Received <day> <Month>, <year>; | Revised <day> <Month>, <year>; | Accepted <day> <Month>, <year>

DOI: xxx/xxxx

RESEARCH ARTICLE

The Funnel Pre-Compensator†

Thomas Berger[1,*] | Timo Reis[1]

[1]Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany

Correspondence
*Email: thomas.berger@uni-hamburg.de

Abstract

We introduce the funnel pre-compensator as a novel and simple adaptive pre-compensator of “high-gain type”. We show that this pre-compensator is feasible for a large class of signal pairs, which satisfy a certain relationship. We show that the funnel pre-compensator guarantees prescribed transient behavior of the compensator error, it is of low complexity and inherently robust since its design is model-free. As an application in adaptive control of nonlinear systems, a cascade of funnel pre-compensators is exploited to obtain an artificial output with explicitly known derivatives which tracks the system output with prescribed transient behavior. In some important cases the interconnection of the system with the pre-compensator cascade is shown to have input-to-state stable internal dynamics. This guarantees feasibility of a novel funnel controller which consists of a funnel pre-compensator cascade in conjunction with a recently developed funnel controller for systems with arbitrary relative degree. We illustrate the application of this interconnection for some mechanical systems.

KEYWORDS:
funnel pre-compensator; high-gain observer; nonlinear systems; funnel control; adaptive control; internal dynamics.

Nomenclature:

$\mathbb{R}_{\geq 0}$	$= [0, \infty)$
$\mathbf{Gl}_n(\mathbb{R})$	the group of invertible matrices in $\mathbb{R}^{n\times n}$
$\sigma(A)$	the spectrum of $A \in \mathbb{R}^{n\times n}$
$\mathcal{L}^\infty_{\text{loc}}(I \to \mathbb{R}^n)$	the set of locally essentially bounded functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval
$\mathcal{W}^{k,\infty}_{\text{loc}}(I \to \mathbb{R}^n)$	the set of k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ with locally essentially bounded first k weak derivatives $f, \dots, f^{(k)}$
$\mathcal{L}^\infty(I \to \mathbb{R}^n)$	the set of essentially bounded functions $f : I \to \mathbb{R}^n$ with norm
$\|f\|_\infty$	$= \operatorname{ess\,sup}_{t\in I} \|f(t)\|$
$\mathcal{W}^{k,\infty}(I \to \mathbb{R}^n)$	the set of k-times weakly differentiable functions $f : I \to \mathbb{R}^n$ such that $f, \dots, f^{(k)} \in \mathcal{L}^\infty(I \to \mathbb{R}^n)$
$\mathcal{C}^k(I \to \mathbb{R}^n)$	the set of k-times continuously differentiable functions $f : I \to \mathbb{R}^n$
$\mathcal{C}(I \to \mathbb{R}^n)$	$= \mathcal{C}^0(I \to \mathbb{R}^n)$
$f\|_J$	restriction of the function $f : I \to \mathbb{R}^n$ to $J \subseteq I$

†This work was supported by the Klaus Tschira Stiftung and the German Research Foundation (Deutsche Forschungsgemeinschaft) via the grant BE 6263/1-1.

1 | INTRODUCTION

In the present paper we propose a novel and simple adaptive pre-compensator of “high-gain type”, the *funnel pre-compensator*. In the case of unknown output derivatives, the funnel pre-compensator may be used to obtain an artificial output, the derivatives of which are known explicitly and which evolves within a prescribed performance funnel around the original output.

In the recent paper [5] a funnel controller for nonlinear systems with arbitrary known relative degree is developed, which resolves the longstanding open problem of how to handle relative degree higher than one in high-gain adaptive control, cf. [22, 24, 35]. Earlier works suggested a “backstepping” procedure in conjunction with an input filter, see [27, 28], or a bang-bang funnel controller, see [34]. Drawbacks are that the backstepping procedure in [27, 28] is quite complicated and impractical since it involves high powers of a gain function which typically takes large values, cf. [20, Sec. 4.4.3], and the approaches in [5, 34] require availability of the output derivatives which means in practice that measurements have to be differentiated. The latter is an ill-posed problem in particular in the presence of noise, see e.g. [20, Sec. 1.4.4].

In view of this, the following control problem is of interest: design a (dynamic) output error feedback $u(t) = F_1\big(t, e(t), z(t)\big)$, $\dot z(t) = F_2\big(t, e(t), z(t)\big)$, where $e(t) = y(t) - y_{\rm ref}(t)$ is the tracking error and $y_{\rm ref}$ the reference signal, such that e has prescribed performance. We stress again that the derivatives of the output may not be known to the controller. In the present paper, we introduce the funnel pre-compensator as a novel tool which may help with the solution of this problem. While we do not focus on the solution itself (and hence, control design is not the main topic of this work), it is our guiding principle and as an application of the funnel pre-compensator we present controllers which achieve the above described control objective for two relevant system classes in Section 4. For an alternative approach to the tracking problem by output feedback using sliding mode controllers see e.g. [36, 37, 38]; however these controllers do not guarantee the prescribed performance of the tracking error.

The funnel pre-compensator resembles (and was inspired by) an (adaptive) high-gain observer and was called “funnel observer” in the preprint [7]; see the classical works [17, 33, 40, 44] and the recent survey [32] for literature on high-gain observers. However, the funnel pre-compensator does not have the properties of a high-gain observer since the derivatives of the output are not approximated. Rather than that, an alternative “artificial output” is derived which evolves within a prescribed performance funnel around the original output, and derivatives of which are computed exactly.

Nevertheless, since the funnel pre-compensator carries the structure of a high-gain observer, some of its benefits are retained. One advantage of high-gain observers is that they can be used to estimate the system states without knowing the exact parameters (in contrast to observer synthesis, see e.g. [13, 16] and the references therein); only some structural assumptions, such as a known relative degree, are necessary. Furthermore, they are robust with respect to input noise. The drawback is that in most cases it is not known a priori how large the high-gain parameter k in the observer must be chosen and appropriate values must be identified by offline simulations. If k is chosen unnecessarily large, the sensitivity to measurement noise increases dramatically.

In order to resolve these problems, the constant high-gain parameter k has been replaced by an adaptation scheme in [11]. The gain $k(t)$ is determined by a differential equation depending on the observation error. This leads to a monotonically increasing $k(t)$ as long as the observation error lies outside a predefined λ-strip $[-\lambda, \lambda]$, and it stops increasing as soon as the error enters the strip. The advantage of this observer is that $k(t)$ is adapted online to the actual needed value, which also leads to lower high-gain parameters in general. However, $k(t)$ is monotonically non-decreasing and hence susceptible to unwarranted increase due to perturbations to the system. Furthermore, while convergence of the observation error to the λ-strip is guaranteed, its transient behavior cannot be influenced.

Another high-gain observer with gain adaptation law is proposed in [41]. Compared to [11] it resolves the drawback of monotonically non-decreasing gain, however a certain condition on the system is necessary which either requires exact knowledge of the high-gain parameter of the system or boundedness of the input $u(t)$. Furthermore, the adaptation law in [41] is not able to influence the transient behavior of the observation error, but only its mean value.

Inspired by the adaptive high-gain observer in [11], we introduce the following funnel pre-compensator:

$$\boxed{\begin{array}{rcll} \dot z_1(t) &=& z_2(t) + \big(q_1 + p_1 k(t)\big)\cdot(y(t) - z_1(t)), & \qquad k(t) = \dfrac{1}{1 - \varphi(t)^2\|y(t) - z_1(t)\|^2}, \\ \dot z_2(t) &=& z_3(t) + \big(q_2 + p_2 k(t)\big)\cdot(y(t) - z_1(t)), & \\ &\vdots& & \\ \dot z_{r-1}(t) &=& z_r(t) + \big(q_{r-1} + p_{r-1} k(t)\big)\cdot(y(t) - z_1(t)), & \\ \dot z_r(t) &=& \big(q_r + p_r k(t)\big)\cdot(y(t) - z_1(t)) + \tilde\Gamma u(t), & \end{array}} \tag{1}$$

where the design parameters $p_i > 0$, $q_i > 0$, $\tilde\Gamma \in \mathbb{R}^{p\times m}$ and the function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ are explained in detail in Section 2.

We like to emphasize that:

- The proposed adaptation scheme for $k(t)$ is simple, non-dynamic, non-monotone,
- and it guarantees prescribed transient behavior of the compensator error;
- the pre-compensator (1) is of low complexity and inherently robust since its design is model-free.

Another advantage of the funnel pre-compensator (1) is that no higher powers of the gain function k are involved in (1), thus typical challenges in the numerical implementation are avoided without the need for any estimates of the underlying model as discussed for high-gain observers in [1, 31].

In contrast to other approaches, the signals u and y given to the funnel pre-compensator (1) are not necessarily the input and output corresponding to some system or plant. We only assume that they are some signals belonging to the following, very large set:

$$\mathcal{P}_r := \left\{ (u,y) \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \times \mathcal{W}^{r,\infty}_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^p) \;\middle|\; \begin{array}{l} \exists\, \Gamma \in \mathcal{C}^1(\mathbb{R}_{\geq 0} \to \mathbb{R}^{m\times p}) : \\ \Gamma y^{(r-1)} \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \\ \frac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma y^{(r-1)}\right) - u \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m) \end{array} \right\},$$

where $r \in \mathbb{N}$. The situation is depicted in Figure 1 . We stress that knowledge of the matrix-valued function Γ is *not* assumed, only that of the signals u and y (which can be viewed as the external signals corresponding to some plant) and the number $r \in \mathbb{N}$ (which can be viewed as the "relative degree" of the possibly underlying plant). For instance, if $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is the input and $y \in \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is the output of the system

$$y^{(r)}(t) = f\left(d(t), y(t), \dots, y^{(r-1)}(t)\right) + Bu(t),$$

where f is a suitable continuous function, d is a bounded disturbance, $B \in \mathbf{Gl}_m(\mathbb{R})$ and $y, \dot{y}, \dots, y^{(r-1)}$ are bounded, then $(u,y) \in \mathcal{P}_r$ with $\Gamma = B^{-1}$. Clearly, the signal set $\mathcal{P}_r$ allows for much larger classes of systems involving functional-differential, partial differential and/or differential-algebraic equations, see e.g. [5, 6, 26] and Section 3. We will show that for signals $(u,y) \in \mathcal{P}_r$ with $r \geq 2$, the funnel pre-compensator (1) has a weakly differentiable and bounded solution $(z_1, \dots, z_r)$ such that k is bounded and

$$\exists\, \varepsilon > 0\; \forall\, t > 0 : \; \|y(t) - z_1(t)\| < \varphi(t)^{-1} - \varepsilon. \tag{2}$$

Furthermore, the derivative $\dot{z}_1$ is known explicitly.

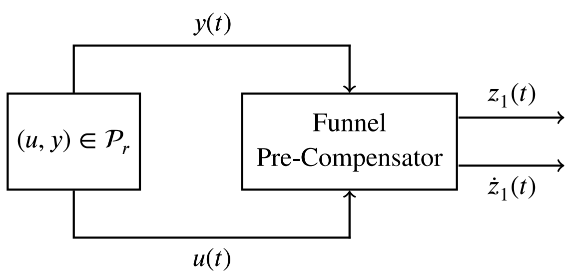

FIGURE 1 Application of the funnel pre-compensator (1) to signals $(u,y) \in \mathcal{P}_r$.

We stress that condition (2) means prescribed transient behavior of the compensator error $e_1(t) := y(t) - z_1(t)$ in the sense that it is pointwise below a given funnel function $1/\varphi$, see Figure 2 . To achieve this, the compensator gain will be increased whenever $\|e_1(t)\|$ approaches the funnel boundary. High values of the gain function lead to a faster decay of the compensator error.

While the funnel pre-compensator yields $\dot{z}_1$ explicitly, $\ddot{z}_1$ depends on $\dot{y}$ and hence higher derivatives remain unknown. To resolve this problem we show that an application of a cascade of funnel pre-compensators yields

- an estimate z for the signal y with prescribed transient behavior and
- the derivatives $\dot{z}, \dots, z^{(r-1)}$ are known explicitly.

As an application of this cascade in adaptive control we investigate its use for output trajectory tracking by funnel control. Given a certain class of systems with input-to-state stable internal dynamics, we show that the interconnection of the system with the

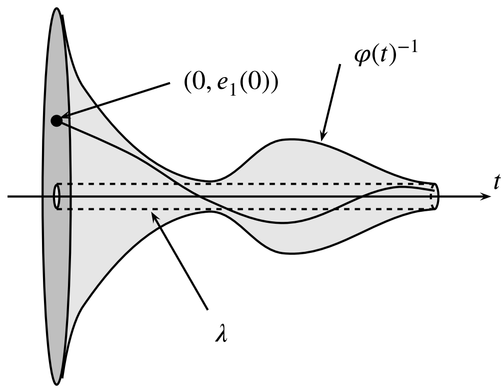

FIGURE 2 Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$ for $t > 0$.

pre-compensator cascade has again input-to-state stable internal dynamics. This allows for the application of available funnel control techniques to the interconnection in order to achieve tracking with prescribed transient behavior of the tracking error without the requirement to compute derivatives of the system output as in [5]. However, this result is limited to systems with relative degree two or three; for higher relative degree it remains an open problem.

The present paper is organized as follows: The funnel pre-compensator is introduced in Section 2 and feasibility is proved. Furthermore, we show that the funnel pre-compensator cascade achieves the desired properties. The application in output trajectory tracking is discussed in Section 3. The interconnection of the funnel pre-compensator with the funnel controller from [5] as a funnel controller for systems with higher relative degree which does not require the output derivatives, is presented in Section 4 for relative degree two and three. A simulation of this interconnection for a mass-spring system mounted on a car is provided in Section 5. Some conclusions are given in Section 6.

2 | THE FUNNEL PRE-COMPENSATOR

In this section we consider the funnel pre-compensator (1) as a new adaptive pre-compensator of high-gain type. Following the methodology of funnel control, see [24, 26] and the references therein, it is our aim to construct a dynamical system which receives signals $(u, y) \in \mathcal{P}_r$ and has output z such that the derivatives of z up to order $r-1$, where $r \in \mathbb{N}$, are known explicitly and the error $e = y - z$ evolves within a prescribed performance funnel

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^p \;\middle|\; \varphi(t)\|e\| < 1 \right\}. \tag{3}$$

This performance funnel is determined by a function φ belonging to

$$\Phi := \left\{ \varphi \in C^1(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \varphi, \dot{\varphi} \text{ are bounded, } \varphi(s) > 0 \text{ for all } s > 0, \text{ and } \liminf_{s\to\infty} \varphi(s) > 0 \right\}.$$

Note that the funnel boundary is given by the reciprocal of φ, see Figure 2 . The case $\varphi(0) = 0$ is explicitly allowed and puts no restriction on the initial value since $\varphi(0)\|e(0)\| < 1$; in this case the funnel boundary $1/\varphi$ has a pole at $t = 0$.

An important property of the funnel class Φ is that each performance funnel $\mathcal{F}_\varphi$ with $\varphi \in \Phi$ is bounded away from zero, i.e., due to boundedness of φ there exists $\lambda > 0$ such that $1/\varphi(t) \geq \lambda$ for all $t > 0$. The funnel boundary is not necessarily monotonically decreasing, while in most situations it is convenient to choose a monotone funnel. However, there are situations where widening the funnel over some later time interval might be beneficial, e.g., in the presence of periodic disturbances or when the signal y changes strongly.

Our first objective is robust estimation of the signal y so that the derivative of the compensator state z_1 in (1) is known explicitly, the compensator error $e_1 = y - z_1$ evolves within the funnel $\mathcal{F}_\varphi$ and all variables are bounded. To achieve this objective we consider the funnel pre-compensator (1) with initial conditions

$$z_i(0) = z_i^0 \in \mathbb{R}^p, \quad i = 1, \dots, r, \tag{4}$$

where $\varphi \in \Phi$, $\tilde{\Gamma} \in \mathbb{R}^{p\times m}$ and $q_i > 0$, $p_i > 0$ for all $i = 1, \dots, r$. The functions $z_i : \mathbb{R}_{\geq 0} \to \mathbb{R}^p$, $i = 1, \dots, r$, are the compensator states and $k : \mathbb{R}_{\geq 0} \to [1, \infty)$ is the compensator gain. The constants $q_i > 0$ are such that the matrix

$$A = \begin{bmatrix} -q_1 & 1 & \\ \vdots & & \ddots \\ -q_{r-1} & & & 1 \\ -q_r & & & 0 \end{bmatrix} \in \mathbb{R}^{r\times r}$$

is Hurwitz, i.e., $\operatorname{Re} \lambda < 0$ for all $\lambda \in \sigma(A)$. The constants p_i depend on the choice of the q_i in the following way: Let $Q = Q^\top > 0$ and

$$P = \begin{bmatrix} P_1 & P_2 \\ P_2^\top & P_4 \end{bmatrix}, \quad P_1 \in \mathbb{R},\ P_2 \in \mathbb{R}^{1\times(r-1)},\ P_4 \in \mathbb{R}^{(r-1)\times(r-1)}$$

be such that

$$A^\top P + PA + Q = 0, \quad P = P^\top > 0.$$

The matrix P depends only on the choice of the constants q_i and the matrix Q. The constants p_i must then satisfy

$$\begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} = P^{-1} \begin{pmatrix} P_1 - P_2 P_4^{-1} P_2^\top \\ 0 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -P_4^{-1} P_2^\top \end{pmatrix}. \tag{5}$$

In passing, we note for later use that any such P satisfies

$$(1, -P_2 P_4^{-1})\, P \begin{pmatrix} 1 \\ (-P_2 P_4^{-1})^\top \end{pmatrix} = P_1 - P_2 P_4^{-1} P_2^\top > 0. \tag{6}$$

We will see later that the above condition guarantees that P defines a quadratic Lyapunov function for the error dynamics of the funnel pre-compensator.

While the funnel pre-compensator (1) resembles a high-gain observer, it is different in its structure when compared to the high-gain observers in [11, 45], where the gain enters with power k^i into the equation for $\dot{z}_i$. Furthermore, the constants q_i are not present in [11, 45], but we show that they are important to ensure boundedness of the error dynamics even when $k(t)$ is small.

Although the pre-compensator (1) is a nonlinear and time-varying system, it is simple in its structure and its dimension depends only on the "relative degree" r given by $\mathcal{P}_r$. The set $\mathcal{P}_r$ of signals u and y ensures error evolution within the funnel: by the design of the pre-compensator (1), the gain $k(t)$ increases if the norm of the error $\|y(t) - z_1(t)\|$ approaches the funnel boundary $1/\varphi(t)$, and decreases if a high gain is not necessary.

For a sketch of the construction of the funnel pre-compensator (1) see also Figure 3 .

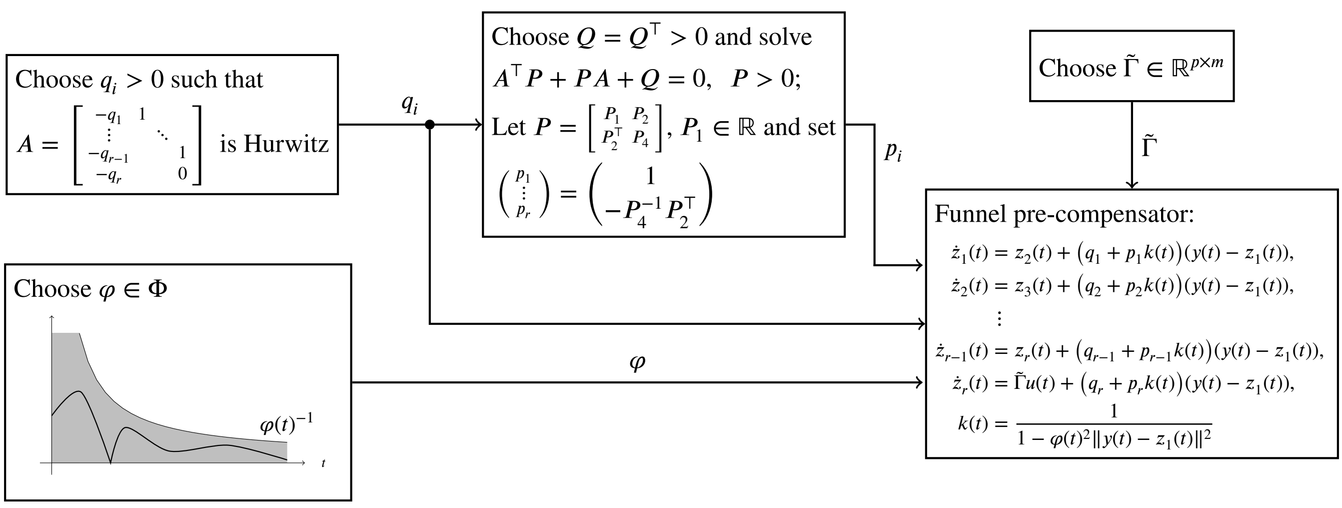

FIGURE 3 Construction of the funnel pre-compensator (1) depending on its design parameters.

We now show that the funnel pre-compensator achieves its objective; note that we only consider the relevant case $r \geq 2$.

Proposition 2.1
Consider $(u, y) \in \mathcal{P}_r$ so that $r \geq 2$, and the funnel pre-compensator (1), (4) with $\varphi \in \Phi$ such that

$$\varphi(0)\|y(0) - z_1^0\| < 1,$$

$\tilde{\Gamma} \in \mathbb{R}^{p\times m}$ and $q_i > 0$, $p_i > 0$ such that (5) is satisfied for corresponding matrices A, P, Q.
Then (1), (4) has a weakly differentiable solution $z = (z_1, \dots, z_r) : \mathbb{R}_{\geq 0} \to (\mathbb{R}^p)^r$ with $k \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to [1, \infty))$ and

$$\exists\, \varepsilon > 0\; \forall\, t > 0 : \; \|y(t) - z_1(t)\| < \varphi(t)^{-1} - \varepsilon. \tag{7}$$

Using the errors

$$\begin{aligned} e_i &:= y^{(i-1)} - z_i, \qquad i = 1, \dots, r-1 \\ e_r &:= \tilde{\Gamma}\Gamma y^{(r-1)} - z_r, \end{aligned} \tag{8}$$

and the constants

$$M_1 := \|(I - \tilde{\Gamma}\Gamma)y^{(r-1)}\|_\infty, \qquad M_2 := \|\tilde{\Gamma}(\dot{\Gamma}y^{(r-1)} + \Gamma y^{(r)} - u)\|_\infty, \tag{9}$$

which are well-defined by $(u, y) \in \mathcal{P}_r$, with $M = (M_1^2 + M_2^2)^{\frac{1}{2}}$ we have

$$\limsup_{t\to\infty} \|e(t)\| \leq \frac{2\, M\, \lambda_{\max}(P)^2}{\lambda_{\min}(Q)\, \lambda_{\min}(P)}. \tag{10}$$

Here $\lambda_{\max}(P)$ denotes the largest eigenvalue of the positive definite matrix P, and $\lambda_{\min}(P)$ denotes its smallest eigenvalue. Furthermore, if $y, \dot{y}, \dots, y^{(r-1)}$ are bounded, then $z_1, \dots, z_r$ are bounded.

Proof. We proceed in several steps.
Step 1: We show existence of a local solution of (1), (4). *Set*

$$\mathcal{D} := \left\{ (t, e_1, \dots, e_r) \in \mathbb{R}_{\geq 0} \times (\mathbb{R}^p)^r \;\middle|\; \varphi(t)\|e_1\| < 1 \right\}$$

and

$$\begin{aligned} f(t) &:= \tilde{\Gamma}\big(\dot{\Gamma}(t)y^{(r-1)}(t) + \Gamma(t)y^{(r)}(t) - u(t)\big) \\ g(t) &:= \big(I - \tilde{\Gamma}\Gamma(t)\big)y^{(r-1)}(t), \qquad t \geq 0. \end{aligned}$$

Invoking $r \geq 2$ we find that the errors (8) *satisfy*

$$\begin{aligned} \dot{e}_1(t) &= e_2(t) - \big(q_1 + p_1 k(t)\big) \cdot e_1(t), & k(t) = \frac{1}{1 - \varphi(t)^2\|e_1(t)\|^2}. \\ &\;\;\vdots \\ \dot{e}_{r-2}(t) &= e_{r-1}(t) - \big(q_{r-2} + p_{r-2}k(t)\big) \cdot e_1(t), \\ \dot{e}_{r-1}(t) &= e_r(t) - \big(q_{r-1} + p_{r-1}k(t)\big) \cdot e_1(t) + g(t), \\ \dot{e}_r(t) &= - \big(q_r + p_r k(t)\big) \cdot e_1(t) + f(t), \end{aligned} \tag{11}$$

By the existence theorem for ordinary differential equations (see e.g. [46, § 10, Thm. XX]), there exists a maximal weakly differentiable solution $e = (e_1, \dots, e_r) : [0, \omega) \to (\mathbb{R}^p)^r$, $\omega \in (0, \infty]$, of (11) *satisfying the initial conditions*

$$\begin{aligned} e_i(0) &= y^{(i-1)}(0) - z_i^0, \quad i = 1, \dots, r, \\ e_r(0) &= \tilde{\Gamma}\Gamma(0)y^{(r-1)}(0) - z_r^0, \end{aligned}$$

and $(t, e(t)) \in \mathcal{D}$ for all $t \in [0, \omega)$. Furthermore, the closure of the graph of e, i.e., the set

$$\overline{\operatorname{graph} e} := \overline{\{ (t, e(t)) \mid t \in [0, \omega) \}},$$

is not a compact subset of $\mathcal{D}$. Thus, a maximal solution $(z_1, \dots, z_r)$ of (1), (4) *can be reconstructed.*
Step 2: We show that $e \in \mathcal{L}^\infty([0, \omega) \to (\mathbb{R}^p)^r)$. Recalling that the Kronecker product of two matrices $V \in \mathbb{R}^{l\times n}$ and $W \in \mathbb{R}^{j\times q}$ is given by

$$V \otimes W = \begin{bmatrix} v_{11}W & \cdots & v_{1n}W \\ \vdots & & \vdots \\ v_{l1}W & \cdots & v_{ln}W \end{bmatrix} \in \mathbb{R}^{lj\times nq}, \tag{12}$$

let

$$\hat{A} := A \otimes I_p = \begin{bmatrix} -q_1 I_p & I_p & & \\ \vdots & & \ddots & \\ -q_{r-1} I_p & & & I_p \\ -q_r I_p & & & 0 \end{bmatrix} \in \mathbb{R}^{rp\times rp}, \quad \hat{P} := P \otimes I_p \in \mathbb{R}^{rp\times rp}, \quad \text{and} \quad \hat{Q} = Q \otimes I_p \in \mathbb{R}^{rp\times rp}.$$

From [9, Fact 7.4.34] it follows that

$$\sigma(\hat{A}) = \sigma(A), \quad \sigma(\hat{Q}) = \sigma(Q), \quad \sigma(\hat{P}) = \sigma(P) \tag{13}$$

and so $A^\top P + PA + Q = 0$ *gives that* $\hat{P} = \hat{P}^\top > 0$, $\hat{Q} = \hat{Q}^\top > 0$ *and*

$$\hat{A}^\top \hat{P} + \hat{P}\hat{A} + \hat{Q} = 0.$$

Since $P_2^\top + P_4 \begin{pmatrix} p_2 \\ \vdots \\ p_r \end{pmatrix} = 0$ *by* (5) *we find*

$$\hat{P} \begin{bmatrix} p_1 I_p \\ \vdots \\ p_r I_p \end{bmatrix} = \begin{bmatrix} (P_1 - P_2 P_4^{-1} P_2^\top) I_p \\ 0 \\ \vdots \\ 0 \end{bmatrix},$$

where $P_1 - P_2 P_4^{-1} P_2^\top \overset{(6)}{>} 0$. *Observe that we may write* (11) *in the form*

$$\dot{e}(t) = \hat{A} e(t) - k(t) \begin{bmatrix} p_1 I_p \\ \vdots \\ p_r I_p \end{bmatrix} e_1(t) + \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(t) \\ f(t) \end{pmatrix}.$$

Since $(u, y) \in \mathcal{P}_r$, *the constants* M_1, M_2 *in* (9) *are well-defined and we have* $\|g(t)\| \le M_1$ *and* $\|f(t)\| \le M_2$ *for almost all* $t \in [0, \omega)$. *With* $M = (M_1^2 + M_2^2)^{\frac{1}{2}}$ *we may now calculate that, for almost all* $t \in [0, \omega)$,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} e(t)^\top \hat{P} e(t) &= e(t)^\top \hat{A}^\top \hat{P} e(t) + e(t)^\top \hat{P} \hat{A} e(t) - 2k(t) e(t)^\top \hat{P} \begin{bmatrix} p_1 I_p \\ \vdots \\ p_r I_p \end{bmatrix} e_1(t) + 2 e(t)^\top \hat{P} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ g(t) \\ f(t) \end{pmatrix} \\ &\le -e(t)^\top \hat{Q} e(t) - 2k(t)(P_1 - P_2 P_4^{-1} P_2^\top) \|e_1(t)\|^2 + 2M \|\hat{P}\| \|e(t)\| \\ &\le -\mu e(t)^\top \hat{P} e(t) + 2M \|\hat{P}\| \|e(t)\|, \end{aligned}$$

where $\mu = \lambda_{\min}(\hat{Q}) / \lambda_{\max}(\hat{P})$. *With* $\delta := \frac{1}{2} \mu \lambda_{\min}(\hat{P})$ *and using that* $ab \le \frac{1}{2}(a^2 + b^2)$ *for all* $a, b \ge 0$, *it follows that*

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} e(t)^\top \hat{P} e(t) &\le -\mu e(t)^\top \hat{P} e(t) + \left(\sqrt{2\delta} \|e(t)\| \right) \left(\frac{2M \|\hat{P}\|}{\sqrt{2\delta}} \right) \\ &\le -\mu e(t)^\top \hat{P} e(t) + \delta \|e(t)\|^2 + \frac{M^2 \|\hat{P}\|^2}{\delta} \\ &\le -\tfrac{\mu}{2} e(t)^\top \hat{P} e(t) + \frac{2M^2 \|\hat{P}\|^2}{\mu \lambda_{\min}(\hat{P})} \end{aligned}$$

for almost all $t \in [0, \omega)$. *Gronwall's lemma now implies that*

$$e(t)^\top \hat{P} e(t) \le e(0)^\top \hat{P} e(0) e^{-\frac{\mu}{2} t} + \frac{2M^2 \|\hat{P}\|^2}{\mu^2 \lambda_{\min}(\hat{P})},$$

and hence

$$\|e(t)\|^2 \le \frac{\lambda_{\max}(\hat{P})}{\lambda_{\min}(\hat{P})} e^{-\frac{\mu}{2} t} \|e(0)\|^2 + \frac{2M^2 \|\hat{P}\|^2}{\mu^2 \lambda_{\min}(\hat{P})^2} \tag{14}$$

for all $t \in [0, \omega)$. *Equation* (14) *in particular implies that* $e \in \mathcal{L}^\infty([0, \omega) \to (\mathbb{R}^p)^r)$. *Therefore, by* (8) *and* $(u, y) \in \mathcal{P}_r$ *we have that* $z_1, \ldots, z_r$ *are bounded, provided that* $y, \dot{y}, \ldots, y^{(r-1)}$ *are bounded.*

Step 3: We show that $k \in \mathcal{L}^\infty([0,\omega) \to \mathbb{R})$. *Let* $\kappa \in (0,\omega)$ *be arbitrary but fixed and* $\lambda := \inf_{t\in(0,\omega)} \varphi(t)^{-1} > 0$. *Since* $\dot\varphi$ *is bounded and* $\liminf_{t\to\infty} \varphi(t) > 0$ *we find that* $\frac{\mathrm{d}}{\mathrm{d}t} \varphi|_{[\kappa,\infty)}(\cdot)^{-1}$ *is bounded and hence there exists a Lipschitz bound* $L > 0$ *of* $\varphi|_{[\kappa,\infty)}(\cdot)^{-1}$. *By Step 2,* e_2 *is bounded and we may choose* $\varepsilon > 0$ *small enough so that*

$$\varepsilon \le \min\left\{ \frac{\lambda}{2}, \inf_{t\in(0,\kappa]} (\varphi(t)^{-1} - \|e_1(t)\|) \right\}$$

and

$$L \le -\sup_{t\in[0,\omega)} \|e_2(t)\| - M_1 + \frac{q_1\lambda}{2} + \frac{\lambda^2}{4\varepsilon}; \tag{15}$$

feasibility of this choice is guaranteed by $r \ge 2$. *Using a standard argument in funnel control, see e.g. [23, pp. 241–242], it is then straightforward to show that*

$$\forall\, t \in (0,\omega): \ \ \varphi(t)^{-1} - \|e_1(t)\| \ge \varepsilon. \tag{16}$$

The estimate (16) *clearly implies boundedness of* k.

Step 4: We show $\omega = \infty$. *Assume that* $\omega < \infty$. *Then, since* e *and* k *are bounded by Steps 2 and 3, it follows that* $\overline{\text{graph}\, e}$ *is a compact subset of* $\mathcal{D}$, *a contradiction. Therefore,* $\omega = \infty$. *Together with Step 3, this in particular implies* (7). *Inequality* (10) *is an immediate consequence of* (14) *together with the observation that by* (13) *we have* $\lambda_{\min}(\hat P) = \lambda_{\min}(P)$, $\lambda_{\max}(\hat P) = \lambda_{\max}(P)$, $\lambda_{\min}(\hat Q) = \lambda_{\min}(Q)$ *and, since* $\hat P$ *is positive definite,* $\|\hat P\| = \lambda_{\max}(\hat P) = \lambda_{\max}(P)$. $\square$

In [41, Thm. 2.2], using the adaptive high-gain observer proposed therein, bounds for the mean value of the observation error e_i (defined similar to (8)) are given; we stress that both the bounds in [41, (14)] and in (10) cannot be made arbitrarily small in general, they depend on the given signals.

Remark 2.2. We consider two special cases for signals $(u, y) \in \mathcal{P}_r$, the funnel pre-compensator (1) and the resulting estimate (10).

(i) $\tilde\Gamma = 0$. It is immediate from (9) that in this case $M_1 = \|y^{(r-1)}\|_\infty$ and $M_2 = 0$, thus $M = \|y^{(r-1)}\|_\infty$ in (10). Note that the choice of $\tilde\Gamma$ is independent of the signals u and y.

(ii) $p = m$, $\Gamma \in \mathbf{Gl}_m(\mathbb{R})$, $\tilde\Gamma = \Gamma^{-1}$ and we have $\Gamma y^{(r)} = u$. This means the signals satisfy a very simple relation and we have exact knowledge of the invertible matrix Γ. Then $M_1 = M_2 = 0$ in (9) and hence $M = 0$ in (10). In particular, this implies that $e(t) \to 0$ and $k(t) \to 1$ for $t \to \infty$.

Remark 2.3. If the signal y is subject to noise, i.e., the funnel pre-compensator (1) receives $y + n$ instead of y, where $n \in C^r([-h,\infty) \to \mathbb{R}^p)$ is such that, for Γ as in $\mathcal{P}_r$, $n, \dot n, \dots, n^{(r-2)}$, $\Gamma n^{(r-1)}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma n^{(r-1)}\right)$ are bounded, then $(u, y+n) \in \mathcal{P}_r$ with Γ. Therefore, Proposition 2.1 yields that the funnel pre-compensator may also be applied to u and $y + n$ and achieves that

$$\forall\, t > 0: \ \ \varphi(t)\|y(t) + n(t) - z_1(t)\| < 1,$$

which implies

$$\forall\, t > 0: \ \ \frac{\varphi(t)}{1 + \varphi(t)\|n(t)\|}\|y(t) - z_1(t)\| < 1,$$

i.e., $y - z_1$ evolves in the funnel $\mathcal{F}_\psi$, where $\psi = \frac{\varphi(t)}{1+\varphi(t)\|n(t)\|}$. If an upper bound for n is known, say $\|n(t)\| \le \nu$ for all $t \ge 0$, then

$$\forall\, t > 0: \ \ \|y(t) - z_1(t)\| < \varphi(t)^{-1} + \nu.$$

Hence, the actual error remains in the wider funnel obtained by adding the corresponding bound of the noise to the funnel bounds used for the pre-compensator. The bound in (10) changes as follows: Modify M_1 and M_2 from (9) to

$$\tilde M_1 := \|g + (I - \tilde\Gamma\Gamma)n^{(r-1)}\|_\infty, \qquad \tilde M_2 := \left\| f + \tilde\Gamma \tfrac{\mathrm{d}}{\mathrm{d}t}\left(\Gamma n^{(r-1)}\right)\right\|_\infty.$$

Then, with $\tilde M := (\tilde M_1^2 + \tilde M_2^2)^{\frac{1}{2}}$, we have that

$$\limsup_{t\to\infty} \|e(t)\| \le \frac{2\,\tilde M\,\lambda_{\max}(P)^2}{\lambda_{\min}(Q)\,\lambda_{\min}(P)} + \left\|\left(n, \dot n, \dots, n^{(r-2)}, \tilde\Gamma\Gamma n^{(r-1)}\right)\right\|_\infty.$$

If the signal u is subject to noise before the funnel pre-compensator receives it, i.e., $u + v$ enters the pre-compensator (1), where $v \in \mathcal{L}^\infty(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$, then obviously $(u + v, y) \in \mathcal{P}_r$ and hence Proposition 2.1 yields that the funnel pre-compensator may also be applied to $u + v$ and y; in particular, (7) is achieved.

While Proposition 2.1 shows that the funnel pre-compensator is able to achieve prescribed transient behavior of the compensator error $e_1 = y - z_1$ and that the errors $e_2, \dots, e_r$ as in (8) converge to a certain strip, we like to stress that no transient behavior can be prescribed for $e_2, \dots, e_r$ since $\dot{y}, \dots, y^{(r-1)}$ are not known. Therefore, $z_2, \dots, z_r$ from the funnel pre-compensator cannot be viewed as estimates for $\dot{y}, \dots, y^{(r-1)}$. In the following we show that a successive application of the funnel pre-compensator to the signals u and z_1 results in a cascade of pre-compensators which yields, as desired,

- an estimate z for the signal y with prescribed transient behavior (i.e., $(t, y(t) - z(t)) \in \mathcal{F}_\varphi$) and
- the derivatives $\dot{z}, \dots, z^{(r-1)}$ are known explicitly.

We stress that a single funnel pre-compensator is not able to achieve the above requirements when $r \geq 3$ since $\ddot{z}_1$ for z_1 as in (1) depends on $\dot{y}$ and the latter is unknown. However, applying another funnel pre-compensator to the signal pair (u, z_1), i.e., z_1 plays the role of the output now, yields a signal $\bar{z}_1$ such that $\dot{\bar{z}}_1$ is known. Furthermore, $\ddot{\bar{z}}_1$ depends on $\dot{z}_1$ which is known as well; it only depends on z_1, z_2 and the measured output y. As a result, $y - \bar{z}_1 = (y - z_1) + (z_1 - \bar{z}_1)$ evolves in a performance funnel and $\dot{\bar{z}}_1, \ddot{\bar{z}}_1$ are known explicitly. This argument may be applied successively.

To this end, we introduce a *cascade of funnel pre-compensators*

$$FP_{r-1} \circ \dots \circ FP_1$$

where the funnel pre-compensators

$$FP_i(p_i, q_i, \tilde{\Gamma}_i, \varphi_i) : \; (u, z_{i-1,1}) \mapsto z_{i,1},$$

are specified, for $i = 1, \dots, r-1$, as follows:

$$\begin{aligned}
\dot{z}_{i,1}(t) &= z_{i,2}(t) + \left(q_{i,1} + p_{i,1} k_i(t)\right) \cdot (z_{i-1,1}(t) - z_{i,1}(t)), \\
\dot{z}_{i,2}(t) &= z_{i,3}(t) + \left(q_{i,2} + p_{i,2} k_i(t)\right) \cdot (z_{i-1,1}(t) - z_{i,1}(t)), \\
&\;\;\vdots \\
\dot{z}_{i,r-1}(t) &= z_{i,r}(t) + \left(q_{i,r-1} + p_{i,r-1} k_i(t)\right) \cdot (z_{i-1,1}(t) - z_{i,1}(t)), \\
\dot{z}_{i,r}(t) &= \phantom{z_{i,r}(t)} + \left(q_{i,r} + p_{i,r} k_i(t)\right) \cdot (z_{i-1,1}(t) - z_{i,1}(t)) + \tilde{\Gamma}_i u(t),
\end{aligned}
\qquad k_i(t) = \frac{1}{1 - \varphi_i(t)^2 \|z_{i-1,1}(t) - z_{i,1}(t)\|^2}. \tag{17}$$

where $z_{0,1} := y$, $\tilde{\Gamma}_i \in \mathbb{R}^{p \times m}$,

$$\varphi_i \in \Phi_{r-1} := \Phi \cap \left\{ \varphi \in C^{r-1}(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \dot{\varphi}, \dots, \varphi^{(r-1)} \text{ bounded } \right\}$$

and $q_{i,j} > 0$, $p_{i,j} > 0$ are such that (5) is satisfied for corresponding matrices A_i, P_i, Q_i for $i = 1, \dots, r-1$. The initial values are

$$z_{i,j}(0) = z_{i,j}^0 \in \mathbb{R}^p, \quad i = 1, \dots, r-1, \; j = 1, \dots, r. \tag{18}$$

The situation is illustrated in Figure 4 .

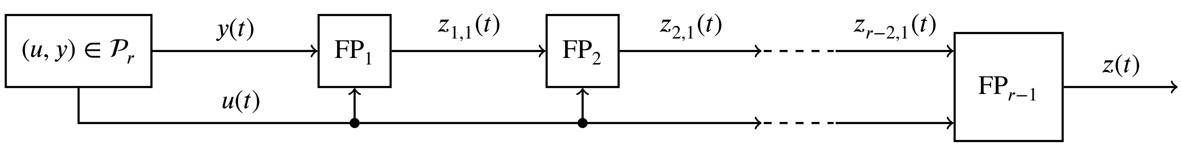

FIGURE 4 Cascade of funnel pre-compensators (1) applied to signals $(u, y) \in \mathcal{P}_r$.

We show that the cascade (17) with $\operatorname{rk} \tilde{\Gamma}_i = m$ applied to signals $(u, y) \in \mathcal{P}_r$, where additionally $y, \dot{y}, \dots, y^{(r-1)}$ are bounded (but $\dot{y}, \dots, y^{(r-1)}$ are unknown), yields an interconnection with output $z = z_{r-1,1}$ such that $y - z$ has prescribed transient behavior and $\dot{z}, \dots, z^{(r-1)}$ are known explicitly.

Theorem 2.4
Consider $(u, y) \in \mathcal{P}_r$ so that $r \geq 2$, and assume that $y, \dot{y}, \dots, y^{(r-1)}$ are bounded. Consider the cascade of funnel pre-compensators $FP_{r-1} \circ \dots \circ FP_1$ defined in (17) for $\varphi_i \in \Phi_{r-1}$, $\tilde{\Gamma}_i \in \mathbb{R}^{p \times m}$ with $\operatorname{rk} \tilde{\Gamma}_i = m$ and $q_{i,j} > 0$, $p_{i,j} > 0$ are such that (5) is satisfied for corresponding matrices A_i, P_i, Q_i, and initial data (18) such that

$$\varphi_i(0) \|z_{i-1,1}(0) - z_{i,1}^0\| < 1, \quad i = 1, \dots, r-1,$$

where $z_{0,1} := y$. Then (17), (18) has weakly differentiable solutions $z_{i,j} \in \mathcal{L}^\infty\left(\mathbb{R}_{\geq 0} \to \mathbb{R}^p\right)$ with $k_i \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to [1,\infty))$ for $i = 1,\dots,r-1$, $j = 1,\dots,r$ and

$$\forall\, i \in \{1,\dots,r-1\}\ \exists\, \varepsilon_i > 0\ \forall\, t > 0: \quad \|z_{i-1,1}(t) - z_{i,1}(t)\| < \varphi_i(t)^{-1} - \varepsilon_i. \tag{19}$$

Furthermore, for $z := z_{r-1,1}$ we have that

$$\forall\, t > 0: \ \|y(t) - z(t)\| < \sum_{i=1}^{r-1} \varphi_i(t)^{-1} - \varepsilon_i. \tag{20}$$

Proof. We show existence of bounded weakly differentiable solutions for each pre-compensator in (17) *and the property* (19) *by induction. Note that* (20) *is a consequence of* (19).

For $i = 1$ we have $z_{0,1} = y$ and hence the existence of bounded global solutions follows from Proposition 2.1. We may calculate that

$$z_{i,1}^{(j)}(t) = z_{i,j+1}(t) + \sum_{l=0}^{j-1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^l \left(q_{i,j-l} + p_{i,j-l}k_i(t)\right)\left(z_{i-1,1}(t) - z_{i,1}(t)\right) \tag{21}$$

for $i = 1,\dots,r-1$ and $j = 0,\dots,r$, where $z_{i,r+1} := \tilde{\Gamma}_i u$. With $w_i(t) := z_{i-1,1}(t) - z_{i,1}(t)$ we calculate

$$\dot{k}_i(t) = 2k_i(t)^2\left(\varphi_i(t)\dot{\varphi}_i(t)w_i(t)^\top w_i(t) + \varphi_i(t)^2 w_i(t)^\top \dot{w}_i(t)\right) \tag{22}$$

for all $i = 1,\dots,r-1$. In particular, for $i = 1$ we obtain that $\dot{z}_{1,1},\dots,z_{1,1}^{(r-1)}$ are bounded since $y,\dots,y^{(r-1)},\varphi_1,\dots,\varphi_1^{(r-1)}$ are bounded and $z_{1,1},\dots,z_{1,r}$, and k_1 are bounded by Proposition 2.1. Now assume that the statement is true for $i \in \{1,\dots,r-2\}$ such that $\dot{z}_{i,1},\dots,z_{i,1}^{(r-1)}$ are bounded. Choosing $\Gamma_i \in \mathbb{R}^{m\times p}$ such that $\Gamma_i\tilde{\Gamma}_i = I_m$ it follows from (21) *with $j = r$ that*

$$\Gamma_i z_{i,1}^{(r)} - u = \sum_{l=0}^{r-1} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^l \left(q_{i,r-l-1} + p_{i,r-l-1}k_i\right)\left(z_{i-1,1} - z_{i,1}\right) \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^m),$$

and hence $(u, z_{i,1}) \in \mathcal{P}_r$, by which an application of Proposition 2.1 is feasible and yields existence of bounded global solutions such that k_{i+1} is bounded. Again invoking (21) *yields boundedness of $\dot{z}_{i+1,1},\dots,z_{i+1,1}^{(r-1)}$.* □

Remark 2.5. Use the notation and assumptions from Theorem 2.4. Then the derivatives $\dot{z},\dots,z^{(r-1)}$ are known explicitly as

$$z^{(j)}(t) = z_{r-1,j+1}(t) + \tilde{P}_j^{r-1}(t), \qquad j = 0,\dots,r-1,$$

where the functions $\tilde{P}_j^i$ are defined in a recursive way:

$$P_0^{a,b}(k,\varphi_0,e_0) := (q_{a,b} + p_{a,b}k)e_0,$$

$$\begin{aligned} P_{i+i}^{a,b}(k,\varphi_0,\dots,\varphi_{i+1},e_0,\dots,e_{i+1}) := {}& \frac{\partial P_i^{a,b}}{\partial k}\left(2k^2(\varphi_0\varphi_1 e_0^\top e_0 + \varphi_0^2 e_0^\top e_1)\right) + \frac{\partial P_i^{a,b}}{\partial \varphi_0}\varphi_1 + \dots + \frac{\partial P_i^{a,b}}{\partial \varphi_i}\varphi_{i+1} \\ & + \frac{\partial P_i^{a,b}}{\partial e_0}e_1 + \dots + \frac{\partial P_i^{a,b}}{\partial e_i}e_{i+1} \end{aligned}$$

for $a, b \in \{1,\dots,r-1\}$ and $i \geq 0$, where $k, \varphi_i \in \mathbb{R}$ and $e_i \in \mathbb{R}^p$ for each $i \geq 0$. Further define, using (17),

$$\tilde{P}_j^i(t) := \sum_{l=0}^{j-1} P_l^{i,j-l}\left(k_i(t),\varphi_i(t),\dots,\varphi_i^{(l)}(t), z_{i-1,1}(t) - z_{i,1}(t),\dots,z_{i-1,1}^{(l)}(t) - z_{i,1}^{(l)}(t)\right)$$

for $i = 1,\dots,r-1$ and $j = 0,\dots,r-1$. We will show that

$$z_{i,1}^{(j)}(t) = z_{i,j+1}(t) + \tilde{P}_j^i(t), \qquad i = 1,\dots,r-1, \ \ j = 0,\dots,r-1. \tag{23}$$

To this end, observe that it follows from (22) and a simple induction that

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^l \left(q_{i,j-l} + p_{i,j-l}k_i(t)\right)w_i(t) = P_l^{i,j-l}\left(k_i(t),\varphi_i(t),\dot{\varphi}_i(t),\dots,\varphi_i^{(l)}(t),w_i(t),\dot{w}_i(t),\dots,w_i^{(l)}(t)\right)$$

for $i = 1,\dots,r-1$, $j = 0,\dots,r-1$ and $l = 0,\dots,j-1$. Then (21) immediately implies (23).

By definition, $\tilde{P}_j^{r-1}(t)$ depends on the derivatives of $z_{r-2,1}$ and $z_{r-1,1} = z$ up to order $j-1$. The dependencies on $\dot{z},\dots,z^{(j-1)}$ may be immediately resolved by applying the same formula again, thus $z^{(j)}$ depends on $z_{r-1,1},\dots,z_{r-1,j+1}$ and on $z_{r-2,1},\dot{z}_{r-2,1},\dots,z_{r-2,1}^{(j-1)}$. Applying (23) in a recursive way to $\dot{z}_{r-2,1},\dots,z_{r-2,1}^{(j-1)}$ we obtain dependencies as depicted in Figure 5 .

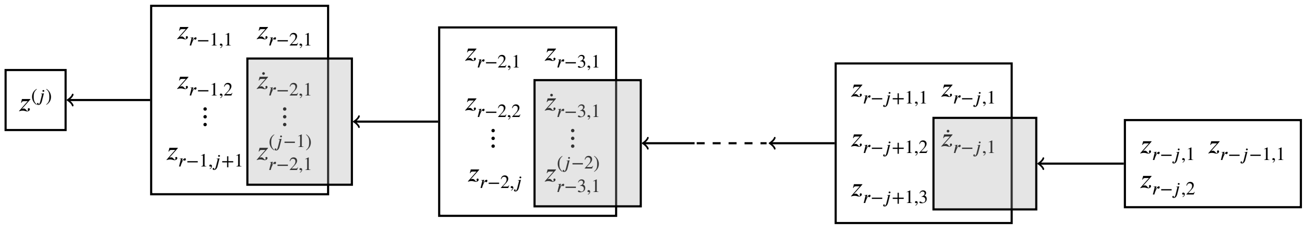

FIGURE 5 Dependency of $z^{(j)}$ on the compensator states. Note that $z_{r-j-1,1} = z_{0,1} = y$ for $j = r - 1$.

3 | APPLICATION TO MINIMUM PHASE SYSTEMS

A possible application of the funnel pre-compensator cascade developed in Section 2 is in high-gain adaptive control in order to solve the longstanding open question of how systems with relative degree larger than one may be appropriately treated, see [22, 24, 35]. Recently, a funnel controller has been designed in [5] which is able to achieve tracking with prescribed transient performance for nonlinear systems of arbitrary relative degree. However, the derivatives of the output must be available for the controller. In practice this means that measurements must be differentiated, which is an ill-posed problem, in particular in the presence of noise, see e.g. [20, Sec. 1.4.4]. In order to resolve this problem, the funnel pre-compensator cascade may be applied to the system which results in an interconnection with new output z satisfying (20), and the derivatives of which are known. Then the funnel controller from [5] may be applied to the interconnection in order to achieve tracking with prescribed transient behavior without the need to calculate output derivatives; for linear minimum phase systems with relative degree two this configuration was successfully implemented in [8]. The situation is depicted in Figure 6 .

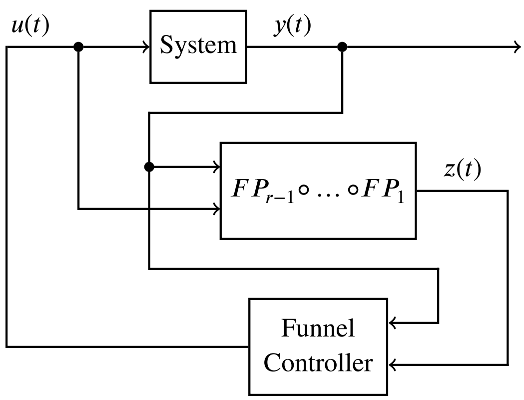

FIGURE 6 Interconnection of a system with funnel pre-compensator cascade and funnel controller.

We stress that, while we consider the combination of the funnel pre-compensator with a funnel controller in the present paper, one may also combine the funnel pre-compensator with other controllers such as a prescribed performance controller, see e.g. [3].

For the solution of tracking problems, a crucial condition is the input-to-state stability of the internal dynamics (the minimum phase property in case of linear systems), cf. [12, 24, 42]. The funnel controller in [5] requires this as well, and hence we need to ensure that for a minimum phase system, the interconnection with the funnel pre-compensator cascade is again minimum phase. In the following we show that this can be achieved for a special class of systems which are linear up to the influence of an operator T and have relative degree two or three. For relative degree larger than three this remains an open problem; we show explicitly where our proof does not work in this case and conjecture that some kind of small gain condition is needed then.

In the following we consider systems described by functional differential equations of the form

$$\begin{aligned} y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + f\left(d(t), T(y, \dot y, \ldots, y^{(r-1)})(t)\right) + \Gamma u(t), \\ y|_{[-h,0]} &= y^0 \in \mathcal{W}^{r-1,\infty}([-h,0] \to \mathbb{R}^m), \end{aligned} \tag{24}$$

where $h > 0$ is the "memory" of the system, $r \in \mathbb{N}$ is the strict relative degree, and

- $d \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^p)$, $p \in \mathbb{N}$, is a disturbance;
- $f \in \mathcal{C}(\mathbb{R}^p \times \mathbb{R}^q \to \mathbb{R}^m)$, $q \in \mathbb{N}$;
- $\Gamma \in \mathbf{Gl}_m(\mathbb{R})$ is the high-frequency gain matrix;
- $T : \mathcal{C}([-h, \infty) \to \mathbb{R}^m)^r \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^q)$ is an operator with the following properties:

  a) T maps bounded trajectories to bounded trajectories, i.e., for all $c_1 > 0$ there exists $c_2 > 0$ such that for all $\zeta \in \mathcal{C}([-h, \infty) \to \mathbb{R}^m)^r$:
  $$\sup_{t \in [-h,\infty)} \|\zeta(t)\| \leq c_1 \;\; \Rightarrow \;\; \sup_{t \in [0,\infty)} \|T(\zeta)(t)\| \leq c_2;$$
  b) T is causal, i.e., for all $t \geq 0$ and all $\zeta, \xi \in \mathcal{C}([-h, \infty) \to \mathbb{R}^m)^r$:
  $$\zeta|_{[-h,t)} = \xi|_{[-h,t)} \;\; \Rightarrow \;\; T(\zeta)|_{[0,t]} \overset{\text{a.e.}}{=} T(\xi)|_{[0,t]}\,;$$
  c) T is "locally Lipschitz" continuous in the following sense: for all $t \geq 0$ there exist $\tau, \delta, c > 0$ such that for all $\zeta, \Delta\zeta \in \mathcal{C}([-h, \infty) \to \mathbb{R}^m)^r$ with $\Delta\zeta|_{[-h,t]} = 0$ and $\| \Delta\zeta|_{[t,t+\tau]} \|_\infty < \delta$ we have
  $$\left\| \left(T(\zeta + \Delta\zeta) - T(\zeta)\right)\Big|_{[t,t+\tau]} \right\|_\infty \leq c\| \Delta\zeta|_{[t,t+\tau]} \|_\infty.$$

The functions $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and $y : [-h, \infty) \to \mathbb{R}^m$ are called *input* and *output* of the system (24), respectively. For fixed $u \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ we call $y \in \mathcal{C}^{r-1}([-h, \omega) \to \mathbb{R}^m)$ a solution of (24) on $[-h, \omega)$, $\omega \in (0, \infty]$, if $y|_{[-h,0]} = y^0$ and $y^{(r-1)}|_{[0,\omega)}$ is weakly differentiable and satisfies the differential equation in (24) for almost all $t \in [0, \omega)$; y is called *maximal*, if it has no right extension that is also a solution. Existence of maximal solutions of (24) for every $y^0 \in \mathcal{W}^{r-1,\infty}([-h, 0] \to \mathbb{R}^m)$ and every $u \in \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ is guaranteed by [26, Thm. 5]; if $y, \dot{y}, \ldots, y^{(r-1)}$ are bounded, then $\omega = \infty$. In this case we clearly have $(u, y|_{[0,\infty)}) \in \mathcal{P}_r$.

The input-to-state stability of the internal dynamics of (24), i.e., the minimum phase property, is modelled by the property a) of the operator T in (24). It is shown in [5] that funnel control is feasible for systems of the class (24), provided that Γ is positive or negative definite. In the case of relative degree one, i.e., $r = 1$, systems similar to (24) are well studied, see [25, 26, 29, 39]. For relative degree two systems see [21], and for higher relative degree see [28]. In the aforementioned references it is shown that the class of systems (24) encompasses linear and nonlinear systems with existing strict relative degree and input-to-state stable internal dynamics and the operator T allows for infinite-dimensional linear systems, systems with hysteretic effects or nonlinear delay elements and combinations thereof. In particular, the class (24) contains the system classes discussed in [19, 27, 28] and the nonlinear systems in [30] provided that the internal dynamics are input-to-state stable.

In order to show that the minimum phase property of systems (24) is preserved by the cascade of funnel pre-compensators, we additionally need that the operator T is bounded whenever the output y is bounded, i.e., we replace property a) with the stronger condition

a') for all $c_1 > 0$ there exists $c_2 > 0$ such that for all $\zeta_1, \ldots, \zeta_r \in \mathcal{C}([-h, \infty) \to \mathbb{R}^m)$:
$$\sup_{t \in [-h,\infty)} \|\zeta_1(t)\| \leq c_1 \;\; \Rightarrow \;\; \sup_{t \in [0,\infty)} \|T(\zeta_1, \ldots, \zeta_r)(t)\| \leq c_2.$$

The class of systems (24) where T satisfies a') in particular contains the class of nonlinear systems in input-normalized Byrnes-Isidori form with exponentially stable zero dynamics as considered in [10], provided the high-frequency gain matrix is constant. We show that, if $r = 2$ or $r = 3$, the interconnection of (24) with the cascade of funnel pre-compensators, where $\tilde{\Gamma}_i = \tilde{\Gamma}$ is invertible, has again relative degree r and input-to-state stable internal dynamics in the sense that it can be rewritten as
$$z^{(r)}(t) = F\left(\tilde{d}(t), \tilde{T}(z, \dot{z}, \ldots, z^{(r-1)})(t)\right) + \tilde{\Gamma} u(t),$$
where $\tilde{T}$ is an operator with the properties a)–c).

Theorem 3.1
Consider a system (24) with $r \in \{2, 3\}$, $y^0 \in \mathcal{W}^{r-1,\infty}([-h, 0] \to \mathbb{R}^m)$ and assume that $\Gamma = \Gamma^\top > 0$ and the operator T

satisfies a'). Further consider the cascade of funnel pre-compensators $FP_{r-1} \circ \ldots \circ FP_1$ defined by (17), (18) with $\varphi_i \in \Phi_{r-1}$ such that

$$\varphi_i(0)\|z_{i-1,1}(0) - z^0_{i,1}\| < 1,$$

where $z_{0,1} := y$ and $q_{i,j} = q_j > 0$, $p_{i,j} = p_j > 0$ are such that (5) is satisfied for corresponding matrices A, P, Q for all $i = 1, \ldots, r-1$, $j = 1, \ldots, r$. Moreover, assume that $\tilde{\Gamma}_i = \tilde{\Gamma} \in \mathbb{R}^{m\times m}$, $i = 1, \ldots, r-1$, such that $\tilde{\Gamma} = \tilde{\Gamma}^\top > 0$ and,

$$\text{if } r = 3, \text{ then } \; I - \Gamma\tilde{\Gamma}^{-1} = \left(I - \Gamma\tilde{\Gamma}^{-1}\right)^\top > 0. \tag{25}$$

Then the conjunction of (24) and (17) with input u and output $z := z_{r-1,1}$ can be equivalently written as

$$z^{(r)}(t) = F\left(\tilde{d}(t), \tilde{T}(z, \dot{z}, \ldots, z^{(r-1)})(t)\right) + \tilde{\Gamma}u(t), \quad z(0) = z^0_{r-1,1}, \tag{26}$$

for $\tilde{d}(t) := \left(\varphi_{r-1}(t), \dot{\varphi}_{r-1}(t), \ldots, \varphi^{(r-1)}_{r-1}(t)\right)^\top \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^r)$, some $F \in \mathcal{C}(\mathbb{R}^r \times \mathbb{R}^{\tilde{q}} \to \mathbb{R}^m)$ and an operator $\tilde{T} : \mathcal{C}([-h, \infty) \to \mathbb{R}^m)^r \to \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{\tilde{q}})$ which satisfies the properties a)–c). Furthermore, for any $u \in \mathcal{L}^\infty_{\text{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m)$ and any solution of (17), (24) we have (20) and the derivatives of the compensator states satisfy (23).

Proof. Step 1: We start with several transformations of the error dynamics between two successive systems.

Step 1a: Define $v_{i,j} := z_{i-1,j} - z_{i,j}$ for $i = 2, \ldots, r-1$ and $j = 1, \ldots, r$. Then

$$\begin{aligned}
\dot{v}_{i,1}(t) &= v_{i,2}(t) - \left(q_1 + p_1 k_i(t)\right) \cdot v_{i,1}(t) + \left(q_1 + p_1 k_{i-1}(t)\right) \cdot v_{i-1,1}(t),\\
&\;\;\vdots\\
\dot{v}_{i,r-1}(t) &= v_{i,r}(t) - \left(q_{r-1} + p_{r-1} k_i(t)\right) \cdot v_{i,1}(t) + \left(q_{r-1} + p_{r-1} k_{i-1}(t)\right) \cdot v_{i-1,1}(t),\\
\dot{v}_{i,r}(t) &= - \left(q_r + p_r k_i(t)\right) \cdot v_{i,1}(t) + \left(q_r + p_r k_{i-1}(t)\right) \cdot v_{i-1,1}(t).
\end{aligned}$$

Step 1b: Defining $e_{1,j}(t) := y^{(j-1)}(t) - z_{1,j}(t)$ for $j = 1, \ldots, r-1$ and $e_{1,r}(t) := y^{(r-1)}(t) - \Gamma\tilde{\Gamma}^{-1} z_{1,r}(t)$ we obtain

$$\begin{aligned}
\dot{e}_{1,1}(t) &= e_{1,2}(t) - \left(q_1 + p_1 k_1(t)\right) \cdot e_{1,1}(t),\\
&\;\;\vdots\\
\dot{e}_{1,r-2}(t) &= e_{1,r-1}(t) - \left(q_{r-2} + p_{r-2} k_1(t)\right) \cdot e_{1,1}(t),\\
\dot{e}_{1,r-1}(t) &= e_{1,r}(t) - \left(q_{r-1} + p_{r-1} k_1(t)\right) \cdot e_{1,1}(t) + (\Gamma\tilde{\Gamma}^{-1} - I) \cdot z_{1,r}(t),\\
\dot{e}_{1,r}(t) &= - \Gamma\tilde{\Gamma}^{-1}\left(q_r + p_r k_1(t)\right) \cdot e_{1,1}(t) + \sum_{i=1}^{r} R_i y^{(i-1)}(t) + f\left(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\right).
\end{aligned}$$

Set $v_{1,1}(t) := e_{1,1}(t)$ and $\tilde{v}(t) := \sum_{i=1}^{r-1} v_{i,1}(t)$, then we may define $v_{1,j}(t) := e_{1,j}(t) - \sum_{k=1}^{j-1} R_{r-j+k+1}\tilde{v}^{(k-1)}(t)$ and obtain

$$\begin{aligned}
\dot{v}_{1,1}(t) &= v_{1,2}(t) - \left(q_1 + p_1 k_1(t)\right) \cdot v_{1,1}(t) + R_r\tilde{v}(t),\\
\dot{v}_{1,2}(t) &= v_{1,3}(t) - \left(q_2 + p_2 k_1(t)\right) \cdot v_{1,1}(t) + R_{r-1}\tilde{v}(t),\\
&\;\;\vdots\\
\dot{v}_{1,r-2}(t) &= v_{1,r-1}(t) - \left(q_{r-2} + p_{r-2} k_1(t)\right) \cdot v_{1,1}(t) + R_3\tilde{v}(t),\\
\dot{v}_{1,r-1}(t) &= v_{1,r}(t) - \left(q_{r-1} + p_{r-1} k_1(t)\right) \cdot v_{1,1}(t) + R_2\tilde{v}(t) + (\Gamma\tilde{\Gamma}^{-1} - I) z_{1,r}(t),\\
\dot{v}_{1,r}(t) &= - \Gamma\tilde{\Gamma}^{-1}\left(q_r + p_r k_1(t)\right) \cdot v_{1,1}(t) + R_1\tilde{v}(t) + \sum_{i=1}^{r} R_i \left(y^{(i-1)}(t) - \tilde{v}^{(i-1)}(t)\right)\\
&\quad + f\left(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\right).
\end{aligned}$$

Now we observe that

$$\begin{aligned}
y(t) - \tilde{v}(t) &= y(t) - v_{1,1}(t) - v_{2,1}(t) - \ldots - v_{r-1,1}(t)\\
&= y(t) - \left(y(t) - z_{1,1}(t)\right) - \left(z_{1,1}(t) - z_{2,1}(t)\right) - \ldots - \left(z_{r-2,1}(t) - z_{r-1,1}(t)\right) = z_{r-1,1}(t) = z(t).
\end{aligned}$$

Furthermore,

$$z_{1,r}(t) = z^{(r-1)}_{1,1}(t) - \sum_{i=0}^{r-2} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^i \left[\left(q_{r-i-1} + p_{r-i-1} k_1(t)\right) v_{1,1}(t)\right]$$

and

$$z_{1,1}(t) = y(t) - v_{1,1}(t) = z(t) + \tilde{v}(t) - v_{1,1}(t) = z(t) + \sum_{i=2}^{r-1} v_{i,1}(t),$$

hence

$$z_{1,r}(t) = z^{(r-1)}(t) + \sum_{i=2}^{r-1} v_{i,1}(t) - \sum_{i=0}^{r-2} \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^i \left[\left(q_{r-i-1} + p_{r-i-1} k_1(t)\right) v_{1,1}(t)\right].$$

Step 1c: Define $w_{i,j}(t) := v_{i,j}(t)$ *for* $i = 2, \dots, r-1$ *and* $j = 1, \dots, r$ *and* $w_{1,r}(t) := v_{1,r}(t)$,

$$w_{1,r-j}(t) := v_{1,r-j}(t) + G\left[\sum_{i=2}^{r-1} v_{i,1}^{(r-j-1)}(t) - \sum_{i=j}^{r-2}\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^{i-j}\left[\big(q_{r-i-1} + p_{r-i-1}k_1(t)\big)v_{1,1}(t)\right]\right]$$

for $j = 1, \dots, r-1$*, where* $G := (I - \Gamma\tilde{\Gamma}^{-1})$*; in particular we have*

$$w_{1,1}(t) = v_{1,1}(t) + G\sum_{i=2}^{r-1} v_{i,1}(t).$$

With $\tilde{w}(t) := \sum_{i=2}^{r-1} w_{i,1}(t)$ *we find*

$$\begin{aligned}
\dot{w}_{1,1}(t) &= w_{1,2}(t) - \Gamma\tilde{\Gamma}^{-1}\big(q_1 + p_1 k_1(t)\big)\cdot\big(w_{1,1}(t) - G\tilde{w}(t)\big) + R_r w_{1,1}(t) + R_r\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t),\\
\dot{w}_{1,2}(t) &= w_{1,3}(t) - \Gamma\tilde{\Gamma}^{-1}\big(q_2 + p_2 k_1(t)\big)\cdot\big(w_{1,1}(t) - G\tilde{w}(t)\big) + R_{r-1} w_{1,1}(t) + R_{r-1}\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t),\\
&\vdots\\
\dot{w}_{1,r-2}(t) &= w_{1,r-1}(t) - \Gamma\tilde{\Gamma}^{-1}\big(q_{r-2} + p_{r-2} k_1(t)\big)\cdot\big(w_{1,1}(t) - G\tilde{w}(t)\big) + R_3 w_{1,1}(t) + R_3\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t),\\
\dot{w}_{1,r-1}(t) &= w_{1,r}(t) - \Gamma\tilde{\Gamma}^{-1}\big(q_{r-1} + p_{r-1} k_1(t)\big)\cdot\big(w_{1,1}(t) - G\tilde{w}(t)\big) + R_2 w_{1,1}(t) + R_2\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t) - Gz^{(r-1)}(t),\\
\dot{w}_{1,r}(t) &= - \Gamma\tilde{\Gamma}^{-1}\big(q_r + p_r k_1(t)\big)\cdot\big(w_{1,1}(t) - G\tilde{w}(t)\big) + R_1 w_{1,1}(t) + R_1\Gamma\tilde{\Gamma}^{-1}\tilde{w}(t)\\
&\qquad + \sum_{i=1}^{r} R_i z^{(i-1)}(t) + f\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)\big),\\
k_1(t) &= \frac{1}{1 - \varphi_1(t)^2\|w_{1,1}(t) - G\tilde{w}(t)\|^2}.
\end{aligned} \tag{27a}$$

and, for $i = 2, \dots, r-1$,

$$\begin{aligned}
\dot{w}_{i,1}(t) &= w_{i,2}(t) - \big(q_1 + p_1 k_i(t)\big)\cdot w_{i,1}(t) + \big(q_1 + p_1 k_{i-1}(t)\big)\cdot\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t)-G\tilde{w}(t)\right)\ \text{if } i=2},\\
&\vdots\\
\dot{w}_{i,r-1}(t) &= w_{i,r}(t) - \big(q_{r-1} + p_{r-1} k_i(t)\big)\cdot w_{i,1}(t) + \big(q_{r-1} + p_{r-1} k_{i-1}(t)\big)\cdot\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t)-G\tilde{w}(t)\right)\ \text{if } i=2},\\
\dot{w}_{i,r}(t) &= - \big(q_r + p_r k_i(t)\big)\cdot w_{i,1}(t) + \big(q_r + p_r k_{i-1}(t)\big)\cdot\underbrace{w_{i-1,1}(t)}_{=\left(w_{1,1}(t)-G\tilde{w}(t)\right)\ \text{if } i=2},\\
k_i(t) &= \frac{1}{1 - \varphi_i(t)^2\|w_{i,1}(t)\|^2}.
\end{aligned} \tag{27b}$$

Step 2: We define the operator $\tilde{T} : C([-h,\infty)\to\mathbb{R}^m)^r \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0}\to\mathbb{R}^{\tilde{q}})$*, where* $\tilde{q} = (r-1)rm + r$*, (essentially) as the solution operator of* (27)*, i.e., for* $\zeta_1, \dots, \zeta_r \in C([-h,\infty)\to\mathbb{R}^m)$ *let* $w_{ij} : [0,\beta)\to\mathbb{R}^m$, $\beta\in(0,\infty]$*, be the unique maximal solution of* (27) *for* $z = \zeta_1, \dot{z} = \zeta_2, \dots, z^{(r-1)} = \zeta_r$ *with appropriate initial values according to the transformation which leads to* (27)*, and define*

$$\tilde{T}(\zeta_1, \dots, \zeta_r)(t) := \big(w_{1,1}(t), \dots, w_{1,r}(t), w_{2,1}(t), \dots, w_{r-1,r}(t), k_1(t), \dots, k_r(t)\big)^\top, \quad t\in[0,\beta).$$

We stress that $y, \dot{y}, \dots, y^{(r-1)}$ *in* (27a) *can be replaced by* $w_{i,j}$ *and* $z, \dot{z}, \dots, z^{(r-1)}$ *using* $y^{(i)} = z^{(i)} + w_{1,1}^{(i)} + \Gamma\tilde{\Gamma}^{-1}\tilde{w}^{(i)}$ *and the differential equations* (27)*. Furthermore, the operator* $\tilde{T}$ *depends on the disturbance* d *and several initial values. In the following we show that* $\tilde{T}$ *is well-defined, i.e.,* $\beta = \infty$*, and has the properties a)–c). Note that for*

$$\mathcal{D} := \left\{ \big(t, w_{1,1}, \dots, w_{1,r}, w_{2,1}, \dots, w_{r-1,r}\big) \in \mathbb{R}_{\geq 0}\times\mathbb{R}^{rm} \;\middle|\; \varphi_1(t)\left\|w_{1,1} - G\sum_{i=2}^{r-1} w_{i,1}\right\| < 1,\ \varphi_i(t)\|w_{i,1}\| < 1,\ i = 2, \dots, r-1 \right\}$$

we have $\big(t, w_{1,1}(t), \dots, w_{1,r}(t), w_{2,1}(t), \dots, w_{r-1,r}(t)\big) \in \mathcal{D}$ *for all* $t \in [0,\beta)$*, and the closure of the graph of the solution* $\big(w_{1,1}, \dots, w_{1,r}, w_{2,1}, \dots, w_{r-1,r}\big)$ *is not a compact subset of* $\mathcal{D}$*.*

Step 2a: First assume that $\zeta_1, \dots, \zeta_r$ are bounded on $[0,\beta)$. We show that $w_{i,j}$ and k_i are bounded as well. As the solution evolves in $\mathcal{D}$, it is clear that $w_{1,1} - G\tilde{w}$, $w_{2,1}, \dots, w_{r-1,1}$ are bounded, and thus also $w_{1,1}$ is bounded. Since $y = z + w_{1,1} + \Gamma\tilde{\Gamma}^{-1}\tilde{w}$, it follows that y is bounded and hence $T(y, \dot{y}, \dots, y^{(r-1)})$ is bounded by property a'). Boundedness of d and continuity of f then imply that $f\big(d(\cdot), T(y, \dot{y}, \dots, y^{(r-1)})(\cdot)\big)$ is bounded.

Now let $w_i := (w_{i,1}^\top, \dots, w_{i,r}^\top)^\top$, then it follows from (27) that

$$\begin{aligned} \dot{w}_1(t) &= \hat{A}w_1(t) - k_1(t)\bar{P}\Gamma\tilde{\Gamma}^{-1}\big(w_{1,1}(t) - G\tilde{w}(t)\big) + B_1(t), \\ \dot{w}_2(t) &= \hat{A}w_2(t) - k_2(t)\bar{P}w_{2,1}(t) + k_1(t)\bar{P}\big(w_{1,1}(t) - G\tilde{w}(t)\big) + B_2(t), \\ \dot{w}_i(t) &= \hat{A}w_i(t) - k_i(t)\bar{P}w_{i,1}(t) + k_{i-1}(t)\bar{P}w_{i-1,1}(t) + B_i(t) \end{aligned} \tag{28}$$

for $i = 3, \dots, r-1$, where $\hat{A}$ is as in the proof of Proposition 2.1, B_i is some suitable bounded function and

$$\bar{P} := \begin{bmatrix} p_1 I_m \\ \vdots \\ p_r I_m \end{bmatrix}.$$

Recall that $\hat{A}^\top\hat{P} + \hat{P}\hat{A} + \hat{Q} = 0$, where $\hat{P} > 0$ and $\hat{Q} > 0$, and that

$$\bar{P}^\top\hat{P} = [\tilde{p}I_m, 0, \dots, 0], \quad \tilde{p} := (P_1 - P_2P_4^{-1}P_2^\top) > 0.$$

We consider the cases $r = 2$ and $r = 3$ separately.

Step 2b: Assume that $r = 2$. Then (28) reads

$$\dot{w}_1(t) = \hat{A}w_1(t) - k_1(t)\bar{P}\Gamma\tilde{\Gamma}^{-1}w_{1,1}(t) + B_1(t).$$

Using the Lyapunov function $V(w_1) = w_1^\top\hat{P}w_1$ one can then show, as in the proof of Proposition 2.1, that w_1 and k_1 are bounded on $[0,\beta)$.

Step 2c: Assume that $r = 3$. Then (28) reads

$$\begin{aligned} \dot{w}_1(t) &= \hat{A}w_1(t) - k_1(t)\bar{P}\Gamma\tilde{\Gamma}^{-1}\big(w_{1,1}(t) - Gw_{2,1}(t)\big) + B_1(t), \\ \dot{w}_2(t) &= \hat{A}w_2(t) - k_2(t)\bar{P}w_{2,1}(t) + k_1(t)\bar{P}\big(w_{1,1}(t) - Gw_{2,1}(t)\big) + B_2(t). \end{aligned}$$

From condition (25) we obtain that $G = G^\top > 0$, hence $G\Gamma\tilde{\Gamma}^{-1} = \big(G\Gamma\tilde{\Gamma}^{-1}\big)^\top > 0$ has a unique matrix square root. Let $K := I_m \otimes \big(G\Gamma\tilde{\Gamma}^{-1}\big)^{\frac{1}{2}} > 0$ (recall the Kronecker product $\otimes$ from the proof of Proposition 2.1) and define the Lyapunov function $V(w_1, w_2) := w_1^\top\hat{P}w_1 + w_2^\top K^\top\hat{P}Kw_2$ for $w_1, w_2 \in \mathbb{R}^{3m}$. Then, for all $t \in [0,\beta)$,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}V\big(w_1(t), w_2(t)\big) &= w_1(t)^\top\big(\hat{A}^\top\hat{P} + \hat{P}\hat{A}\big)w_1(t) - 2k_1(t)w_1(t)^\top\hat{P}\bar{P}\Gamma\tilde{\Gamma}^{-1}\big(w_{1,1}(t) - G\tilde{w}(t)\big) \\ &\quad + 2w_1(t)^\top B_1(t) + w_2(t)^\top\big(\hat{A}^\top K^\top\hat{P}K + K^\top\hat{P}K\hat{A}\big)w_2(t) - 2k_2(t)w_2(t)^\top K^\top\hat{P}K\bar{P}w_{2,1}(t) \\ &\quad + 2w_2(t)^\top K^\top\hat{P}KB_2(t) + 2k_1(t)w_2(t)^\top K^\top\hat{P}K\bar{P}\big(w_{1,1}(t) - Gw_{2,1}(t)\big), \end{aligned}$$

and since it is easy to see that $\hat{A}$ and K commute and $K^\top\hat{P}K\bar{P} = \tilde{p}[I_m, 0, \dots, 0]^\top G\Gamma\tilde{\Gamma}^{-1}$, it follows that, for some positive $\alpha_1, \alpha_2, M_1, M_2$,

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}V\big(w_1(t), w_2(t)\big) &\le -\alpha_1\|w_1(t)\|^2 - \alpha_2\|w_2(t)\|^2 - 2k_1(t)\left(\tilde{p}w_{1,1}^\top\Gamma\tilde{\Gamma}^{-1} - \tilde{p}w_{2,1}^\top G\Gamma\tilde{\Gamma}^{-1}\right)\big(w_{1,1}(t) - Gw_{2,1}(t)\big) \\ &\quad + M_1\|w_1(t)\| + M_2\|w_2(t)\| \\ &= -\alpha_1\|w_1(t)\|^2 - \alpha_2\|w_2(t)\|^2 + M_1\|w_1(t)\| + M_2\|w_2(t)\| \\ &\quad - 2\tilde{p}k_1(t)\left(w_{1,1} - Gw_{2,1}\right)^\top\Gamma\tilde{\Gamma}^{-1}\big(w_{1,1}(t) - Gw_{2,1}(t)\big) \\ &\le -\alpha_1\|w_1(t)\|^2 - \alpha_2\|w_2(t)\|^2 + M_1\|w_1(t)\| + M_2\|w_2(t)\|. \end{aligned}$$

As in the proof of Proposition 2.1 we may now show that w_1 and w_2 are bounded and that k_1 and k_2 are bounded as well on $[0,\beta)$.

Step 2d: We show $\beta = \infty$ (not assuming boundedness of $\zeta_1, \dots, \zeta_r$). Assume that $\beta < \infty$. Then $\zeta_1, \dots, \zeta_r$ are bounded on $[0,\beta)$ and hence $w_{i,j}$ and k_i are bounded by Steps 2a–2c. Therefore, it follows that the closure of the graph of the solution $\big(w_{1,1}, \dots, w_{1,r}, w_{2,1}, \dots, w_{r-1,r}\big)$ is a compact subset of $\mathcal{D}$, a contradiction, thus $\beta = \infty$.

Step 2e: It remains to show that $\tilde{T}$ has the properties a)–c). Properties b) and c) are clear and property a) is an immediate consequence of Steps 2a–2c.

Step 3: By Step 2 we may write the conjunction of (24) *and* (17) *with input* u *and output* $z = z_{r-1,1}$ *in the form*

$$z^{(r)}(t) = \tilde{\Gamma} u(t) + \sum_{j=0}^{r-1} \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^j \left[\left(q_{r-j} + p_{r-j} k_{r-1}(t)\right) w_{r-1,1}(t)\right]$$

and hence

$$z^{(r)}(t) = F\left(\tilde{d}(t), \tilde{T}(z, \dot{z}, \ldots, z^{(r-1)})(t)\right) + \tilde{\Gamma} u(t)$$

for $\tilde{d}(t) := \left(\varphi_{r-1}(t), \dot{\varphi}_{r-1}(t), \ldots, \varphi_{r-1}^{(r-1)}(t)\right)^\top \in \mathcal{L}^\infty(\mathbb{R}_{\geq 0} \to \mathbb{R}^r)$, *some* $F \in \mathcal{C}(\mathbb{R}^r \times \mathbb{R}^{\tilde{q}} \to \mathbb{R}^m)$ *and the operator* $\tilde{T} : \mathcal{C}([-h, \infty) \to \mathbb{R}^m)^r \to \mathcal{L}^\infty_{\mathrm{loc}}(\mathbb{R}_{\geq 0} \to \mathbb{R}^{\tilde{q}})$ *which satisfies the properties a)–c). It is clear that any solution of* (17), (24) *satisfies the properties* (23) *and* (20). □

Remark 3.2. A careful inspection of the proof of Theorem 3.1 reveals that in order for Theorem 3.1 to hold true for $r \geq 4$ we would need to show that (28) has bounded solutions. However, we were only able to find suitable Lyapunov functions in the cases $r = 2$ and $r = 3$, thus the proof for $r \geq 4$ remains an open problem; in particular, a recursive Lyapunov function of the form $V_i(w_1, \ldots, w_i) = V_{i-1}(w_1, \ldots, w_{i-1}) + w_i^\top K_i^\top \hat{P} K_i w_i$ does not exist in the latter case. It is worth noting that in the case $r = 2$ no condition on $\tilde{\Gamma}$ is present and for $r = 3$ condition (25) means, roughly speaking, that we need to choose $\tilde{\Gamma}$ "larger than" Γ, which resembles a small gain condition, cf. [15]. We conjecture that some kind of small gain condition is needed in the case $r \geq 4$.

4 | FUNNEL CONTROL VIA FUNNEL PRE-COMPENSATOR

As discussed in Section 3, in virtue of Theorem 3.1 we may apply the funnel controller from [5] to the interconnection of system (24) with the funnel pre-compensator cascade in the cases $r = 2$ and $r = 3$, cf. Figure 6 . For completeness we state the resulting controller structure and the corresponding feasibility result. The funnel controller as in [5] is given by

$$\boxed{\begin{aligned} e_0(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t), \\ e_1(t) &= \dot{e}_0(t) + k_0(t) \cdot e_0(t), \\ e_2(t) &= \dot{e}_1(t) + k_1(t) \cdot e_1(t), \\ &\vdots \\ e_{r-1}(t) &= \dot{e}_{r-2}(t) + k_{r-2}(t) \cdot e_{r-2}(t), \end{aligned} \qquad \begin{aligned} k_i(t) &= \frac{1}{1 - \varphi_i(t)^2 \|e_i(t)\|^2}, \quad i = 0, \ldots, r-1, \\ u(t) &= -k_{r-1}(t) e_{r-1}(t), \end{aligned}} \tag{29}$$

where $r \in \mathbb{N}$ is the relative degree and the reference signal and funnel functions satisfy

$$y_{\mathrm{ref}} \in \mathcal{W}^{r,\infty}(\mathbb{R}_{\geq 0} \to \mathbb{R}^m), \qquad \varphi_0 \in \Phi_r, \ \ \varphi_1 \in \Phi_{r-1}, \ \ldots, \ \ \varphi_{r-1} \in \Phi_1. \tag{30}$$

In [5], the existence of solutions of the initial value problem resulting from the application of the funnel controller (29) to a system (24) is investigated (actually, a much larger class of nonlinear systems is allowed in [5]). We stress that (29) requires availability of $\dot{y}, \ldots, y^{(r-1)}$. This can be avoided using the funnel pre-compensator cascade.

The combination of the funnel controller (29) with the cascade of funnel pre-compensators $FP_{r-1} \circ \ldots \circ FP_1$ defined by (17), (18) reads as follows, where we only consider the two cases $r = 2$ and $r = 3$:

$$\boxed{\begin{aligned} &\textbf{Case } r = 2\textbf{:} \\ \dot{z}_1(t) &= z_2(t) + \left(q_1 + p_1 k_2(t)\right) \cdot (y(t) - z_1(t)), \\ \dot{z}_2(t) &= \left(q_2 + p_2 k_2(t)\right) \cdot (y(t) - z_1(t)) + \tilde{\Gamma} u(t), \\ e_0(t) &= z_1(t) - y_{\mathrm{ref}}(t), \\ e_1(t) &= \dot{e}_0(t) + k_0(t) e_0(t), \end{aligned} \qquad \begin{aligned} k_0(t) &= \tfrac{1}{1 - \varphi_0(t)^2 \|e_0(t)\|^2}, \\ k_1(t) &= \tfrac{1}{1 - \varphi_1(t)^2 \|e_1(t)\|^2}, \\ k_2(t) &= \tfrac{1}{1 - \varphi_2(t)^2 \|y(t) - z_1(t)\|^2}, \\ u(t) &= -k_1(t) \cdot e_1(t), \end{aligned}} \tag{31}$$

Case $r = 3$:

$$
\begin{aligned}
\dot z_{1,1}(t) &= z_{1,2}(t) + \big(q_1 + p_1 k_3(t)\big)\cdot(y(t) - z_{1,1}(t)), & k_0(t) &= \tfrac{1}{1-\varphi_0(t)^2\|e_0(t)\|^2},\\
\dot z_{1,2}(t) &= z_{1,3}(t) + \big(q_2 + p_2 k_3(t)\big)\cdot(y(t) - z_{1,1}(t)), & k_1(t) &= \tfrac{1}{1-\varphi_1(t)^2\|e_1(t)\|^2},\\
\dot z_{1,3}(t) &= \big(q_3 + p_3 k_3(t)\big)\cdot(y(t) - z_{1,1}(t)) + \tilde\Gamma u(t), & k_2(t) &= \tfrac{1}{1-\varphi_2(t)^2\|e_2(t)\|^2},\\
\dot z_{2,1}(t) &= z_{2,2}(t) + \big(q_1 + p_1 k_4(t)\big)\cdot(z_{1,1}(t) - z_{2,1}(t)), & k_3(t) &= \tfrac{1}{1-\varphi_3(t)^2\|y(t)-z_{1,1}(t)\|^2},\\
\dot z_{2,2}(t) &= z_{2,3}(t) + \big(q_2 + p_2 k_4(t)\big)\cdot(z_{1,1}(t) - z_{2,1}(t)), & k_4(t) &= \tfrac{1}{1-\varphi_4(t)^2\|z_{1,1}(t)-z_{2,1}(t)\|^2},\\
\dot z_{2,3}(t) &= \big(q_3 + p_3 k_4(t)\big)\cdot(z_{1,1}(t) - z_{2,1}(t)) + \tilde\Gamma u(t), & u(t) &= -k_2(t)\cdot e_2(t),\\
e_0(t) &= z_{2,1}(t) - y_{\mathrm{ref}}(t),\\
e_1(t) &= \dot e_0(t) + k_0(t)\cdot e_0(t),\\
e_2(t) &= \dot e_1(t) + k_1(t)\cdot e_1(t),
\end{aligned}
\tag{32}
$$

where y_{ref} and $\varphi_0, \dots, \varphi_{r-1}$ satisfy (30), $\tilde\Gamma = \tilde\Gamma^\top > 0$, $\varphi_r, \dots, \varphi_{2r-2} \in \Phi_{r-1}$ and $q_1, \dots, q_r, p_1, \dots, p_r > 0$ are such that (5) is satisfied for corresponding matrices P and Q. In a slightly different structure, the controller (31) for the case $r = 2$ was already successfully implemented in [8], see also the discussion therein.

Note that the derivatives $\dot e_0$ and $\dot e_1$ that appear in (31) and (32) only serve as short-hand notations and may be resolved in terms of the pre-compensator states and the output y using the differential equations in (31) and (32) as follows:

$$
\begin{aligned}
r = 2\textbf{:}\quad & \dot e_0(t) = z_2(t) + \big(q_1 + p_1 k_2(t)\big)(y(t) - z_1(t)) - \dot y_{\mathrm{ref}}(t),\\
r = 3\textbf{:}\quad & \dot e_0(t) = z_{2,2}(t) + \big(q_1 + p_1 k_4(t)\big)(z_{1,1}(t) - z_{2,1}(t)) - \dot y_{\mathrm{ref}}(t),\\
& \dot e_1(t) = z_{2,3}(t) + \big(q_2 + p_2 k_4(t)\big)(z_{1,1}(t) - z_{2,1}(t)) + p_1 k_4(t)^2\Big(\varphi_4(t)\dot\varphi_4(t)\|z_{1,1}(t) - z_{2,1}(t)\|^2\\
&\qquad + \varphi_4(t)^2(z_{1,1}(t) - z_{2,1}(t))^\top\big(z_{1,2}(t) + (q_1 + p_1 k_3(t))(y(t) - z_{1,1}(t))\\
&\qquad - z_{2,2}(t) - (q_1 + p_1 k_4(t))(z_{1,1}(t) - z_{2,1}(t))\big)\Big)(z_{1,1}(t) - z_{2,1}(t))\\
&\qquad + \big(q_1 + p_1 k_2(t)\big)\big(z_{1,2}(t) + (q_1 + p_1 k_3(t))(y(t) - z_{1,1}(t)) - z_{2,2}(t) - (q_1 + p_1 k_4(t))(z_{1,1}(t) - z_{2,1}(t))\big)\\
&\qquad - \ddot y_{\mathrm{ref}}(t) + k_0(t)^2\Big(\varphi_0(t)\dot\varphi_0(t)\|e_0(t)\|^2 + \varphi_0(t)^2 e_0(t)^\top \dot e_0(t)\Big)e_0(t) + k_0(t)\dot e_0(t).
\end{aligned}
$$

We stress that the dynamic output error feedback controllers (31) and (32) are model-free, of (comparatively) low complexity, robust with respect to modeling errors, disturbances and uncertainties, and they achieve prescribed performance of the tracking error. Feasibility of (31) and (32) in the respective cases is a direct consequence of Theorem 3.1 and [5, Thm. 3.1].

Corollary 4.1
Consider a system (24) with $r \in \{2, 3\}$, $y^0 \in \mathcal{W}^{r-1,\infty}([-h, 0] \to \mathbb{R}^m)$ and assume that $\Gamma = \Gamma^\top > 0$ and the operator $\boldsymbol{T}$ satisfies a'). Let y_{ref} and $\varphi_0, \dots, \varphi_{r-1}$ be such that (30) holds and $\varphi_r, \dots, \varphi_{2r-2} \in \Phi_{r-1}$ be such that z_1, e_0, e_1 as defined in (31) or $z_{1,1}, z_{2,1}, e_0, e_1, e_2$ as defined in (32), resp., with initial data (18) satisfy

$$\varphi_i(0)\|e_i(0)\| < 1, \quad \text{for all } i = 0, \dots, r-1,$$

and

$$
\begin{aligned}
\varphi_2(0)\|y(0) - z_1(0)\| < 1, &\quad \text{if } r = 2,\\
\varphi_3(0)\|y(0) - z_{1,1}(0)\| < 1 \quad\text{and}\quad \varphi_4(0)\|z_{1,1}(0) - z_{2,1}(0)\| < 1, &\quad \text{if } r = 3.
\end{aligned}
$$

Further let $q_1, \dots, q_r, p_1, \dots, p_r > 0$ be such that (5) is satisfied for corresponding matrices A, P, Q, and let $\tilde\Gamma = \tilde\Gamma^\top > 0$ be such that (25) is satisfied.

Then the application of the funnel controller (31) (if $r = 2$) or (32) (if $r = 3$), resp., to (24) yields an initial-value problem, which has a solution, and every solution can be extended to a maximal solution $(y, z) : [-h, \omega) \to \mathbb{R}^m$, $\omega \in (0, \infty]$, where $z = (z_1, z_2)$ if $r = 2$ and $z = (z_{1,1}, z_{1,2}, z_{1,3}, z_{2,1}, z_{2,2}, z_{2,3})$ if $r = 3$, which has the following properties:

(i) The solution is global (i.e., $\omega = \infty$).

(ii) The input $u : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$, the gain functions $k_0, \dots, k_{2r-2} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, the compensator states z and $y, \dots, y^{(r-1)} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ are bounded.

(iii) The functions $e_0, \dots, e_{r-1} : \mathbb{R}_{\geq 0} \to \mathbb{R}^m$ and the compensator errors $y - z_1$ or $y - z_{1,1}, z_{1,1} - z_{2,1}$, resp., evolve in their respective performance funnels in the sense

$$\begin{aligned} \exists\, \varepsilon_0, \dots, \varepsilon_{2r-2} > 0\ \forall\, t > 0 : \qquad\qquad & \|e_i(t)\| \leq \varphi_i(t)^{-1} - \varepsilon_i, \quad i = 0, \dots, r-1, \\ & \|y(t) - z_1(t)\| \leq \varphi_2(t)^{-1} - \varepsilon_2, \quad \text{if } r = 2, \\ \|y(t) - z_{1,1}(t)\| \leq \varphi_3(t)^{-1} - \varepsilon_3 \quad \text{and} \quad & \|z_{1,1}(t) - z_{2,1}(t)\| \leq \varphi_4(t)^{-1} - \varepsilon_4, \quad \text{if } r = 3. \end{aligned}$$

In particular, the tracking error $e(t) = y(t) - y_{\text{ref}}(t)$ satisfies, for all $t > 0$,

$$\begin{aligned} \|e(t)\| &\leq \varphi_0(t)^{-1} + \varphi_2(t)^{-1} - \varepsilon_0 - \varepsilon_2, \quad \text{if } r = 2, \\ \|e(t)\| &\leq \varphi_0(t)^{-1} + \varphi_3(t)^{-1} + \varphi_4(t)^{-1} - \varepsilon_0 - \varepsilon_3 - \varepsilon_4, \quad \text{if } r = 3. \end{aligned}$$

Remark 4.2. The controllers (31) and (32) contain a lot of design parameters and there is a lot of freedom in choosing them. First of all, the parameters q_i must be chosen as the coefficients of a Hurwitz polynomial and then the p_i are defined in terms of the q_i, see Section 2 and Figure 3 . While the choice of the q_i influences the convergence speed of the pre-compensator states, in our simulations it turned out that varying the q_i had only little effect on the overall controller performance. Larger effects can be achieved by appropriately designing the funnel functions φ_i of the controllers. While the performance of the tracking error is usually prescribed by the problem or application (and hence, if $r = 2$, φ_0 and φ_2 or, if $r = 3$, φ_0, φ_3 and φ_4 must be chosen accordingly), there is still a lot of freedom in choosing the funnel functions for e_i with $i \geq 1$. A brief analysis of appropriate choices and some rules of thumb can be found in [5, Sec. 4.2].

Remark 4.3. We compare the controllers (31) and (32) to an alternative approach presented in the recent conference paper [14]; a similar approach, based on prescribed performance control, can be found in [4]. In the work [14] the funnel controller from [26] is combined with a high-gain observer. For nonlinear SISO systems with higher relative degree a virtual (weighted) output is defined as

$$s(t) = e(t) + k_2 \mu \dot{e}(t) + \ldots + k_r \mu^{r-1} e^{(r-1)}(t)$$

for some design parameters $k_i > 0$ and a scaling parameter $\mu > 0$. Then the system has relative degree one with respect to the virtual output s. In a first step, it is shown that funnel control is feasible for the system with output s, and for sufficiently small scaling parameter μ the original tracking error e is close to s and hence evolves in a prescribed performance funnel. However, tuning of the scaling parameter μ has to be done a posteriori and hence depends on the system parameters and the chosen reference trajectory; once a parameter value is fixed, error evolution in the performance funnel cannot be guaranteed when the reference signal or system parameters are changed. Suitable values for μ need to be identified by offline simulations which contrasts the identification-free methodology of funnel control. In particular, this approach is not model-free like standard funnel control approaches and the controller is not robust in the sense that, when all its parameters are fixed, it works for a class of systems satisfying some structural assumption.
Since the above approach still involves output derivatives, in a second step presented in [14], the output derivatives are estimated using a high-gain observer as follows:

$$\begin{aligned} \dot{z}_1(t) &= z_2(t) + \tfrac{\alpha_1}{\varepsilon}(e(t) - z_1(t)), \\ \dot{z}_2(t) &= z_3(t) + \tfrac{\alpha_2}{\varepsilon^2}(e(t) - z_1(t)), \\ &\vdots \\ \dot{z}_{r-1}(t) &= z_r(t) + \tfrac{\alpha_{r-1}}{\varepsilon^{r-1}}(e(t) - z_1(t)), \\ \dot{z}_r(t) &= F\big(t, z_1(t), \dots, z_r(t)\big) + \Gamma u(t) + \tfrac{\alpha_r}{\varepsilon^r}(e(t) - z_1(t)), \end{aligned} \tag{33}$$

where $\alpha_1, \dots, \alpha_r$ are such that the matrix $\begin{bmatrix} -\alpha_1 & 1 & & \\ \vdots & & \ddots & \\ -\alpha_{r-1} & & & 1 \\ -\alpha_r & & & 0 \end{bmatrix} \in \mathbb{R}^{r \times r}$ is Hurwitz (similar to the q_i chosen for (1)) and ε is a small tuning parameter; actually $k = 1/\varepsilon$ is the high-gain parameter of the high-gain observer (33). The function F is a nominal model for the dynamics of (24) when $u = 0$. We stress that for (33) the high-frequency gain matrix Γ needs to be known exactly. It is shown in [14] that, for fixed system data, μ and ε can be chosen sufficiently small so that the funnel controller combined with the high-gain observer yields a tracking error e which evolves in a prescribed performance funnel. However, the drawbacks described above for the controller without high-gain observer remain.
An alternative to using the virtual output s with scaling parameter μ may be to use the funnel controller (29) instead and

to combine it with the high-gain observer (33) so that $e^{(i)}$ is replaced by its estimate z_i. For relative degree $r = 3$ and the choice $F = 0$ the resulting controller takes the form

$$\boxed{\begin{array}{ll} \dot z_1(t) = z_2(t) + \frac{\alpha_1}{\varepsilon}(e(t) - z_1(t)), & e_0(t) = z_1(t), \\ \dot z_2(t) = z_3(t) + \frac{\alpha_2}{\varepsilon^2}(e(t) - z_1(t)), & e_1(t) = z_2(t) + k_0(t)e_0(t), \\ \dot z_3(t) = \Gamma u(t) + \frac{\alpha_3}{\varepsilon^3}(e(t) - z_1(t)), & e_2(t) = z_3(t) + \frac{\mathrm{d}}{\mathrm{d}t}\big(k_0(t)e_0(t)\big) + k_1(t)e_1(t), \\ k_i(t) = \dfrac{1}{1 - \varphi_i(t)^2\|e_i(t)\|^2}, \quad i = 0, 1, 2, & \\ u(t) = -k_2(t)e_2(t). & \end{array}} \tag{34}$$

More explicitly, we may write

$$e_2(t) = z_3(t) + 2k_0(t)^2\big(\varphi_0(t)\dot\varphi_0(t)\|e_0(t)\|^2 + \varphi_0(t)^2 e_0(t)^\top z_2(t)\big)e_0(t) + k_0(t)z_2(t) + k_1(t)e_1(t).$$

The controller (34) is very sensitive with respect to the choice of the initial values. If $z_1(0) \neq e(0)$, then enormous peaks in the generated control u at the beginning are to be expected because of the peaking phenomenon of the high-gain observer, cf. [2, 18, 32]. To circumvent the peaking, saturation is commonly used. However, we stress that a saturation of the observer states as suggested in [14] is not possible here, since this only leads to infeasibility of the controller (the closed-loop system does not have a global solution). Instead, the control input must be directly saturated using

$$u(t) = \operatorname{sat}\big(-k_2(t)e_2(t), \bar u\big), \tag{35}$$

where $\bar u > 0$ is the saturation level and

$$\operatorname{sat} : \mathbb{R} \times \mathbb{R} \to \mathbb{R},\ (v, \bar v) \mapsto \begin{cases} v, & |v| \le \bar v, \\ \operatorname{sign}(v)\,\bar v, & |v| > \bar v. \end{cases}$$

While the peaking phenomenon can be avoided by the saturation of the input, care must be taken with the choice of the saturation level, because values which are too small may lead to infeasibility of the controller in the sense that the solution leaves the domain of the closed-loop differential equation in finite time. Appropriate saturation levels must be identified, and we stress that a formal feasibility proof for the controller (34) with and without saturation (35) does not exist yet. Therefore, we consider (34), (35) only for reasons of comparison.

In (34) only the scaling parameter ε from the high-gain observer remains. Nevertheless, the choice of ε still depends on the system parameters, thus most of the drawbacks remain. Note also that the high-frequency gain matrix Γ still appears in (34), however, due to input saturation, it is possible to relax this and use some nominal model instead, cf. [31]. We compare the controller (34), (35) to the controller (32) in a simulation in Section 5.2. For the exact same simulation setup, the controller (34) with saturation (35) (when ε is sufficiently small) performs better than (32). However, we note that (34), (35) requires a lot of computational effort, which dramatically increases when $|z_1(0) - e(0)|$ increases; more often than not the numerical methods were not able to provide a solution in our simulations. Such a behavior does not happen for the controller (32) which uses the funnel pre-compensator, which hence seems preferable to (34) in practical applications, where real-time capability is required. Let us summarize the drawbacks of the controller (34), (35) compared to (32):

- no feasibility proof is available for (34) (with or without (35));
- (34) is not model-free and suitable values for ε need to be identified by offline simulations;
- the choice of ε depends on the system parameters and the reference trajectory, hence, once it has been fixed, changes in the system parameters may lead to infeasibility;
- the computational effort for (34), (35) is very sensitive with respect to the choice of initial values and dramatically increases when $|z_1(0) - e(0)|$ increases.

5 | SIMULATIONS

5.1 | Mass on car system

We illustrate the combined funnel controller and funnel pre-compensator in (31) and (32) by a simulation for a mass-spring system mounted on a car from [43], see Fig. 7 , and compare it to the simulation of the funnel controller (29) for this system as performed in [5]. As depicted in Fig. 7 , the mass m_2 (in kg) moves on a ramp which is inclined by the angle α (in rad) and mounted on a car with mass m_1 (in kg). We assume that we may control the force $u = F$ (in N) acting on it. The equations of motion are given by

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{x}(t) \\ \ddot{s}(t) \end{pmatrix} + \begin{pmatrix} 0 \\ ks(t) + d\dot{s}(t) \end{pmatrix} = \begin{pmatrix} u(t) \\ 0 \end{pmatrix}, \tag{36}$$

where x (in m) is the horizontal car position and s (in m) the relative position of the mass on the ramp. The constants k (in N/m), d (in Ns/m) are the coefficients of the spring and damper, resp. The output is the horizontal position of the mass on the ramp,

$$y(t) = x(t) + s(t)\cos\alpha.$$

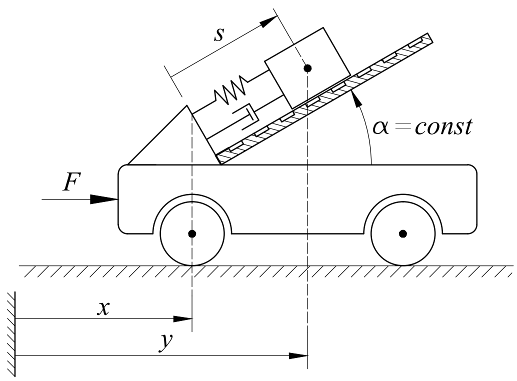

FIGURE 7 Mass on car system.

The system (36) can be reformulated such that it belongs to the class (24), see [43], with a relative degree r depending on the angle α and the damping d. We distinguish three cases, where the first two correspond to the same experimental setup as in [5].

Case 1: If $0\,\mathrm{rad} < \alpha < \frac{\pi}{2}\,\mathrm{rad}$, see Fig. 7 , then system (36) has relative degree $r = 2$ and the high-frequency gain matrix reads $\Gamma = \frac{\sin^2\alpha}{m_1+m_2\sin^2\alpha} > 0\,/\mathrm{kg}$. For the simulation, we choose the reference trajectory $y_{\mathrm{ref}}(t) = \cos t$ m, the parameters $m_1 = 4\,\mathrm{kg}, m_2 = 1\,\mathrm{kg}, k = 2\,\mathrm{N/m}, d = 1\,\mathrm{Ns/m}$, the initial values $x(0) = s(0) = 0\,\mathrm{m}$, $\dot{x}(0) = \dot{s}(0) = 0\,\mathrm{m/s}$, and $\alpha = \frac{\pi}{4}$ rad. For the controller (31) we choose the initial values $z_1(0) = z_2(0) = 0$, the funnel functions

$$\varphi_0(t) = \varphi_2(t) = 2(5e^{-2t} + 0.1)^{-1}, \qquad \varphi_1(t) = (1.3e^{-4t} + 0.01)^{-1},$$

and $\tilde{\Gamma} = \frac{1}{4}/\mathrm{kg} > \frac{1}{9}/\mathrm{kg} = \Gamma$. The parameters q_i, p_i are determined by the coefficients of the Hurwitz polynomial

$$(s+5)^2 = s^2 + 10s + 25,$$

by which $q_1 = 10$ and $q_2 = 25$. Therefore, $A = \left[\begin{smallmatrix} -10 & 1 \\ -25 & 0 \end{smallmatrix}\right]$ and the Lyapunov equation $A^\top P + PA = -I_2$ has the solution

$$P = \begin{bmatrix} \frac{13}{10} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{63}{250} \end{bmatrix},$$

by which $p_1 = 1$ and $p_2 = \frac{125}{63}$. Obviously the initial errors lie within the respective funnel boundaries and all assumptions of Corollary 4.1 are satisfied, thus it yields that funnel control is feasible. The sum $\varphi_0^{-1} + \varphi_2^{-1}$ equals the funnel boundary as chosen for the simulation in [5], hence the results may be compared.

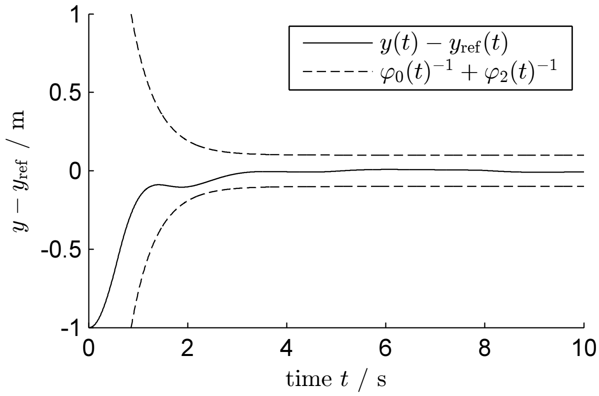

Fig. 8 a: Funnel and tracking error

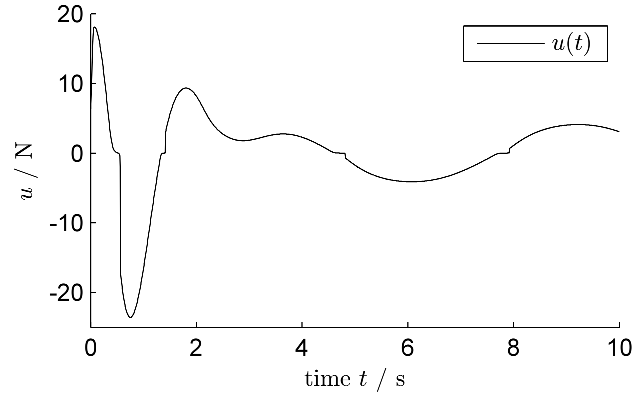

Fig. 8 b: Input function

FIGURE 8 Simulation of the controller (31) for the mass on car system (36) with $\alpha = \frac{\pi}{4}$ rad.

The simulation of the controller (31) applied to (36) over the time interval $0 - 10$ s has been performed in MATLAB (solver: `ode15s`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 8 . Fig. 8 a shows the tracking error and the funnel boundary, while Fig. 8 b shows the corresponding input function generated by the controller. It can be seen that the proposed funnel controller (31) guarantees that the tracking error evolves within the prescribed performance funnel and it yields a similar performance of the input as the controller (29) when we compare it to the simulation results in [5]. A video clip of the simulation can be found in the supplementary material.

Case 2: If $\alpha = 0$ rad and $d \neq 0$ Ns/m, see Fig. 9 , then system (36) has relative degree $r = 3$ and high-frequency gain matrix $\Gamma = \frac{d}{m_1 m_2} > 0$ /kgs. For the simulation, we choose the reference trajectory $y_{\mathrm{ref}}(t) = \cos t$ m, the parameters $m_1 = 4$ kg, $m_2 = 1$ kg, $k = 2$ N/m, $d = 1$ Ns/m and the initial values $x(0) = s(0) = 0$ m, $\dot{x}(0) = \dot{s}(0) = 0$ m/s.

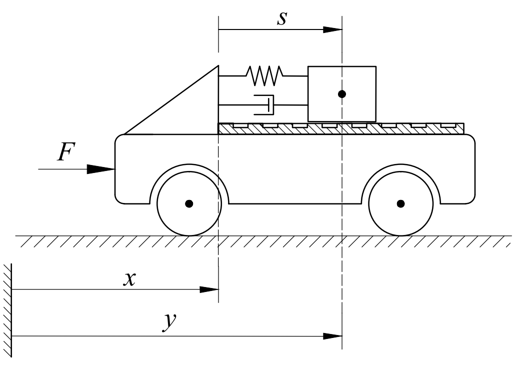

FIGURE 9 Mass on car system with $\alpha = 0$ rad.

For the controller (32) we choose the initial values $z_{i,j}(0) = 0$, $i = 1, 2$, $j = 1, 2, 3$, the funnel functions

$$\varphi_0(t) = \varphi_3(t) = \varphi_4(t) = 3(10e^{-2t} + 0.1)^{-1}, \quad \varphi_1(t) = (2.5e^{-3t} + 0.01)^{-1}, \quad \varphi_2(t) = (15e^{-20t} + 0.01)^{-1}$$

and $\tilde{\Gamma} = 0.8$/kgs $> \frac{1}{4}$/kgs $= \Gamma$ such that (25) is satisfied. The parameters q_i, p_i are determined by the coefficients of the Hurwitz polynomial

$$(s + 5)^3 = s^3 + 15s^2 + 75s + 125,$$

by which $q_1 = 15$, $q_2 = 75$ and $q_3 = 125$. Therefore, $A = \begin{bmatrix} -15 & 1 & 0 \\ -75 & 0 & 1 \\ -125 & 0 & 0 \end{bmatrix}$ and the Lyapunov equation $A^\top P + PA = -I_3$ has the solution

$$P = \begin{bmatrix} \frac{58}{5} & -\frac{1}{2} & -\frac{136}{125} \\ -\frac{1}{2} & \frac{136}{125} & -\frac{1}{2} \\ -\frac{136}{125} & -\frac{1}{2} & \frac{1333}{3125} \end{bmatrix},$$

by which $p_1 = 1$, $p_2 = \frac{1383}{391}$ and $p_3 = \frac{2230}{333}$. The initial errors lie within the respective funnel boundaries and all assumptions of Corollary 4.1 are satisfied, thus it yields that funnel control is feasible.

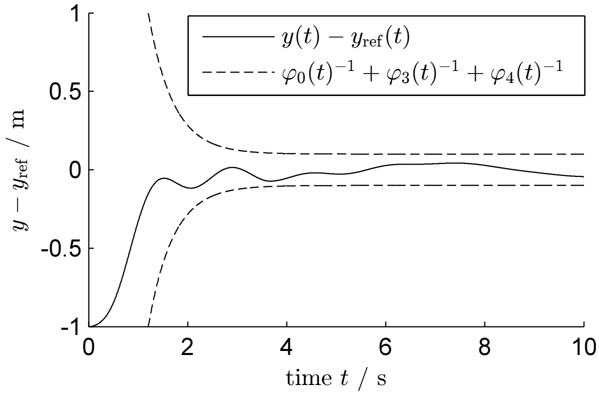

Fig. 10 a: Funnel and tracking error

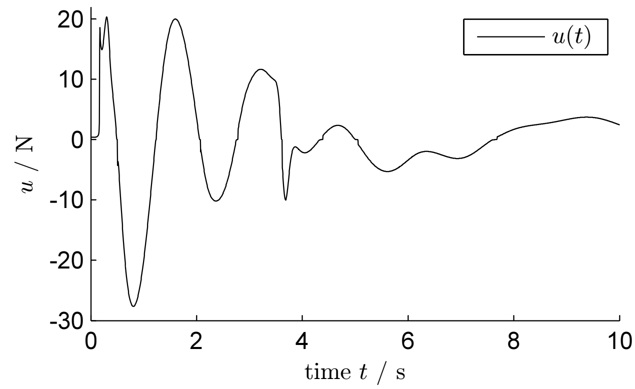

Fig. 10 b: Input function

FIGURE 10 Simulation of the controller (32) for the mass on car system (36) with $\alpha = 0\,\mathrm{rad}$.

The simulation of the controller (32) applied to (36) over the time interval $0 - 10\,\mathrm{s}$ has been performed in MATLAB (solver: `ode15s`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 10 , where the tracking error is shown in Fig. 10 a and the input in Fig. 10 b. We see that the funnel controller (32) is able to guarantee that the tracking error evolves within the prescribed performance funnel. The performance of the control input generated by (32) is comparable to that generated by the controller (29) in the simulation results in [5]; we stress that the controller (32) does not require availability of $\dot{y}$ and $\ddot{y}$. A video clip of the simulation can be found in the supplementary material.

Case 3: If $\alpha = 0\,\mathrm{rad}$, $d = 0\,\mathrm{Ns/m}$ and $k \neq 0\,\mathrm{N/m}$, then system (36) has relative degree $r = 4$ and high-frequency gain matrix $\Gamma = \frac{k}{m_1 m_2} > 0\,/\mathrm{kgs}^2$. For this case we did not state a feasible funnel controller in Section 4 since we were not able to extend the result of Theorem 3.1 to the case $r \geq 4$, cf. also Remark 3.2. Nevertheless, the funnel controller (29) may be combined with the funnel pre-compensator cascade (17), (18) even in the case $r = 4$, however a feasibility proof does not exist for the resulting controller. We omit the statement of the controller (which extends along the lines of (31) and (32)) and only provide a simulation for it which may serve as a motivation for future research for a feasibility proof.

For the simulation, we choose the reference trajectory $y_{\mathrm{ref}}(t) = \sin t$ m, the parameters $m_1 = 4\,\mathrm{kg}, m_2 = 1\,\mathrm{kg}, k = 2\,\mathrm{N/m}, d = 0\,\mathrm{Ns/m}$ and the initial values $x(0) = s(0) = 0\,\mathrm{m}$, $\dot{x}(0) = \dot{s}(0) = 0\,\mathrm{m/s}$. For the controller we choose the initial values $z_{i,j}(0) = 0$, $i = 1, 2, 3$, $j = 1, \dots, 4$, the funnel functions

$$\varphi_0(t) = \varphi_4(t) = \varphi_5(t) = \varphi_6 = 4(10e^{-2t} + 0.1)^{-1}, \quad \varphi_1(t) = \varphi_2(t) = (4e^{-3t} + 0.1)^{-1}, \quad \varphi_3(t) = (10e^{-20t} + 0.1)^{-1}$$

and $\tilde{\Gamma} = 0.6/\mathrm{kgs}^2 > 0.5/\mathrm{kgs}^2 = \Gamma$. The parameters q_i, p_i are determined by the coefficients of the Hurwitz polynomial

$$(s + 5)^4 = s^4 + 20s^3 + 150s^2 + 500s + 625,$$

by which $q_1 = 20$, $q_2 = 150$, $q_3 = 500$ and $q_4 = 625$. Therefore, $A = \begin{bmatrix} -20 & 1 & 0 & 0 \\ -150 & 0 & 1 & 0 \\ -500 & 0 & 0 & 1 \\ -625 & 0 & 0 & 0 \end{bmatrix}$ and the solution of the Lyapunov equation $A^\top P + PA = -I_4$ leads to $p_1 = 1$, $p_2 = \frac{2817}{5771}$, $p_3 = \frac{3029}{150}$ and $p_4 = \frac{6197}{263}$.

The simulation of the funnel controller (29) combined with the funnel pre-compensator cascade (17), (18) for $r = 4$ applied to (36) over the time interval $0 - 10\,\mathrm{s}$ has been performed in MATLAB (solver: `ode15s`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is

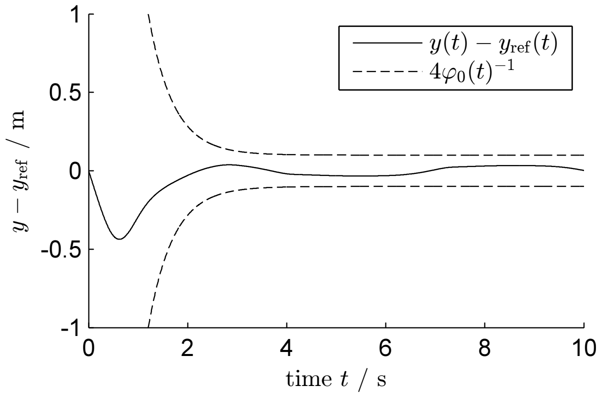

Fig. 11 a: Funnel and tracking error

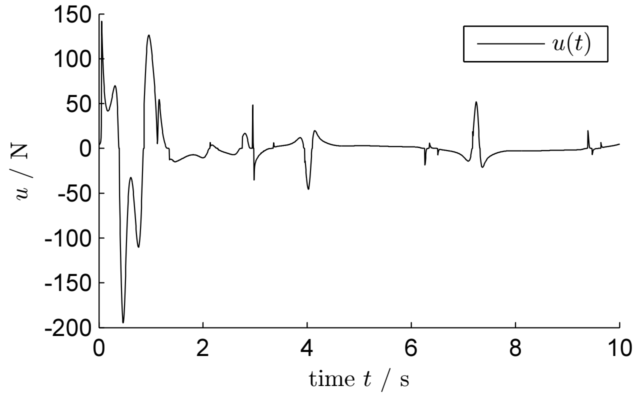

Fig. 11 b: Input function

FIGURE 11 Simulation of the funnel controller (29) combined with the funnel pre-compensator cascade (17), (18) for $r = 4$ for the mass on car system (36) with $\alpha = 0\,\mathrm{rad}$ and $d = 0\,\mathrm{Ns/m}$.

depicted in Fig. 11 . The tracking error is shown in Fig. 11 a and the input in Fig. 11 b. We see that prescribed performance of the tracking error is achieved, although nor formal feasibility proof is available yet.

5.2 | Comparison with controller based on high-gain observer

We compare the combination of funnel controller and funnel pre-compensator cascade to the combination of funnel controller and high-gain observer. To this end, we consider the mass on car system (36) with $\alpha = 0\,\mathrm{rad}$, i.e., Case 2 with relative degree $r = 3$. For the simulation we chose the same setup as described in Case 2 above and the corresponding parameters for the controller (32), however with the different initial values $z_{1,1}(0) = z_{2,1}(0) = 0.02$ for the pre-compensator states. For the controller (34) with saturated input (35) we choose the initial values $z_1(0) = -0.98$, $z_2(0) = z_3(0) = 0$, the funnel functions

$$\varphi_0(t) = (10e^{-2t} + 0.1)^{-1}, \quad \varphi_1(t) = (2.5e^{-3t} + 0.01)^{-1}, \quad \varphi_2(t) = (15e^{-20t} + 0.01)^{-1},$$

$\alpha_i = q_i$ for $i = 1, 2, 3$, and $\varepsilon = 1/20$ as well as saturation level $\bar{u} = 10$.

The simulation of the controllers (32) and (34), (35), resp., applied to (36) over the time interval $0 - 10\,\mathrm{s}$ has been performed in MATLAB (solver: `ode15s`, rel. tol.: 10^{-14}, abs. tol.: 10^{-10}) and is depicted in Fig. 12 , where the tracking errors are shown in Fig. 12 a and c and the corresponding inputs in Fig. 12 b and d. It can be seen that, because of the saturation, the performance of the controller (34), (35) is better than that of the controller (32) depicted in Fig. 10 .

We stress that further increasing $|z_1(0) - e(0)|$ resulted in a dramatic increase of the computational effort: for $z_{1,1}(0) = z_{2,1}(0) = 0.05$ and $z_1(0) = -0.95$ the computations took only a couple of seconds for the controller (32) on our machine (Intel Core i5-3570, 8 GB RAM), while it were about 90 minutes for the controller (34), (35). The huge difference in runtime may be explained by the dimensions of the gains in this case: the maximum of k_2 in (32) is approximately $2.5 \cdot 10^3$ and k_2 in (34) is approximately $3 \cdot 10^4$. Therefore, the controller (34), (35) does not seem to be suitable for real-time control applications and, together with other drawbacks which are discussed in Remark 4.3, this questions its usefulness in real-world applications.

6 | CONCLUSION

In the present paper we have introduced the funnel pre-compensator as a novel and simple adaptive pre-compensator, which resembles the structure of high-gain observers. We showed that the funnel pre-compensator is feasible for the large class of signal pairs $\mathcal{P}_r$. The proposed adaptation scheme for the pre-compensator gain is of low complexity and inherently robust since its design is model-free, and we showed that it guarantees prescribed transient behavior of the compensator error. Using a cascade of funnel pre-compensators, we proved that it is possible to obtain an artificial output with explicitly known derivatives which tracks the system output with prescribed transient behavior. As an application in adaptive control, we showed that for a certain class

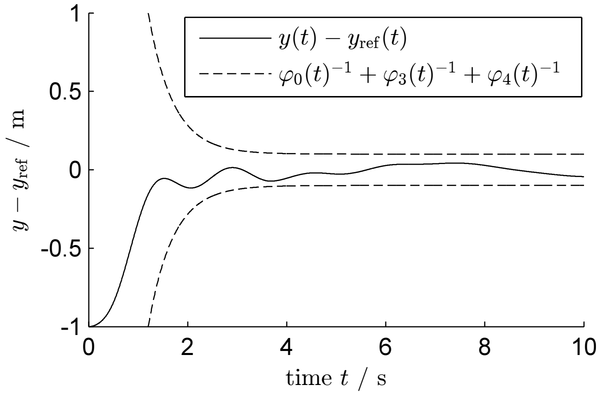

Fig. 12 a: Funnel and tracking error for (32)

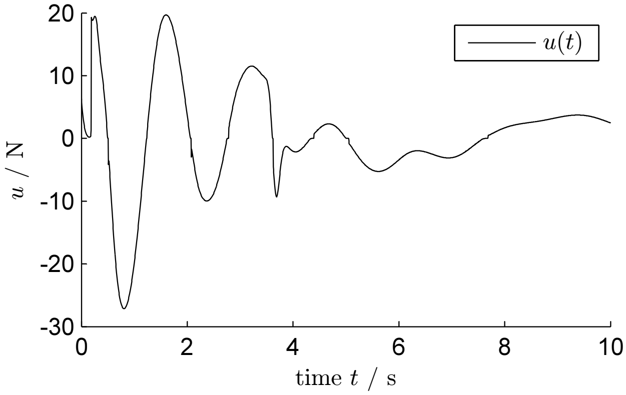

Fig. 12 b: Input function for (32)

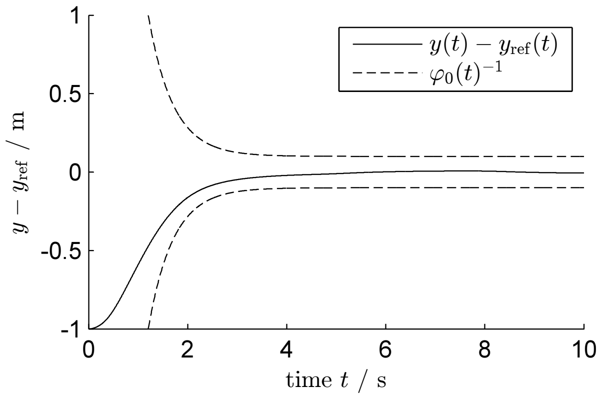

Fig. 12 c: Funnel and tracking error for (34)

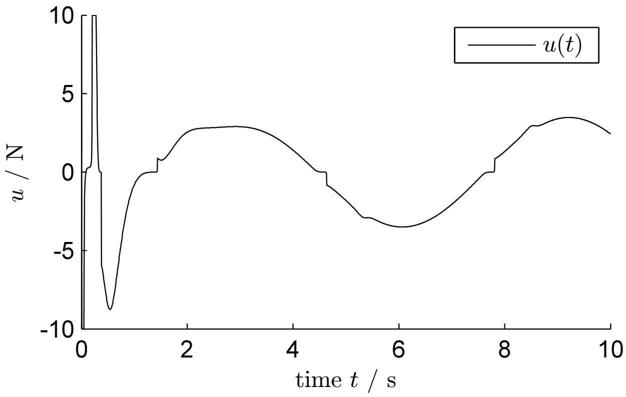

Fig. 12 d: Input function for (34)

FIGURE 12 Simulation of the controllers (32) and (34), (35) for the mass on car system (36) with $\alpha = 0\,\mathrm{rad}$.

of nonlinear systems, the interconnection with the funnel pre-compensator cascade has input-to-state stable internal dynamics provided the relative degree does not exceed three. This guarantees feasibility of a novel funnel controller which consists of a funnel pre-compensator cascade in conjunction with the recently developed funnel controller from [5]; this new controller does not require the derivatives of the output. We have compared this controller to the combination of the funnel controller from [5] with a high-gain observer.

The results that we obtained in Sections 3 and 4 suggest that the funnel pre-compensator is a suitable tool for resolving the problem of higher relative degree in stabilization and tracking problems with prescribed performance. If a system has a higher relative degree and derivatives of the output are not available, then a filter or observer is frequently used to obtain approximations of the output derivatives, see the survey [24] and the references therein. As explained there, the concept of funnel control is usually combined with a back-stepping procedure to overcome the higher relative degree, which however is quite complicated and impractical, cf. [20, Sec. 4.4.3]. Nevertheless, in the last sentence of [24, Sec. 6] it is conjectured that the combination of a high-gain observer with a funnel-type controller might be beneficial for tracking of higher relative degree systems. In Section 4 we have shown that the funnel pre-compensator, which resembles a high-gain observer, may be used to achieve this for systems with relative degree two or three. Systems of higher relative degree are the topic of future research.

ACKNOWLEDGEMENT

We are indebted to Achim Ilchmann (TU Ilmenau) for bringing this problem to our attention and for several constructive discussions. We thank Sergey Dashkovskiy (U Würzburg) for pointing out that condition (25) resembles a small gain condition.

References

[1] Alessandro Astolfi and Lorenzo Marconi. A high-gain nonlinear observer with limited gain power. IEEE Trans. Autom. Control, 60(11):3059–3064, 2015.

[2] A.N. Atassi and Hassan K. Khalil. A separation principle for the stabilization of a class of nonlinear systems. IEEE Trans. Autom. Control, 44(9):1672–1687, 1999.

[3] Charalampos P. Bechlioulis and George A. Rovithakis. A low-complexity global approximation-free control scheme with prescribed performance for unknown pure feedback systems. Automatica, 50(4):1217–1226, 2014.

[4] Charalampos P. Bechlioulis, Achilles Theodorakopoulos, and George A. Rovithakis. Output feedback stabilization with prescribed performance for uncertain nonlinear systems in canonical form. In Proc. 52nd IEEE Conf. Decis. Control, Florence, Italy, pages 5084–5089, 2013.

[5] Thomas Berger, Lê Huy Hoàng, and Timo Reis. Funnel control for nonlinear systems with known strict relative degree. Automatica, 87:345–357, 2018.

[6] Thomas Berger, Achim Ilchmann, and Timo Reis. Funnel control for nonlinear functional differential-algebraic systems. In Proceedings of the MTNS 2014, pages 46–53, Groningen, NL, 2014.

[7] Thomas Berger and Timo Reis. The funnel observer. Universität Hamburg, Hamburger Beiträge zur Angewandten Mathematik 2016-03, 2016.

[8] Thomas Berger and Timo Reis. Funnel control via funnel pre-compensator for minimum phase systems with relative degree two. IEEE Trans. Autom. Control, 2018. To appear.

[9] Dennis S. Bernstein. Matrix Mathematics: Theory, Facts, and Formulas. Princeton University Press, Princeton and Oxford, 2nd edition, 2009.

[10] Eric Bullinger and Frank Allgöwer. Adaptive λ-tracking for nonlinear higher relative degree systems. Automatica, 41(7):1191–1200, 2005.

[11] Eric Bullinger, Achim Ilchmann, and Frank Allgöwer. A simple adaptive observer for nonlinear systems. In H. J. C. Huijberts, editor, Proceedings of the IFAC Symposium Nonlinear Control Systems Design, NOLCOS98, pages 806–811, Enschede, The Netherlands, 1998. Pergamon.

[12] Christopher I. Byrnes and Alberto Isidori. A frequency domain philosophy for nonlinear systems, with application to stabilization and to adaptive control. In Proc. 23rd IEEE Conf. Decis. Control, volume 1, pages 1569–1573, 1984.

[13] Young Man Cho and R. Rajamani. A systematic approach to adaptive observer synthesis for nonlinear systems. IEEE Trans. Autom. Control, 42:534–537, 1997.

[14] Dhrubajit Chowdhury and Hassan K. Khalil. Funnel control of higher relative degree systems. In Proc. American Control Conf. 2017, pages 598–603, Seattle, USA, 2017.

[15] Sergey N. Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. An iss small gain theorem for general networks. Math. Control Signals Syst., 19(2):93–122, 2007.

[16] S.V. Emel'yanov and S.K. Korovin. Observers and output differentiators in uncertain systems. Computational Mathematics and Modeling, 15:195–212, 2004.

[17] F. Esfandiari and Hassan K. Khalil. Observer-based design of uncertain systems: recovering state feedback robustness under matching conditions. In Proceedings of Allerton Conference, Monticello, IL, pages 97–106, 1987.

[18] F. Esfandiari and Hassan K. Khalil. Output feedback stabilization of fully linearizable systems. Int. J. Control, 56(5):1007–1037, 1992.

[19] Christoph M. Hackl. High-gain adaptive position control. Int. J. Control, 84(10):1695–1716, 2011.

[20] Christoph M. Hackl. Contributions to high-gain adaptive control in mechatronics. PhD thesis, Fakultät für Elektrotechnik und Informationstechnik, TU München, 2012.

[21] Christoph M. Hackl, Norman Hopfe, Achim Ilchmann, Markus Mueller, and Stephan Trenn. Funnel control for systems with relative degree two. SIAM J. Control Optim., 51(2):965–995, 2013.

[22] Achim Ilchmann. Non-identifier-based adaptive control of dynamical systems: A survey. IMA J. Math. Control & Information, 8(4):321–366, 1991.

[23] Achim Ilchmann. Decentralized tracking of interconnected systems. In Knut Hüper and Jochen Trumpf, editors, Mathematical System Theory - Festschrift in Honor of Uwe Helmke on the Occasion of his Sixtieth Birthday, pages 229–245. CreateSpace, 2013.

[24] Achim Ilchmann and Eugene P. Ryan. High-gain control without identification: a survey. GAMM Mitt., 31(1):115–125, 2008.

[25] Achim Ilchmann, Eugene P. Ryan, and Christopher J. Sangwin. Systems of controlled functional differential equations and adaptive tracking. SIAM J. Control Optim., 40(6):1746–1764, 2002.

[26] Achim Ilchmann, Eugene P. Ryan, and Christopher J. Sangwin. Tracking with prescribed transient behaviour. ESAIM: Control, Optimisation and Calculus of Variations, 7:471–493, 2002.

[27] Achim Ilchmann, Eugene P. Ryan, and Philip Townsend. Tracking control with prescribed transient behaviour for systems of known relative degree. Syst. Control Lett., 55(5):396–406, 2006.

[28] Achim Ilchmann, Eugene P. Ryan, and Philip Townsend. Tracking with prescribed transient behavior for nonlinear systems of known relative degree. SIAM J. Control Optim., 46(1):210–230, 2007.

[29] Achim Ilchmann, Eugene P. Ryan, and Stephan Trenn. Tracking control: Performance funnels and prescribed transient behaviour. Syst. Control Lett., 54(7):655–670, 2005.

[30] Z.P. Jiang, Iven M. Y. Mareels, David J. Hill, and J. Huang. A unifying framework for global regulation via nonlinear output feedback: From ISS to iISS. IEEE Trans. Autom. Control, 49:549–562, 2004.

[31] Hassan K. Khalil. Cascade high-gain observer for high-dimensional systems. In Proc. 55th IEEE Conf. Decis. Control, Las Vegas, USA, pages 7141–7146, 2016.

[32] Hassan K. Khalil and Laurent Praly. High-gain observers in nonlinear feedback control. Int. J. Robust & Nonlinear Control, 24:993–1015, 2014.

[33] Hassan K. Khalil and Ali Saberi. Adaptive stabilization of a class of nonlinear systems using high-gain feedback. IEEE Trans. Autom. Control, 32:1031–1035, 1987.

[34] Daniel Liberzon and Stephan Trenn. The bang-bang funnel controller for uncertain nonlinear systems with arbitrary relative degree. IEEE Trans. Autom. Control, 58(12):3126–3141, 2013.

[35] A. Stephen Morse. Overcoming the obstacle of high relative degree. European J. Control, 2(1):29–35, 1996.

[36] Tiago Roux Oliveira, Antonio Estrada, and Leonid M. Fridman. Global and exact HOSM differentiator with dynamic gains for output-feedback sliding mode control. Automatica, 81(7):156–163, 2017.

[37] Tiago Roux Oliveira, Alessandro Jacoud Peixoto, and Liu Hsu. Peaking free output-feedback exact tracking of uncertain nonlinear systems via dwell-time and norm observers. Int. J. Robust & Nonlinear Control, 23(5):483–513, 2013.

[38] Alessandro Jacoud Peixoto, Tiago Roux Oliveira, Liu Hsu, Fernando Lizarralde, and Ramon R. Costa. Global tracking sliding mode control for a class of nonlinear systems via variable gain observer. Int. J. Robust & Nonlinear Control, 21(2):177–196, 2011.

[39] Eugene P. Ryan and Christopher J. Sangwin. Controlled functional differential equations and adaptive stabilization. Int. J. Control, 74(1):77–90, 2001.

[40] Ali Saberi and Peddapullaiah Sannuti. Observer design for loop transfer recovery and for uncertain dynamical systems. IEEE Trans. Autom. Control, 35(8):878–897, 1990.

[41] Ricardo G. Sanfelice and Laurent Praly. On the performance of high-gain observers with gain adaptation under measurement noise. Automatica, 47(10):2165–2176, 2011.

[42] S. S. Sastry and Alberto Isidori. Adaptive Control of Linearizable Systems. IEEE Trans. Autom. Control, 34(11):1123–1131, 1989.

[43] Robert Seifried and Wojciech Blajer. Analysis of servo-constraint problems for underactuated multibody systems. Mech. Sci., 4:113–129, 2013.

[44] Antonio Tornambè. Use of asymptotic observers having high gains in the state and parameter estimation. In Proc. 27th IEEE Conf. Decis. Control, Austin, Texas, pages 1791–1794, 1988.

[45] Antonio Tornambè. High-gain observers for non-linear systems. Int. J. Systems Sci., 23:1475–1489, 1992.

[46] Wolfgang Walter. Ordinary Differential Equations. Springer-Verlag, New York, 1998.

$\square$