambda \in \mathbb{C}$. We show that the dimension of $\ker(\lambda I_{n_{\overline{c}}} - A_{\overline{c}})^i$ is given in terms of $\mathscr{W}^*$ and $\mathscr{W}_i^\lambda$, and hence the Jordan canonical form of $A_{\overline{c}}$ is fully determined by these numbers, cf. also a similar result (for $B = 0$) in [5]. Note that it is necessary to treat $\mathscr{V}^*$ and $\mathscr{W}^*$ as subspaces of $\mathbb{C}^n$ here, although they have a basis consisting of real-valued vectors.

Proposition 2.3. *Let $[E, A, B] \in \Sigma_{l,n,m}$ and use the notation from Theorem 2.1 so that* (2) *holds. Let $\mathscr{V}^*$ and $\mathscr{W}^*$ be the limits of the augmented Wong sequences embedded in $\mathbb{C}^n$. Then, for all $\lambda \in \mathbb{C}$,*

$$\det(\lambda I_{n_{\overline{c}}} - A_{\overline{c}}) = 0 \quad \Longleftrightarrow \quad \mathscr{W}_1^\lambda \subseteq \mathscr{W}^*.$$

Furthermore,

$$n_{\overline{c}} = \dim \mathscr{V}^* - \dim(\mathscr{V}^* \cap \mathscr{W}^*),$$
$$\dim \ker(\lambda I_{n_{\overline{c}}} - A_{\overline{c}})^i = \dim(\mathscr{W}^* + \mathscr{W}_i^\lambda) - \dim \mathscr{W}^*, \quad i \geq 1.$$

Proof. As shown in the proof of [3, Thm. 6.4] we may, without loss of generality, assume that $[E, A, B]$ is in feedback form (2) and we have that

$$\mathscr{V}^* = \mathbb{C}^{|\alpha|} \times \mathbb{C}^{|\beta|} \times \{0\}^{|\gamma| - \ell(\gamma)} \times \{0\}^{|\delta| - \ell(\delta)} \times \{0\}^{|\kappa|} \times \mathbb{C}^{n_{\overline{c}}},$$
$$\mathscr{W}^* = \mathbb{C}^{|\alpha|} \times \mathbb{C}^{|\beta|} \times \mathbb{C}^{|\gamma| - \ell(\gamma)} \times \{0\}^{|\delta| - \ell(\delta)} \times \mathbb{C}^{|\kappa|} \times \{0\}^{n_{\overline{c}}}.$$

This implies $n_{\overline{c}} = \dim \mathscr{V}^* - \dim(\mathscr{V}^* \cap \mathscr{W}^*)$. Furthermore, similar to the calculation of the formulas [3, (6.1) & (6.2)], it is straightforward that

$$\mathscr{W}^* + \mathscr{W}_i^\lambda =$$
$$\mathbb{C}^{|\alpha|} \times \mathbb{C}^{|\beta|} \times \mathbb{C}^{|\gamma| - \ell(\gamma)} \times \{0\}^{|\delta| - \ell(\delta)} \times \mathbb{C}^{|\kappa|} \times \ker(\lambda I_{n_{\overline{c}}} - A_{\overline{c}})^i$$

for $i \geq 0$. This implies the remaining statements. □

3. Regularization by feedback

In this subsection we investigate how regularity can be gained or lost under the action of proportional state feedback, that is the addition of the proportional feedback law $u(t) = Fx(t)$ to the given system $[E, A, B] \in \Sigma_{l,n,m}$ for some matrix $F \in \mathbb{R}^{m \times n}$. The resulting system has the form $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = (A + BF)x(t)$. The set of all homogenous DAEs

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t), \tag{3}$$

where $E, A \in \mathbb{R}^{l \times n}$ is denoted by $\Sigma_{l,n}$ and we write $[E, A] \in \Sigma_{l,n}$. The behavior of $[E, A] \in \Sigma_{l,n}$ is given by

$$\mathfrak{B}_{[E,A]} := \{ x \in \mathscr{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n) \ \mid Ex \in \mathscr{AC}(\mathbb{R}; \mathbb{R}^l),$$
$$x \text{ satisfies (3) for almost all } t \in \mathbb{R} \ \},$$

A DAE $[E, A] \in \Sigma_{l,n}$ is regular if, and only if, $sE - A$ is regular or, equivalently, $l = n$ and for every $f \in \mathscr{C}^\infty(\mathbb{R}; \mathbb{R}^n)$ there exists $x \in \mathscr{C}^\infty(\mathbb{R}; \mathbb{R}^n)$ such that $E\dot{x}(t) = Ax(t) + f(t)$ for all $t \in \mathbb{R}$ and x is uniquely determined by $x(0)$, see [27]. If existence of a solution is not guaranteed for every smooth inhomogeneity, but solutions are still unique, the considered DAE is called autonomous.

Definition 3.1. A DAE $[E, A] \in \Sigma_{l,n}$ is called *autonomous*, if

$$\forall x_1, x_2 \in \mathfrak{B}_{[E,A]} : \ x_1|_{(-\infty,0)} \overset{\text{a.e.}}{=} x_2|_{(-\infty,0)} \implies x_1 \overset{\text{a.e.}}{=} x_2.$$

The notion of autonomy has been introduced by Polderman and Willems in [24, Def. 3.2.1] for general behaviors. Autonomy has been algebraically characterized for systems in $\Sigma_{l,n}$ in [3, Cor. 5.2]; the following result is an immediate consequence of this.

Lemma 3.2. *A DAE $[E, A] \in \Sigma_{l,n}$ is*

$$\textit{autonomous} \quad \Longleftrightarrow \quad \mathrm{rk}_{\mathbb{R}(s)}(sE - A) = n,$$
$$\textit{regular} \quad \Longleftrightarrow \quad l = n \ \wedge \ [E, A] \textit{ is autonomous}.$$

Remark 3.3. Using Lemma 3.2 and the quasi-Kronecker form from [4, Thm. 2.6] it is straightforward to derive the following equivalences for $[E, A] \in \Sigma_{l,n}$:

$[E, A]$ is autonomous

$$\Leftrightarrow \exists \text{ intvl. } I \subseteq \mathbb{R}\ \forall x_1, x_2 \in \mathfrak{B}_{[E,A]} : \ \left(x_1|_I \overset{\text{a.e.}}{=} x_2|_I \ \Rightarrow \ x_1 \overset{\text{a.e.}}{=} x_2 \right)$$
$$\Leftrightarrow \forall \text{ intvls. } I \subseteq \mathbb{R}\ \forall x_1, x_2 \in \mathfrak{B}_{[E,A]} : \ \left(x_1|_I \overset{\text{a.e.}}{=} x_2|_I \ \Rightarrow \ x_1 \overset{\text{a.e.}}{=} x_2 \right)$$
$$\Leftrightarrow \forall x \in \mathfrak{B}_{[E,A]} : \ \left(Ex(0) = 0 \ \Rightarrow \ x \overset{\text{a.e.}}{=} 0 \right).$$

We introduce the concepts of autonomizability and regularizability as follows.

Definition 3.4. A system $[E, A, B] \in \Sigma_{l,n,m}$ is called

(i) *autonomizable*, if there exists some $F \in \mathbb{R}^{m \times n}$ such that the DAE $[E, A + BF]$ is autonomous.

(ii) *regularizable*, if there exists some $F \in \mathbb{R}^{m \times n}$ such that the pencil $sE - (A + BF)$ is regular.

Note that regularizability implies autonomizability. In the following we derive algebraic and geometric criteria for autonomizability and regularizability, resp., as well as criteria in terms of the feedback form.

Theorem 3.5 (Characterization of autonomizability)**.** *For $[E,A,B] \in \Sigma_{l,n,m}$ and the limits $\mathscr{V}^*$ and $\mathscr{W}^*$ of the augmented Wong sequences the following conditions are equivalent:*

a) $[E,A,B]$ *is autonomizable.*

b) $\mathrm{rk}_{\mathbb{R}(s)}[sE-A,B] \geq n$.

c) In any feedback form (2) *of* $[E,A,B]$, *we have* $\ell(\beta) \leq \ell(\gamma)$.

d) $\dim(E\mathscr{V}^* + \operatorname{im} B) \geq \dim \mathscr{V}^*$.

e) $\dim(A\mathscr{W}^* + \operatorname{im} B) \geq \dim \mathscr{W}^*$.

Proof. a)⇒b): Let $F \in \mathbb{R}^{m\times n}$ be such that $[E,A+BF]$ is autonomous. Then $\mathrm{rk}_{\mathbb{R}(s)}\left[sE-(A+BF)\right] = n$ by Lemma 3.2, and thus

$$n = \mathrm{rk}_{\mathbb{R}(s)}\left[sE-(A+BF)\right] = \mathrm{rk}_{\mathbb{R}(s)}[sE-A,B]\begin{bmatrix} I_n \\ -F \end{bmatrix} \leq \mathrm{rk}_{\mathbb{R}(s)}[sE-A,B].$$

b)⇒c): This follows from

$$n \leq \mathrm{rk}_{\mathbb{R}(s)}[sE-A,B] = \mathrm{rk}_{\mathbb{R}(s)}\, S\,[sE-A,B]\begin{bmatrix} T & 0 \\ -F & V \end{bmatrix} \overset{(2)}{=} n - \ell(\beta) + \ell(\gamma).$$

c)⇒a): Without loss of generality we may assume that $[E,A,B]$ is in feedback form (2). By $\ell(\beta) \leq \ell(\gamma)$ we may define

$$\tilde{\gamma} := (\gamma_1,\ldots,\gamma_{\ell(\beta)}), \quad \hat{\gamma} := (\gamma_{\ell(\gamma)-\ell(\beta)},\ldots,\gamma_{\ell(\gamma)}).$$

Furthermore, let

$$F_\beta := \operatorname{diag}(e_1^{[\beta_1]},\ldots,e_1^{[\beta_{\ell(\beta)}]}) \in \mathbb{R}^{|\beta|\times\ell(\beta)},$$

and choose the feedback matrix

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & F_\beta^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{m\times n}.$$

Then we have

$$A+BF = \begin{bmatrix} N_\alpha^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & E_{\tilde{\gamma}}F_\beta^\top & L_{\tilde{\gamma}}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & L_{\hat{\gamma}}^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & A_{\overline{c}} \end{bmatrix},$$

so it remains to show that the subsystem

$$\left[\begin{bmatrix} L_\beta & 0 \\ 0 & K_{\tilde{\gamma}}^\top \end{bmatrix}, \begin{bmatrix} K_\beta & 0 \\ E_{\tilde{\gamma}}F_\beta^\top & L_{\tilde{\gamma}}^\top \end{bmatrix}\right]$$

is autonomous. The latter follows from the observation that by simple row permutations, where $\eta - 1 = (\eta_1 - 1,\ldots,\eta_k - 1) \in \mathbb{N}_0^k$ for any multi-index $\eta \in \mathbb{N}^k$, we obtain

$$\begin{aligned} & s\begin{bmatrix} L_\beta & 0 \\ 0 & K_{\tilde{\gamma}}^\top \end{bmatrix} - \begin{bmatrix} K_\beta & 0 \\ E_{\tilde{\gamma}}F_\beta^\top & L_{\tilde{\gamma}}^\top \end{bmatrix} \\ &= P_1\left(s\begin{bmatrix} L_\beta & 0 \\ 0 & N_{\tilde{\gamma}-1} \\ 0 & E_{\tilde{\gamma}}^\top \end{bmatrix} - \begin{bmatrix} K_\beta & 0 \\ 0 & I_{|\tilde{\gamma}-1|} \\ F_\beta^\top & 0 \end{bmatrix}\right) \\ &= P_2P_1\left(s\begin{bmatrix} N_\beta & * \\ 0 & N_{\tilde{\gamma}-1} \end{bmatrix} - \begin{bmatrix} I_{|\beta|} & 0 \\ 0 & I_{|\tilde{\gamma}-1|} \end{bmatrix}\right), \end{aligned}$$

where P_1, P_2 are appropriate permutation matrices, and the system $\left(\left[\begin{smallmatrix} N_\beta & * \\ 0 & N_{\tilde{\gamma}-1} \end{smallmatrix}\right], \left[\begin{smallmatrix} I_{|\beta|} & 0 \\ 0 & I_{|\tilde{\gamma}-1|} \end{smallmatrix}\right]\right)$ is clearly autonomous.

c)⇔d): Again, we may assume that $[E,A,B]$ is in feedback form (2). Then, using the formulas [3, (6.1) & (6.2)], we have that

$$E\mathscr{V}^* + \operatorname{im} B = \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|-\ell(\beta)} \times \operatorname{im} E_\gamma \times \{0\}^{|\delta|} \times \{0\}^{|\kappa|} \times \mathbb{R}^{n_{\overline{c}}},$$

and therefore

$$\begin{aligned} \dim \mathscr{V}^* &= |\alpha| + |\beta| + n_{\overline{c}}, \\ \dim(E\mathscr{V}^* + \operatorname{im} B) &= |\alpha| + |\beta| - \ell(\beta) + \ell(\gamma) + n_{\overline{c}}. \end{aligned}$$

This implies that $\ell(\gamma) \geq \ell(\beta)$ if, and only if, $\dim(E\mathscr{V}^* + \operatorname{im} B) \geq \dim \mathscr{V}^*$.

c)⇔e): The proof is analogous to c)⇔d) after observing that

$$A\mathscr{W}^* + \operatorname{im} B = \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|-\ell(\beta)} \times \mathbb{R}^{|\gamma|} \times \{0\}^{|\delta|} \times \mathbb{R}^{|\kappa|} \times \{0\}^{n_{\overline{c}}}.$$

□

Autonomizability along with its algebraic characterization in Theorem 3.5 b) has been first investigated in [17] (autonomy has been called *uniqueness regularity property* in this paper), however our proof based on the feedback form (2) is much simpler and we have stated it for completeness.

Criteria for regularizability can now immediately be obtained from Theorem 3.5 and Lemma 3.2, invoking that in the feedback form (2)

$$l = n \quad \Longleftrightarrow \quad \ell(\beta) = \ell(\gamma) + \ell(\delta).$$

Corollary 3.6 (Characterization of regularizability)**.** *For $[E,A,B] \in \Sigma_{l,n,m}$ and the limits $\mathscr{V}^*$ and $\mathscr{W}^*$ of the augmented Wong sequences the following conditions are equivalent:*

a) $[E,A,B]$ *is regularizable.*

b) $l = n$ *and* $\mathrm{rk}_{\mathbb{R}(s)}[sE-A,B] = n$.

c) $l = n$ *and* $\dim(E\mathscr{V}^* + \operatorname{im} B) = \dim \mathscr{V}^*$.

d) $l = n$ *and* $\dim(A\mathscr{W}^* + \operatorname{im} B) = \dim \mathscr{W}^*$.

e) In any feedback form (2) *of* $[E,A,B]$, *we have* $\ell(\delta) = 0$ *and* $\ell(\beta) = \ell(\gamma)$.

The algebraic characterization in Corollary 3.6 b) can also be found in [16] in a more general context and the geometric characterizations c) and d) have already been obtained in [23].

Remark 3.7 (PD-autonomizability and -regularizability)**.** In the present section we have considered proportional state feedback. A generalization to the case of proportional state and derivative feedback, i.e., $u(t) = F_P x(t) - F_D \dot{x}(t)$, is straightforward using the PD-feedback form from [21, Sec. 2]. We call a system $[E,A,B] \in \Sigma_{l,n,m}$ PD-autonomizable (PD-regularizable), if there exist $F_D, F_P \in \mathbb{R}^{m\times n}$ such that $[E+BF_D, A+BF_P]$ is autonomous (the pencil $s(E+BF_D)-(A+BF_P)$ is regular). In fact, it is straightforward to show that $[E,A,B]$ is PD-autonomizable (PD-regularizable) if, and only if, it is autonomizable (regularizable). For the case of regularizability this has been shown in [23].

4. Minimality

In this section we provide the framework for the regularization of DAE systems presented in Section 5. We study the concepts of behavioral equivalence and minimality which have been introduced for general behaviors in [24]. Essentially, we call two systems *behaviorally equivalent*, if their behaviors coincide. This means that we do not allow for state space or input space transformations or for feedback transformations, but on the other hand differentiation of the equations is permitted.

Definition 4.1. Two systems $[E_i,A_i,B_i] \in \Sigma_{l,n,m}$, $i=1,2$, are called *behaviorally equivalent*, if

$$\mathfrak{B}_{[E_1,A_1,B_1]} \cap \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m) = \mathfrak{B}_{[E_2,A_2,B_2]} \cap \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m);$$

we write

$$[E_1,A_1,B_1] \simeq_{\mathfrak{B}} [E_2,A_2,B_2].$$

Remark 4.2. Our definition of the behavior $\mathfrak{B}_{[E,A,B]}$ of $[E,A,B] \in \Sigma_{l,n,m}$ satisfies that, for all $(x,u) \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m)$,

$$(x,u) \in \mathfrak{B}_{[E,A,B]}$$
$$\iff \forall t \in \mathbb{R}:\ Ex(t) = Ex(0) + \int_0^t Ax(s) + Bu(s)\, \mathrm{d}s.$$

Therefore, our definition of the behavior coincides with [24, Def. 2.4.1] in the sense that for any weak solution $(\tilde{x},\tilde{u})$ according to [24, Def. 2.3.7] there exists $(x,u) \in \mathfrak{B}_{[E,A,B]}$ such that $(\tilde{x},\tilde{u}) \overset{\text{a.e.}}{=} (x,u)$. In Definition 4.1 it is necessary to restrict the behaviors to $\mathscr{C}^\infty$ because otherwise the two systems

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} x(t) = I_2 x(t) \quad \text{and} \quad 0 = I_2 x(t)$$

would not be behaviorally equivalent since their behaviors differ in the second variable which is zero everywhere for the first system, but only zero almost everywhere for the second system (thus containing any function in $\mathscr{L}^1_{\text{loc}}$ with this property).

Note that it is no restriction to consider behavioral equivalence only for systems with the same number of rows. If $[E_i,A_i,B_i] \in \Sigma_{l_i,n,m}$ with $l_1 > l_2$, then we may simply add $l_1 - l_2$ zero rows to E_2, A_2 and B_2 which does not change the behavior of $[E_2,A_2,B_2]$.

As we have mentioned before, for behavioral equivalence we allow that some of the equations in (1) are differentiated. This leads to a transformation of the form $U(\frac{\mathrm{d}}{\mathrm{d}t})(\frac{\mathrm{d}}{\mathrm{d}t}E - A)x(t) - U(\frac{\mathrm{d}}{\mathrm{d}t})Bu(t) = 0$ with some $U(s) \in \mathbb{R}[s]^{l\times l}$. Furthermore, since the behaviors must coincide (on $\mathscr{C}^\infty$) the transformation $U(s)$ must be reversible, that means $U(s)$ must be invertible over $\mathbb{R}[s]$, i.e., unimodular. As shown in [24, Thms. 2.5.4 & 3.6.2] this is exactly the set of transformations that characterizes behavioral equivalence; this is summarized in the following lemma.

Lemma 4.3 (Behavioral equivalence by unimodular transformation)**.** *Let* $[E_i,A_i,B_i] \in \Sigma_{l,n,m}$*,* $i=1,2$*. Then* $[E_1,A_1,B_1] \simeq_{\mathfrak{B}} [E_2,A_2,B_2]$ *if, and only if, there exists* $U(s) \in \mathbf{Gl}_l(\mathbb{R}[s])$ *such that*

$$[sE_1 - A_1, -B_1] = U(s)[sE_2 - A_2, -B_1].$$

The concept of behavioral equivalence allows to consider minimality of the description of the given behavior. That is, we seek a DAE (1) with a minimal number of equations that describes the behavior.

Definition 4.4. A system $[E,A,B] \in \Sigma_{l,n,m}$ is called *minimal*, if

$$\forall k \in \{1,\dots,l\}\ \forall [\tilde{E},\tilde{A},\tilde{B}] \in \Sigma_{k,n,m}:$$
$$\left([E,A,B] \simeq_{\mathfrak{B}} \left[\begin{bmatrix}\tilde{E}\\ 0_{l-k,n}\end{bmatrix}, \begin{bmatrix}\tilde{A}\\ 0_{l-k,n}\end{bmatrix}, \begin{bmatrix}\tilde{B}\\ 0_{l-k,m}\end{bmatrix}\right] \implies k = l\right).$$

In the usual definition given in [24, Def. 2.5.24], minimality is defined via polynomial matrix descriptions $R(s) \in \mathbb{R}[s]^{g\times q}$ of a given system, which satisfies the higher order DAE $R(\frac{\mathrm{d}}{\mathrm{d}t})w(t) = 0$. Then, loosely speaking, $R(s)$ is called minimal, if for any $k \in \{1,\dots,g\}$ and any $\tilde{R}(s) \in \mathbb{R}[s]^{k\times q}$ the polynomial matrices $R(s)$ and $\left[\begin{smallmatrix}\tilde{R}(s)\\ 0_{g-k,q}\end{smallmatrix}\right]$ induce the same behavior only if $k = g$. Therefore, the definition of minimality of $[E,A,B] \in \Sigma_{l,n,m}$ as in Definition 4.4 seems to be weaker than minimality of the polynomial matrix $[sE - A, -B]$ in the sense of [24]. However, in Theorem 4.5 we show that the two notions are indeed equivalent.

It has been shown in [24] that minimality of a polynomial matrix $R(s)$ is equivalent to full rank of $R(s)$ over $\mathbb{R}[s]$. We show that the same statement is true for systems in $\Sigma_{l,n,m}$. Furthermore, we derive a new geometric characterization and a characterization in terms of the feedback form.

Theorem 4.5 (Characterization of minimality)**.** *For* $[E,A,B] \in \Sigma_{l,n,m}$ *and the limits* $\mathscr{V}^*$ *and* $\mathscr{W}^*$ *of the augmented Wong sequences the following conditions are equivalent:*

a) $[E,A,B]$ *is minimal.*

b) $\mathrm{rk}_{\mathbb{R}[s]}[sE - A, B] = l$ *(that is,* $[E,A,B]$ *is minimal in the sense of [24]).*

c) In any feedback form (2) *of* $[E,A,B]$*, we have* $\ell(\delta) = 0$*.*

d) $E\mathscr{V}^* + A\mathscr{W}^* + \operatorname{im} B = \mathbb{R}^l$*.*

Proof. a)⇒b): By [4, Thm. 2.6] there exist $S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that

$$S[E,0]T = \begin{bmatrix} E_P & 0 & 0 \\ 0 & E_R & 0 \\ 0 & 0 & E_Q \end{bmatrix}, \quad S[A,B]T = \begin{bmatrix} A_P & 0 & 0 \\ 0 & A_R & 0 \\ 0 & 0 & A_Q \end{bmatrix},$$

where $E_P, A_P \in \mathbb{R}^{l_P \times n_P}, E_R, A_R \in \mathbb{R}^{n_R \times n_R}, E_Q, A_Q \in \mathbb{R}^{l_Q \times n_Q}$ are such that $l_P < n_P$ (or $l_P = n_P = 0$), $l_Q > n_Q$ (or $l_Q = n_Q = 0$), $sE_R - A_R$ is regular and, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned} \operatorname{rk}_{\mathbb{C}}(\lambda E_P - A_P) &= l_P, \quad \operatorname{rk} E_P = l_P \\ \operatorname{rk}_{\mathbb{C}}(\lambda E_Q - A_Q) &= n_Q, \quad \operatorname{rk} E_Q = n_Q. \end{aligned}$$

It follows that $\operatorname{rk}_{\mathbb{R}[s]}(sE_P - A_P) = l_P$, $\operatorname{rk}_{\mathbb{R}[s]}(sE_R - A_R) = n_R$ and $\operatorname{rk}_{\mathbb{R}[s]}(sE_Q - A_Q) = n_Q$. Suppose that $[sE - A, -B]$ does not have full row rank, then it follows that $l_Q > 0$. By [4, Lem. 3.1] there exist $M(s) \in \mathbb{R}[s]^{n_Q \times l_Q}, K(s) \in \mathbb{R}[s]^{(l_Q - n_Q) \times l_Q}$ such that $U_3(s) := \begin{bmatrix} M(s) \\ K(s) \end{bmatrix} \in \mathbf{Gl}_{l_Q}(\mathbb{R}[s])$ and

$$U_3(s)(sE_Q - A_Q) = \begin{bmatrix} I_{n_Q} \\ 0 \end{bmatrix}.$$

Then, for $U(s) := \operatorname{diag}(I_{l_P}, I_{n_R}, U_3(s))S \in \mathbf{Gl}_l(\mathbb{R}[s])$ we find that

$$U(s)[sE - A, -B] = \begin{bmatrix} sE_P - A_P & 0 & 0 \\ 0 & sE_R - A_R & 0 \\ 0 & 0 & I_{n_Q} \\ 0 & 0 & 0 \end{bmatrix} T^{-1},$$

and hence, invoking Lemma 4.3, $[E,A,B]$ is not minimal, a contradiction.

b)⇒a): This is a direct consequence of the definition of minimality and Lemma 4.3.

b)⇔c): This follows from

$$\operatorname{rk}_{\mathbb{R}(s)}[sE - A, B] = \operatorname{rk}_{\mathbb{R}(s)} S\,[sE - A, B] \begin{bmatrix} T & 0 \\ -F & V \end{bmatrix} \overset{(2)}{=} l - \ell(\delta).$$

c)⇔d): As shown in the proof of Theorem 3.5 it is no loss of generality to assume that $[E,A,B]$ is in feedback form (2) and that then

$$\begin{aligned} E\mathscr{V}^* + A\mathscr{W}^* + \operatorname{im} B \\ = \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta| - \ell(\beta)} \times \mathbb{R}^{|\gamma|} \times \{0\}^{|\delta|} \times \mathbb{R}^{|\kappa|} \times \mathbb{R}^{n_{\overline{c}}}. \end{aligned}$$

The assertion is now an immediate consequence of the above. □

A careful inspection of the proof of Theorem 4.5 also shows how, for given $[E,A,B] \in \Sigma_{l,n,m}$, a minimal system $[\tilde{E}, \tilde{A}, \tilde{B}] \in \Sigma_{k,n,m}$ can be found such that

$$[E,A,B] \simeq_{\mathfrak{B}} \left[\begin{bmatrix} \tilde{E} \\ 0_{l-k,n} \end{bmatrix}, \begin{bmatrix} \tilde{A} \\ 0_{l-k,n} \end{bmatrix}, \begin{bmatrix} \tilde{B} \\ 0_{l-k,m} \end{bmatrix} \right].$$

5. Regularization by reinterpretation of variables

In this subsection we investigate the regularization and index reduction of a control system $[E,A,B] \in \Sigma_{l,n,m}$ by a combination of behavioral equivalence transformation, proportional state feedback and reinterpretation of variables (i.e., states as inputs and/or inputs as states). The aim is to obtain a new system $[\hat{E}, \hat{A}, \hat{B}] \in \Sigma_{\hat{n},\hat{n},\hat{m}}$ such that $s\hat{E} - \hat{A}$ is regular and has index at most 1. The index $\nu \in \mathbb{N}_0$ of a regular matrix pencil $sE - A \in \mathbb{R}[s]^{n \times n}$ is defined via its (quasi-)Weierstraß form [2, 18, 20]: if for some $S, T \in \mathbf{Gl}_n(\mathbb{R})$

$$S(sE - A)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & sN - I_{n-r} \end{bmatrix},$$

$$\text{then} \quad \nu := \begin{cases} 0, & \text{if } r = n, \\ \min\{\, k \in \mathbb{N}_0 \mid N^k = 0 \,\}, & \text{if } r < n. \end{cases}$$

The index is independent of the choice of S, T and can be computed via the Wong sequences corresponding to $sE - A$ as shown in [2].

In order to obtain a regular index-1 system we allow for feedback transformations as discussed in Section 3 and for behavioral equivalence transformations as discussed in Section 4; invoking Lemma 4.3, the latter corresponds to a unimodular left transformation. Additionally, we allow for a (possible) permutation of state and input variables. To be precise, we seek for $U(s) \in \mathbf{Gl}_l(\mathbb{R}[s])$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$, $F \in \mathbb{R}^{m \times n}$ and a permutation matrix $P \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that

$$U(s)\,[sE - A, -B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} P = \begin{bmatrix} s\hat{E} - \hat{A} & -\hat{B} \\ 0 & 0 \end{bmatrix},$$

where $s\hat{E} - \hat{A} \in \mathbb{R}[s]^{\hat{n} \times \hat{n}}$ is regular and has index at most 1.

The motivation for the additional permutation of variables (which is also deemed *reinterpretation* [13]) is as follows: Assume that we are given a minimal system $[E,A,B] \in \Sigma_{l,n,m}$ in feedback form (2). Then $\ell(\delta) = 0$ by Theorem 4.5. In this system the pencil $sE - A$ is regular and of index at most 1 except for the nonregular block

$$\begin{bmatrix} sL_\beta - K_\beta & 0 \\ 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix}.$$

The block $sL_\beta - K_\beta$ corresponds to an underdetermined DAE and contains $\ell(\beta)$ free variables. In the context of the behavioral approach, where it is not distinguished between the meaning of the variables, the free variables are typically treated as inputs, since “they can be viewed as unexplained by the model and imposed on the system by the environment” [24]. It is part of the methodology of the behavioral approach that the interpretation of variables (i.e., which variables are states and which ones are inputs) should be done after the analysis of the model reveals the free variables. This approach obeys the physical meaning of the system variables and it may turn out that in the original model the choice of states and inputs was inappropriate.

In contrast to the β-block which contains free variables, the γ-block contains constraints on some of the input variables, since the corresponding DAE reads

$$\tfrac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top x_3(t) = L_\gamma^\top x_3(t) + E_\gamma u_2(t).$$

Any solution (x_3, u_2) of this DAE is almost everywhere zero. In particular, the variables u_2 are no free variables in the system, hence in the context of the behavioral approach they cannot be viewed as inputs, but must be viewed as states. The "reinterpretation" can be achieved by a multiplication of the augmented pencil $[sE-A, -B]$ from the right with a permutation matrix. It is remarkable that after this reinterpretation the new system is regular and of index at most 1.

The general case is treated in the following theorem which is the main result of this section.

Theorem 5.1 (Regularization)**.** *Let $[E,A,B] \in \Sigma_{l,n,m}$ and use the notation from Theorem 2.1. Then there exist $U(s) \in \mathbf{Gl}_l(\mathbb{R}[s])$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$, $F \in \mathbb{R}^{m\times n}$ and a permutation matrix $P \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that, with $\mu := \ell(\beta) - \ell(\gamma) \in \mathbb{Z}$,*

$$U(s)\,[sE-A, -B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} P = \begin{bmatrix} s\hat{E}-\hat{A} & -\hat{B} \\ 0_{\ell(\delta),n-\mu} & 0_{\ell(\delta),m+\mu} \end{bmatrix},$$

where $s\hat{E}-\hat{A} \in \mathbb{R}[s]^{(n-\mu)\times(n-\mu)}$ is regular and has index at most 1; in particular $[\hat{E},\hat{A},\hat{B}]$ is minimal.

Proof. It is immediate from (2) that $n-\mu \geq 0$ and $m+\mu \geq 0$. Let $S \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$, $F \in \mathbb{R}^{m\times n}$ be such that (2) holds. Then

$$Z(s)S[sE-A,-B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} =$$

$$\left[\begin{bmatrix} sI_{|\alpha|}-N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sL_\beta - K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & -K_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{|\delta|-\ell(\delta)} & 0 & 0 \\ 0 & 0 & 0 & 0_{\ell(\delta),|\delta|-\ell(\delta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}}-A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} -E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right] \tag{4}$$

where

$$Z(s) = \operatorname{diag}\left(I_{|\alpha|},\, I_{|\beta|-\ell(\beta)},\, -\sum_{k=0}^{\nu_\gamma-1} s^k N_\gamma^k,\, P_\delta(s),\, -\sum_{k=0}^{\nu_\kappa-1} s^k N_\kappa^k,\, I_{n_{\overline{c}}} \right)$$

is unimodular with $\nu_\gamma = \max\{\gamma_1,\ldots,\gamma_{\ell(\gamma)}\}$, $\nu_\kappa = \max\{\kappa_1,\ldots,\kappa_{\ell(\kappa)}\}$, and

$$P_\delta(s) = P_1 \cdot \operatorname{diag}\left(\begin{bmatrix} 0_{\delta_i-1,1} & -\sum_{k=0}^{\delta_i-2} s^k \left(N_{\delta_i-1}^\top\right)^k \\ 0_{1,1} & 0_{1,\delta_i-1} \end{bmatrix} \right)_{j=1,\ldots,\ell(\delta)}$$

with the permutation matrix

$$P_1 = \begin{bmatrix} L_\delta \\ E_\delta^\top \end{bmatrix}.$$

By a suitable permutation of the $\ell(\delta)$ zero rows in (4) via a permutation matrix $P_2 \in \mathbf{Gl}_l(\mathbb{R})$ and a permutation of columns via the permutation matrix

$$P_3 := \begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta^\top & 0 & 0 & 0 & 0 & 0 & E_\beta & 0 \\ 0 & 0 & L_\gamma & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{|\delta|-\ell(\delta)} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{c}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & I_{\ell(\alpha)} & 0 & 0 \\ 0 & 0 & E_\gamma^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{m-\ell(\alpha)-\ell(\beta)} \end{bmatrix}$$

we obtain that

$$P_2 Z(s)S[sE-A,-B] \begin{bmatrix} T & 0 \\ F & V \end{bmatrix} P_3 = \begin{bmatrix} s\hat{E}-\hat{A} & -\hat{B} \\ 0_{\ell(\delta),n-\mu} & 0_{\ell(\delta),m+\mu} \end{bmatrix},$$

where

$$s\hat{E}-\hat{A} = \begin{bmatrix} sI_{|\alpha|}-N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sI_{|\beta|-\ell(\beta)}-N_{\beta-1}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & -I_{|\gamma|} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{|\delta|-\ell(\delta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & -I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}}-A_{\overline{c}} \end{bmatrix},$$

$$\hat{B} = \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & E_{\beta-1} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

Recall that $\eta - 1 = (\eta_1 - 1, \ldots, \eta_k - 1) \in \mathbb{N}_0^k$ for some multi-index $\eta \in \mathbb{N}^k$. It remains to observe that $s\hat{E}-\hat{A} \in \mathbb{R}[s]^{(n-\ell(\beta)+\ell(\gamma))\times(n-\ell(\beta)+\ell(\gamma))}$ is regular and has index at most 1 and hence we also have that $[\hat{E},\hat{A},\hat{B}]$ is minimal. ☐

Remark 5.2.

(i) The proof of Theorem 5.1 is constructive. The matrices S,T,V and F which put $[E,A,B]$ in feedback form are constructed in [21]. The unimodular matrix $U(s)$ and the permutation matrix P are given explicitly in the proof of Theorem 5.1.

(ii) Theorem 5.1 yields that, in terms of a feedback form (2) of $[E,A,B]$,

- the number of redundant equations is $\ell(\delta)$,
- the number of free states is $\ell(\beta)$,
- the number of constrained inputs is $\ell(\gamma)$.

Furthermore, the regular index-1 subsystem $[\hat{E},\hat{A},\hat{B}]$ of $[E,A,B]$ in Theorem 5.1 which is given by (5) is fully determined by the augmented Wong sequences up to a permutation of the entries of the multi-indices and up to similarity of $A_{\overline{c}}$; this is a direct consequence of Theorem 2.2 and Proposition 2.3. Therefore, if the transformation leading to this system is not of interest, the augmented Wong sequences provide a simple and easily implementable tool for the calculation of $[\hat{E},\hat{A},\hat{B}]$.

(iii) Regularization plays an important role in optimal control with differential-algebraic constraints: The classical techniques for the solution of the optimal control problem such as, for instance, Riccati equations and Langrange multiplier-based approaches can only be applied to regular systems [19, 22]. A preliminary procedure leading to a regular system therefore enables us to use the classical approaches of optimal control theory.

(iv) Note that in [13] the general procedure of regularization and index reduction has already been developed by means of an iterative numerical procedure. In contrast to these results, Theorem 5.1 provides an explicit construction of the transformations leading to a regular index-1 subsystem. Furthermore, as explained in item (ii), the augmented Wong sequences provide a simple method for the explicit and efficient calculation of this subsystem.

6. Conclusion

We have considered linear time-invariant DAE control systems and addressed the question whether there exists a feedback which renders the closed-loop system regular. We have shown that this property can equivalently be characterized by simple algebraic and geometric conditions. Moreover, we have considered the slightly more general problem of existence of a feedback such that an autonomous closed-loop system is obtained. The proofs are constructive: The feedback matrix can be obtained by using the feedback canonical form [21].

Thereafter we have equivalently characterized minimality of DAE control systems. The latter is a property which, loosely speaking, states that the behavior of a control system cannot be described by a DAE with fewer equations. We have proved that this concept is equivalent to minimality in the sense of [24]. An equivalent geometric condition for minimality has been derived as well.

These results have been the basis for our considerations for systems which are not regularizable by feedback: It has been shown that each system can be reduced to a regular index-1 system after feedback, minimalization and suitable reinterpretation of state and input variables.

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