ODE observers for DAE systems

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Abstract

We consider linear time-invariant differential-algebraic systems which are not necessarily regular. The following question is addressed: When does an (asymptotic) observer which is realized by an ODE system exist? In our main result we characterize the existence of such observers by means of a simple criterion on the system matrices. To be specific, we show that an ODE observer exists if, and only if, the completely controllable part of the system is impulse observable. Extending the observer design from earlier works we provide a procedure for the construction of (asymptotic) ODE observers.

Keywords: Differential-algebraic equations; observers; controllability; observability; Kalman decomposition.

1 Introduction

In our recent work Berger and Reis (2017) we have considered observer design for linear systems described by differential-algebraic equations (DAEs) of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) + Du(t)\,, \end{aligned} \tag{1}$$

which are not necessarily regular in the sense that they may be under- or overdetermined. This approach is based on the observer notions in Valcher and Willems (1999), where equivalent criteria for the existence of (exact, asymptotic) observers have been presented. It has been shown in Berger and Reis (2017) that, with the same criteria as in Valcher and Willems (1999), DAE systems admit (exact, asymptotic) observers which are again DAEs.

In the present article, we address the question when observers for DAE systems can be constructed which are described by ordinary differential equations (ODEs). This observer type is preferable from a practical point of view, since unconstrained observer dynamics do not involve derivatives of the inputs and outputs, which would lead to an ill-posed problem. Another advantage of ODE observers is that they can be initialized without any further restrictions. The aforementioned problem has been well investigated already several decades ago. In El-Tohami, Lovass-Nagy and Mukundan (1983) asymptotic ODE observers are constructed using a singular value decomposition of the matrix E; a similar method has been used in Fahmy and O'Reilly (1989). The construction developed in Verhaegen and Van Dooren (1986) is based on the staircase form and generalized Sylvester equations, while the method in Shafai and Carroll (1987) uses generalized inverses. All the aforementioned approaches involve different, quite restrictive assumptions on the system (1) (for instance, El-Tohami et al. (1983) require restrictive consistency conditions) and additionally regularity of the matrix pencil $sE-A$ and (complete) observability is required. These assumptions have been relaxed in Darouach and Boutayeb (1995), where asymptotic ODE observers are constructed under the assumptions of impulse observability and behavioral detectability; regularity of $sE-A$ is not required. While behavioral detectability is clearly necessary for the existence of asymptotic ODE observers, impulse observability is not. A weaker (but still not necessary) condition than impulse observability has been derived in Müller and Hou (1993); this condition is a rank condition involving E,A,B,C and it is notable that in this work a condition which depends on the matrix B appears for the first time. We will illustrate in Section 3 that indeed the existence of ODE observers depends on the choice of B. The result of Müller and Hou (1993) has again been improved in Hou and Müller (1999b), where so called *causal detectability* has been derived as

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an equivalent condition for the existence of asymptotic ODE observers. However, the proof given in (Hou and Müller, 1999b, Thm. 5) seems a little bit incomplete, as at the crucial point it is simply referred to "similar arguments as in" another result of another paper by the authors. Moreover, the causal detectability condition is not easy to check since the test provided in Hou and Müller (1999b) relies on a certain decomposition resembling the Kalman decomposition and the construction procedure is quite involved (using procedures from two other papers by the authors). More recently, in Darouach (2014) the problem has been considered again, but regular $sE-A$ and complete controllability is required.

In the present paper, we show that for general linear DAE systems, an ODE observer exists if, and only if, the completely controllable part of the system is impulse observable; a physically meaningful condition. The observer is moreover asymptotic if, and only if, the DAE system is additionally behaviorally detectable. We also provide simple alternative characterizations in terms of subspace intersections involving the augmented Wong sequences. These conditions are formulated in terms of the system matrices E,A,B,C and are easy to check using MATLAB for instance; a MATLAB function for the computation of the required pre-image is given in Berger and Trenn (2012). We like to stress that our proofs involve completely different methods than the above mentioned earlier works. Furthermore, we provide a construction procedure for the observer in each case, which extends the method for impulse observable systems presented in Darouach and Boutayeb (1995).

As a first step towards our main result we show, using a simple argument, that the existence of ODE observers is equivalent to the existence of observers with index at most one. The latter class has already been introduced in Nikoukhah (1998) for ODE systems, generalized to semi-explicit index-1 DAEs in Åslund and Frisk (2006), and considered for nonlinear descriptor systems in Labisch and Konigorski (2014), see also the references therein.

Throughout this article, we use the following notation: For $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$ and $A \in \mathbb{K}^{n\times m}$, we use the symbols $\operatorname{im}_{\mathbb{K}} A$, $\ker_{\mathbb{K}} A$ and $\operatorname{rk}_{\mathbb{K}} A$ for the image, kernel and rank of A, resp. The subscripts are omitted when they are clear from context. The group of invertible real matrices of size $n\times n$ is denoted by $\mathbf{Gl}_n(\mathbb{R})$, and $\|x\|$ is the Euclidean norm of $x \in \mathbb{R}^n$. By $\mathbb{N}$ we denote the set of natural numbers including zero. The symbols $\mathbb{C}_+$ and $\overline{\mathbb{C}_+}$ denote the sets of complex numbers with positive and nonnegative real part, resp.

Further, $f|_I$ is the restriction of a function $f:\mathbb{R}\to\mathbb{R}^n$ to $I\subseteq\mathbb{R}$ and $\dot f$ ($f^{(i)}$) is the (i-th) weak derivative of f, see (Adams, 1975, Chap. 1). We further use the following function spaces in this article:

$\mathscr{C}^\infty(\mathbb{R};\mathbb{R}^n)$	set of infinitely-times continuously differentiable $\mathbb{R}^n$-valued functions
$\mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$	set of locally (Lebesgue) integrable $\mathbb{R}^n$-valued functions
$\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$	set of locally integrable $\mathbb{R}^n$-valued functions with locally integrable weak derivative $\dot f$

2 Preliminaries

We study linear time-invariant DAE systems (1) where $E,A\in\mathbb{R}^{l\times n}$, $B\in\mathbb{R}^{l\times m}$, $C\in\mathbb{R}^{p\times n}$, $D\in\mathbb{R}^{p\times m}$. Systems of that type are also called *descriptor systems*. The set of systems (1) is denoted by $\Sigma_{l,n,m,p}$ and we write $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$. DAE systems of the form (1) naturally occur when modeling dynamical systems subject to algebraic constraints, e.g. chemical process systems (see Kumar and Daoutidis (1999)), mechanical systems (see Simeon (2013); Simeon, Führer and Rentrop (1991)), and modified nodal analysis models of electrical circuits (see Riaza (2008)); see also the textbooks Kunkel and Mehrmann (2006); Lamour, März and Tischendorf (2013). In the present paper we do not assume that the matrix pencil $sE-A$ is *regular*, which would mean that $l=n$ and $\det(sE-A)$ is not the zero polynomial.

The functions $u:\mathbb{R}\to\mathbb{R}^m$ and $y:\mathbb{R}\to\mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x,u,y):\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p$ is said to be a *solution* of (1), if $Ex\in\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^l)$ and (x,u,y) solves (1) in the weak sense. Recall that $Ex\in\mathscr{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^l)$ implies continuity of Ex (though x itself may be discontinuous). The *behavior* $\mathfrak{B}_{[E,A,B,C,D]}$ of (1) is defined as the set of all solutions $(x,u,y):\mathbb{R}\to\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p$ of (1). Based on this behavior, DAE systems have been studied in detail e.g. in Berger (2014). For the analysis of DAE systems in $\Sigma_{l,n,m,p}$ we assume that the states, inputs and outputs of the system are fixed a priori by the designer. This is different from other approaches based on the behavioral setting, cf. Berger and Van Dooren (2015); Campbell, Kunkel and Mehrmann (2012); Valcher and Willems (1999).

We consider different notions of controllability and observability for DAE systems. For a rigorous time domain definition and a detailed discussion we refer to the surveys Berger and Reis (2013); Berger, Reis and Trenn (2017). In the following we state their algebraic characterizations.

Proposition 2.1. *A system* $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$ *is*

(i) completely controllable *if, and only if,* $\text{im}_{\mathbb{C}}(\lambda E - A) + \text{im}_{\mathbb{C}} B = \text{im}_{\mathbb{C}} E + \text{im}_{\mathbb{C}} B = \text{im}_{\mathbb{C}} E + \text{im}_{\mathbb{C}} A + \text{im}_{\mathbb{C}} B$ *for all* $\lambda \in \mathbb{C}$.

(ii) impulse observable *if, and only if,* $\ker E \cap A^{-1}(\text{im} E) \cap \ker C = \{0\}$.

(iii) completely observable *if, and only if,* $\ker E \cap \ker C = \{0\}$ *and* $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$ *for all* $\lambda \in \mathbb{C}$.

(iv) behaviorally detectable *if, and only if,* $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$ *for all* $\lambda \in \overline{\mathbb{C}_+}$.

We also need the Kalman controllability decomposition derived in Berger and Trenn (2014).

Theorem 2.2. *For any* $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ *there exist* $T \in \mathbf{Gl}_n(\mathbb{R})$, $S \in \mathbf{Gl}_l(\mathbb{R})$ *such that* $[SET, SAT, SB, CT, D]$ *is in* Kalman controllability decomposition (KCD)*, i.e.,*

$$\begin{aligned} S(sE-A)T &= \begin{bmatrix} sE_{11}-A_{11} & sE_{12}-A_{12} & sE_{13}-A_{13} \\ 0 & sE_{22}-A_{22} & sE_{23}-A_{23} \\ 0 & 0 & sE_{33}-A_{33} \end{bmatrix}, \\ SB &= \begin{bmatrix} B_1 \\ 0 \\ 0 \end{bmatrix}, \quad CT = \begin{bmatrix} C_1 & C_2 & C_3 \end{bmatrix}, \end{aligned} \tag{2}$$

where

(i) $[E_{11}, A_{11}, B_1, C_1, D] \in \Sigma_{l_1,n_1,m,p}$ *with* $l_1 = \text{rk}[E_{11}, B_1] \le n_1 + m$ *is completely controllable,*

(ii) $[E_{22}, A_{22}, 0, C_2, D] \in \Sigma_{l_2,n_2,m,p}$ *with* $l_2 = n_2$ *and* E_{22} *is invertible,*

(iii) $[E_{33}, A_{33}, 0, C_3, D] \in \Sigma_{l_3,n_3,m,p}$ *with* $l_3 \ge n_3$ *satisfies* $\text{rk}_{\mathbb{C}}(\lambda E_{33} - A_{33}) = n_3$ *for all* $\lambda \in \mathbb{C}$.

To introduce the concept of an (asymptotic) observer for a DAE system, we first need to define acceptors.

Definition 2.3. Consider a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$. A system $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,p_o}$ is called an *acceptor for* $[E,A,B,C,D]$, if for all $(x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]}$ there exist $x_o \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{n_o})$, $z \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{p_o})$ such that

$$(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}.$$

The concept of an acceptor has been first introduced by Valcher and Willems (1999) for behaviors. Loosely speaking, an acceptor absorbs the external signals of a given system without influencing the system. A special class of acceptors is that of observers. We use the definition of observers of DAE systems from Berger and Reis (2017). Note that the following definition has also been stated for behavioral systems in Valcher and Willems (1999).

Definition 2.4. Consider a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$. Then a system $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ is called

a) an *observer for* $[E,A,B,C,D]$, if it is an acceptor for $[E,A,B,C,D]$, and

$$\begin{aligned} &\forall\, (x,u,y,x_o,z) \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^n) : \\ &\Big((x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]} \,\wedge\, (x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]} \,\wedge\, Ez(0) = Ex(0) \Big) \quad \Longrightarrow \quad z = x. \end{aligned}$$

b) an *asymptotic observer for* $[E,A,B,C,D]$, if it is an observer for $[E,A,B,C,D]$, and

$$\begin{aligned} &\forall\, (x,u,y,x_o,z) \in \mathscr{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^n) : \\ &\Big((x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]} \,\wedge\, (x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]} \Big) \quad \Longrightarrow \quad \lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|z - x\| = 0, \end{aligned}$$

where "ess sup" denotes the essential supremum.

The definition of an observer means that once the observer matches the state via $Ez(0) = Ex(0)$, it does not lose track, i.e., the whole trajectories have to coincide ($z = x$). Note that, by time-invariance, the condition $Ez(0) = Ex(0)$ may be replaced by the existence of some $t \in \mathbb{R}$ such that $Ez(t) = Ex(t)$.

In order to define index-1 observers we need to introduce the notion of the index: The index $\nu \in \mathbb{N}$ of a regular matrix pencil $sE - A$ is defined via its (quasi-)Weierstraß form, cf. Berger, Ilchmann and Trenn (2012); Gantmacher (1959); Kunkel and Mehrmann (2006); Lamour et al. (2013): if for some $S, T \in \mathbf{Gl}_n(\mathbb{R})$

$$S(sE - A)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & sN - I_{n-r} \end{bmatrix}, \quad N \text{ nilpotent,}$$
$$\text{then} \quad \nu := \begin{cases} 0, & \text{if } r = n, \\ \min\left\{ k \in \mathbb{N} \mid N^k = 0 \right\}, & \text{if } r < n. \end{cases}$$

The index is independent of the choice of S, T and can be computed e.g. via the Wong sequences (defined below) corresponding to $sE - A$ as shown in Berger et al. (2012).

Definition 2.5. Let a system $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ be given and let $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ be an observer for $[E, A, B, C, D]$. Then we call $[E_o, A_o, B_o, C_o, D_o]$

a) *regular*, if $l_o = n_o$ and $sE_o - A_o$ is regular;

b) an *index-1 observer*, if it is regular and the index of $sE_o - A_o$ is at most one;

c) an *ODE observer*, if $E_o = I_{n_o}$.

Clearly, every ODE observer is an index-1 observer. The following result highlights some advantages of index-1 observers. Its proof is straightforward and therefore omitted.

Proposition 2.6. *Let a system $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ be given and let $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ be an observer for $[E, A, B, C, D]$. Then the following two statements are equivalent:*

(i) *$[E_o, A_o, B_o, C_o, D_o]$ is regular and freely initializable in the sense that for all $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ and $x_o^0 \in \mathbb{R}^{n_o}$ there exist $x_o \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^{n_o})$, $z \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ such that*

$$(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]} \quad \text{and} \quad E_o x_o(0) = E_o x_o^0.$$

(ii) *$[E_o, A_o, B_o, C_o, D_o]$ is an index-1 observer.*

In the following we show that the existence of an index-1 observer is equivalent to the existence of an ODE observer.

Proposition 2.7. *Let a system $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ be given. Then the following two statements are equivalent.*

(i) *There exists an (asymptotic) index-1 observer for $[E, A, B, C, D]$.*

(ii) *There exists an (asymptotic) ODE observer for $[E, A, B, C, D]$.*

Proof. It suffices to show (i)⇒(ii): Assume that $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ is an (asymptotic) index-1 observer for $[E, A, B, C, D]$. Then there exist $S \in \mathbf{Gl}_{l_o}(\mathbb{R})$, $T \in \mathbf{Gl}_{n_o}(\mathbb{R})$ such that $S(sE_o - A_o)T = \left[\begin{smallmatrix} sI_r - J & 0 \\ 0 & -I_{n_o-r} \end{smallmatrix}\right]$, $SB_o = \left[\begin{smallmatrix} B_{o,1} \\ B_{o,2} \end{smallmatrix}\right]$, $C_oT = [C_{o,1}, C_{o,1}]$, and hence $(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}$ with $\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = T^{-1}x_o$ if, and only if,

$$\begin{aligned} \dot{x}_1 &= Jx_1 + B_{o,1} \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \\ 0 &= x_2 + B_{o,2} \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \\ z &= C_{o,1}x_1 + C_{o,2}x_2 + D_o \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \end{aligned}$$

which is equivalent to

$$\begin{aligned} \dot{x}_1 &= Jx_1 + B_{o,1} \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), \\ z &= C_{o,1}x_1 + (D_o - C_{o,2}B_{o,2}) \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right). \end{aligned} \tag{3}$$

Therefore, $[I_r, J, B_{o,1}, C_{o,1}, D_o - C_{o,2}B_{o,2}]$ is an (asymptotic) ODE observer for $[E, A, B, C, D]$. ☐

In order to geometrically characterize the existence of an (asymptotic) ODE observer we use the augmented Wong sequences (see Berger and Reis (2013); Berger and Reis (2015); Berger and Trenn (2014) and the references therein) which are defined as follows for $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$:

$$\begin{aligned} &\mathscr{V}^0_{[E,A,B]} := \mathbb{R}^n, && \mathscr{V}^{i+1}_{[E,A,B]} := A^{-1}(E\mathscr{V}^i_{[E,A,B]} + \operatorname{im} B), \\ &\mathscr{W}^0_{[E,A,B]} := \{0\}, && \mathscr{W}^{i+1}_{[E,A,B]} := E^{-1}(A\mathscr{W}^i_{[E,A,B]} + \operatorname{im} B), \\ &\mathscr{V}^*_{[E,A,B]} := \bigcap_{i\in\mathbb{N}} \mathscr{V}^i_{[E,A,B]}, && \mathscr{W}^*_{[E,A,B]} := \bigcup_{i\in\mathbb{N}} \mathscr{W}^i_{[E,A,B]}. \end{aligned} \tag{4}$$

Recall that, for some matrix $M \in \mathbb{R}^{l\times n}$, $M\mathscr{S} = \left\{\, x \in \mathbb{R}^l \;\middle|\; x \in \mathscr{S} \,\right\}$ denotes the image of $\mathscr{S} \subseteq \mathbb{R}^n$ under M and $M^{-1}\mathscr{S} = \{\, x \in \mathbb{R}^n \mid Mx \in \mathscr{S} \,\}$ denotes the preimage of $\mathscr{S} \subseteq \mathbb{R}^l$ under M.

The sequences $(\mathscr{V}^i_{[E,A,B]})_{i\in\mathbb{N}}$ and $(\mathscr{W}^i_{[E,A,B]})_{i\in\mathbb{N}}$ are called *augmented Wong sequences* since they are based on the Wong sequences ($B = 0$) used in Berger et al. (2012); Berger and Trenn (2012); Berger and Trenn (2013) and which have their origin in Wong (1974) who was the first using both sequences (with $B = 0$) for the analysis of matrix pencils.

As shown in Berger and Reis (2013) the augmented Wong sequences allow a characterization of complete controllability as follows.

Lemma 2.8. *$[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ is completely controllable if, and only if,*

$$\mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]} = \mathbb{R}^n.$$

Remark 2.9. The augmented Wong sequences are related to the *reachable space* of a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$, which is defined as

$$\mathscr{R}_{[E,A,B]} := \left\{ x_f \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists t_f > 0\, \exists (x,u,y) \in \mathfrak{B}_{[E,A,B,C,D]} : \\ x \in \mathscr{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \,\wedge\, x(0) = 0 \,\wedge\, x(t_f) = x_f \end{array} \right\},$$

cf. also Berger and Reis (2013). In Berger and Reis (2013) it is shown that

$$\mathscr{R}_{[E,A,B]} = \mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]},$$

hence complete controllability can also be characterized by the intuitive condition $\mathscr{R}_{[E,A,B]} = \mathbb{R}^n$.

3 Main result

In this section we state and prove a characterization of existence of (asymptotic) ODE and index-1 observers. Before this result is shown, we advance to simple examples. We start with an example of a system for which there does not exist any ODE observer (and thus, by Proposition 2.7, there does neither exist any index-1 observer).

Example 3.1. Consider the system

$$[E,A,B,C,D] = \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \end{bmatrix}, 0 \right] \in \Sigma_{2,2,1,1}.$$

Then we have $x_2 = y = -u$ and $x_1 = \dot{y} = -\dot{u}$. Consequently, an observer $[E_o,A_o,B_o,C_o,D_o]$ has to take derivatives of y or u in order to satisfy Definition 2.4 a). Hence, by Proposition 2.7, the construction of ODE or index-1 observers for $[E,A,B,C,D]$ is impossible.

We like to stress that this example also shows that the existence of ODE observers depends on the choice of the matrix B, unlike the existence of DAE observers, which is independent of B, see Berger and Reis (2017). If we choose $B = 0$, then the system only has the trivial solution $x_1 = x_2 = 0$ and it is easy to find an ODE observer. Also note that the system is not impulse observable and nevertheless an ODE observer exists in the case $B = 0$.

The subsequent example shows that there exist systems with higher index which admit the construction of ODE observers.

Example 3.2. Consider the system

$$[E,A,B,C,D] = \left[\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}, \begin{bmatrix}0\\ 0\\ 1\end{bmatrix}, \begin{bmatrix}1 & 0 & 0\end{bmatrix}, 0\right] \in \Sigma_{3,3,1,1}.$$

We show that the ODE system

$$[E_o,A_o,B_o,C_o,D_o] = \left[\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}, \begin{bmatrix}-1 & 1\\ -2 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\\ 0 & 0\end{bmatrix}\right]$$

given by

$$\begin{aligned}\dot{x}_{o1} &= -x_{o2} - u + y, & z_1 &= y,\\ \dot{x}_{o2} &= x_{o1} - 2x_{o2} - 2u, & z_2 &= x_{o1},\\ & & z_3 &= x_{o2}\end{aligned}$$

is an asymptotic observer for $[E,A,B,C,D]$. Denoting the states of $[E,A,B,C,D]$ by x_1, x_2 and x_3, we obtain $z_1 = y = x_1$ and, for $e_2 = z_2 - x_2$ and $e_3 = z_3 - x_3$,

$$\begin{aligned}\dot{e}_2 &= -x_{o2} - u + y - x_1 = -z_3 - u = -e_3,\\ \dot{e}_3 &= x_{o1} - 2x_{o2} - 2u - x_2 = e_2 - 2e_3.\end{aligned}$$

Therefore, we have

$$\begin{pmatrix}\dot{e}_2\\ \dot{e}_3\end{pmatrix} = \begin{bmatrix}0 & -1\\ 1 & -2\end{bmatrix}\begin{pmatrix}e_2\\ e_3\end{pmatrix}$$

and since the matrix $\left[\begin{smallmatrix}0 & -1\\ 1 & -2\end{smallmatrix}\right]$ only has the eigenvalue -1, it follows that $e_2(t) \to 0$ and $e_3(t) \to 0$ for $t \to \infty$. This shows that Definition 2.4 b) is satisfied.

Now we present the main result of the present paper.

Theorem 3.3. *Let a system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ be given. Then the following statements are equivalent:*

1) There exists an ODE observer for $[E,A,B,C,D]$.

2) There exists an index-1 observer for $[E,A,B,C,D]$.

3) For some (and hence any) KCD (2) *the completely controllable part $[E_{11},A_{11},B_1,C_1,D]$ of $[E,A,B,C,D]$ is impulse observable.*

4) The augmented Wong sequences in (4) *satisfy*

$$\mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]} \cap \ker E \cap A^{-1}\left(E\left(\mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]}\right)\right) \cap \ker C = \{0\}. \tag{5}$$

Furthermore, the following statements are equivalent:

1') There exists an asymptotic ODE observer for $[E,A,B,C,D]$.

2') There exists an asymptotic index-1 observer for $[E,A,B,C,D]$.

3') $[E,A,B,C,D]$ is behaviorally detectable and for some (and hence any) KCD (2) *we have that $[E_{11},A_{11},B_1,C_1,D]$ is impulse observable.*

4') $[E,A,B,C,D]$ is behaviorally detectable and the augmented Wong sequences in (4) *satisfy* (5).

Proof. By Proposition 2.7 we have 1) $\Leftrightarrow$ 2).

3) $\Rightarrow$ 2): Since $[E_{11},A_{11},B_1,C_1,D]$ is impulse observable and E_{22} in (2) is invertible it follows that

$$[\tilde{E},\tilde{A},\tilde{B},\tilde{C},D] := \left[\begin{bmatrix}E_{11} & E_{12}\\ 0 & E_{22}\end{bmatrix}, \begin{bmatrix}A_{11} & A_{12}\\ 0 & A_{22}\end{bmatrix}, \begin{bmatrix}B_1\\ 0\end{bmatrix}, [C_1,C_2], D\right] \in \Sigma_{l_1+l_2,n_1+n_2,m,p} \tag{6}$$

is impulse observable. Then (Berger and Reis, 2017, Thm. 3.8) implies that there exists an index-1 observer $[E_o^1, A_o^1, B_o^1, C_o^1, D_o^1] \in \Sigma_{l_o,n_o,m+p,n_1+n_2}$ for $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}, D]$. Define

$$[E_o, A_o, B_o, C_o, D_o] := \left[E_o^1, A_o^1, B_o^1, T\begin{bmatrix} C_o^1 \\ 0 \end{bmatrix}, T\begin{bmatrix} D_o^1 \\ 0 \end{bmatrix}\right] \in \Sigma_{l_o,n_o,m+p,n},$$

where T is as in (2). Since the DAE $\frac{\mathrm{d}}{\mathrm{d}t}E_{33}x_3 = A_{33}x_3$ does only have the trivial solution, cf. Berger and Trenn (2012), it follows that $[E_o, A_o, B_o, C_o, D_o]$ is an observer for $[E, A, B, C, D]$. Since $sE_o^1 - A_o^1$ has index at most one, it follows that $[E_o, A_o, B_o, C_o, D_o]$ is an index-1 observer.

2) $\Rightarrow$ 3): Without loss of generality we may assume that $[E, A, B, C, D]$ is in KCD (2). Consider the completely controllable part $[E_{11}, A_{11}, B_1, C_1, D]$ of $[E, A, B, C, D]$. By the Kalman observability decomposition, see (Berger et al., 2017, Thm. 8.3), there exist $\tilde{S} \in \mathbf{Gl}_{l_1}(\mathbb{R})$ and $\tilde{T} \in \mathbf{Gl}_{n_1}(\mathbb{R})$ such that

$$[\tilde{S}E_{11}\tilde{T}, \tilde{S}A_{11}\tilde{T}, \tilde{S}B_1, C_1\tilde{T}] = \left[\begin{bmatrix} \tilde{E}_{11} & \tilde{E}_{12} & \tilde{E}_{13} \\ 0 & \tilde{E}_{22} & \tilde{E}_{23} \\ 0 & 0 & \tilde{E}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & \tilde{A}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \end{bmatrix}, [0, 0, \tilde{C}_3]\right], \tag{7}$$

where $\tilde{E}_{ij}, \tilde{A}_{ij} \in \mathbb{R}^{r_i \times q_j}$, $\tilde{B}_i \in \mathbb{R}^{r_i \times m}$ for $i, j = 1, \ldots, 3$, $\tilde{C}_3 \in \mathbb{R}^{p \times q_3}$ with

a) $r_1 \leq q_1$ and $\mathrm{rk}_{\mathbb{C}}(\lambda\tilde{E}_{11} - \tilde{A}_{11}) = r_1$ for all $\lambda \in \mathbb{C}$,

b) $r_2 = q_2$ and $\tilde{E}_{22}$ is invertible,

c) $[\tilde{E}_{33}, \tilde{A}_{33}, \tilde{B}_3, \tilde{C}_3, 0]$ is completely observable.

Using the existence of an index-1 observer $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ for $[E, A, B, C, D]$ we derive some consequences for the form (7) and proceed in several steps.

Step 1: We consider the subsystem $[\tilde{E}_{11}, \tilde{A}_{11}, \tilde{B}_1, 0, 0]$ with property a). Using the quasi-Kronecker form (Berger and Trenn, 2013, Cor. 2.3), which is an extension of the Kronecker canonical form as derived Gantmacher (1959), we find $V \in \mathbf{Gl}_{r_1}(\mathbb{R}), W \in \mathbf{Gl}_{q_1}(\mathbb{R})$ such that

$$V(s\tilde{E}_{11} - \tilde{A}_{11})W = \begin{bmatrix} sE_P - A_P & 0 \\ 0 & sN - I_k \end{bmatrix},$$

where $E_P, A_P \in \mathbb{R}^{l_P \times n_P}, l_P < n_P$ (or $l_P = n_P = 0$), such that $\mathrm{rk}_{\mathbb{C}}(\lambda E_P - A_P) = l_P$, $\mathrm{rk}\, E_P = l_P$, and $N \in \mathbb{R}^{k \times k}$ is nilpotent. If $n_P > 0$, then (Berger and Trenn, 2012, Thm. 3.2) implies existence of $x_P \in \mathscr{C}^\infty(\mathbb{R}; \mathbb{R}^{n_P})$, $x_P \neq 0$, with $x_P(0) = 0$ such that $E_P\dot{x}_P(t) = A_P x_P(t)$ for all $t \in \mathbb{R}$. Then

$$x := T\begin{pmatrix} \tilde{T}\begin{pmatrix} Wx_P \\ 0 \end{pmatrix} \\ 0 \end{pmatrix} \text{ satisfies } (x, 0, 0) \in \mathfrak{B}_{[E,A,B,C,D]}.$$

Since $(0, 0, 0) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}$ and $[E_o, A_o, B_o, C_o, D_o]$ is an observer for $[E, A, B, C, D]$ it follows that $x = 0$, whence $x_P = 0$, a contradiction. Therefore, $n_P = 0$ and we may without loss of generality assume that

$$s\tilde{E}_{11} - \tilde{A}_{11} = sN - I_{r_1}, \quad N \text{ nilpotent.} \tag{8}$$

Step 2: We consider the subsystem $[\tilde{E}_{33}, \tilde{A}_{33}, \tilde{B}_3, \tilde{C}_3, 0]$ with property c). Choose $\tilde{W} \in \mathbf{Gl}_{r_3}(\mathbb{R}), \tilde{V} \in \mathbf{Gl}_{q_3}(\mathbb{R})$ such that

$$[\tilde{W}\tilde{E}_{33}\tilde{V}, \tilde{W}\tilde{A}_{33}\tilde{V}, \tilde{C}_3\tilde{V}] = \left[\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, [\tilde{C}_{31}, \tilde{C}_{32}]\right],$$

where $k = \mathrm{rk}\,\tilde{E}_{33}$, $J_{11} \in \mathbb{R}^{k \times k}, J_{12} \in \mathbb{R}^{k \times (q_3 - k)}, J_{21} \in \mathbb{R}^{(r_3 - k) \times k}, J_{22} \in \mathbb{R}^{(r_3 - k) \times (q_3 - k)}, \tilde{C}_{31} \in \mathbb{R}^{p \times k}$ and $\tilde{C}_{32} \in \mathbb{R}^{p \times (q_3 - k)}$. By complete observability due to c) we have $\ker \tilde{E}_{33} \cap \ker \tilde{C}_3 = \{0\}$, which gives

$$\ker \begin{bmatrix} I_k & 0 \\ \tilde{C}_{31} & \tilde{C}_{32} \end{bmatrix} = \{0\},$$

thus $\mathrm{rk}\,\tilde{C}_{32} = q_3 - k$.

Henceforth, without loss of generality we may assume that $\tilde{W} = I_{r_3}$ and $\tilde{V} = I_{q_3}$.

Step 3: We show that $N = 0$. If $r_1 = 0$, then there is nothing to show. Assume that $r_1 > 0$ and let $\nu \in \mathbb{N}$ be such that $N^\nu = 0$ and $N^{\nu-1} \neq 0$. Since $\tilde{E}_{22}$ is invertible and (8) holds, we may without loss of generality assume that $\tilde{E}_{22} = I_{r_2}$ and $\tilde{E}_{12} = \tilde{A}_{12} = 0$. If the latter is not satisfied it can always be achieved by a straightforward transformation. Therefore, (7) takes the form

$$[\tilde{S}E_{11}\tilde{T}, \tilde{S}A_{11}\tilde{T}, \tilde{S}B_1, C_1\tilde{T}] = \left[\begin{bmatrix} N & 0 & \tilde{E}_{13} & \tilde{E}_{14} \\ 0 & I_{r_2} & \tilde{E}_{23} & \tilde{E}_{24} \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} I_{r_1} & 0 & \tilde{A}_{13} & \tilde{A}_{14} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} & \tilde{A}_{24} \\ 0 & 0 & J_{11} & J_{12} \\ 0 & 0 & J_{21} & J_{22} \end{bmatrix}, \begin{bmatrix} \tilde{B}_1 \\ \tilde{B}_2 \\ \tilde{B}_3 \\ \tilde{B}_4 \end{bmatrix}, [0, 0, \tilde{C}_{31}, \tilde{C}_{32}] \right]. \quad (9)$$

Step 3a: Since the observer $[E_o, A_o, B_o, C_o, D_o]$ is index-1, we find that it does not differentiate the input and output of the system $[E, A, B, C, D]$, cf. also (3), that is

$$\forall T > 0 \, \exists C(T) > 0 \, \forall t \in [0, T] \, \forall (x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]} \cap \mathscr{C}^\infty(\mathbb{R}; \mathbb{R}^{n_o+m+p+n}) :$$
$$\|z(t)\| \leq C(T) \left(\|x_o(0)\| + \max_{0 \leq s \leq t} \left\| \left(\begin{smallmatrix} u(s) \\ y(s) \end{smallmatrix}\right) \right\| \right). \quad (10)$$

In the following we relate the solutions of the system to those of the observer to show that the solutions cannot contain derivatives of the input. We consider $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ with the following properties:

(i) $(x, u, y) \in \mathscr{C}^\infty(\mathbb{R}; \mathbb{R}^{n+m+p})$,

(ii) $\left[\begin{smallmatrix} \tilde{T}^{-1} & 0 \\ 0 & I_{n_2+n_3} \end{smallmatrix}\right] T^{-1} x = \begin{pmatrix} \tilde{x} \\ 0 \end{pmatrix}$ and $\tilde{x} = (x_1^\top, \ldots, x_4^\top)^\top$ according to the partitioning in (9),

(iii) $\tilde{x}(0) = 0$ and $u(0) = 0$.

We define the following nonempty subset of $\mathfrak{B}_{[E,A,B,C,D]}$,

$$\overline{\mathfrak{B}}_{[E,A,B,C,D]} := \left\{ (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \;\middle|\; (x, u, y) \text{ satisfies (i)–(iii)} \right\}.$$

Let $(x, u, y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$. By Proposition 2.6 there exist $x_o \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^{n_o})$ and $z \in \mathscr{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ such that

$$(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o, A_o, B_o, C_o, D_o]} \quad \text{and} \quad E_o x_o(0) = 0.$$

As $sE_o - A_o$ has index at most one it further follows that x_o and z are smooth. By $x(0) = 0$ and $u(0) = 0$ it follows that $y(0) = 0$, thus $x_o(0) = 0$ (by the index-1 property) and $z(0) = 0$. We may now conclude from the definition of an observer that $z = x$. Then using $\dot{x}_3 = J_{11}x_3 + J_{12}x_4 + \tilde{B}_3 u$ and $y = \tilde{C}_{31}x_3 + \tilde{C}_{32}x_4 + Du$ we obtain

$$\begin{aligned} \|y(t)\| &= \left\| \int_0^t \tilde{C}_{31} e^{J_{11}(t-s)} \left(J_{12}x_4(s) + \tilde{B}_3 u(s)\right) \mathrm{d}s + \tilde{C}_{32}x_4(t) + Du(t) \right\| \\ &\leq \int_0^T \left\| \tilde{C}_{31} e^{J_{11}(t-s)} [J_{12}, \tilde{B}_3] \right\| \mathrm{d}s \max_{0 \leq s \leq t} \left\| \left(\begin{smallmatrix} x_4(s) \\ u(s) \end{smallmatrix}\right) \right\| + \|[\tilde{C}_{32}, D]\| \left\| \left(\begin{smallmatrix} x_4(t) \\ u(t) \end{smallmatrix}\right) \right\| \end{aligned}$$

for all $t \in [0, T]$, where $T > 0$. Using this as well as $z = x$ and $x_o(0) = 0$ it then follows from (10) that

$$\forall T > 0 \, \exists C(T) > 0 \, \forall (x, u, y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]} \, \forall t \in [0, T] : \; \|x(t)\| \leq C(T) \max_{0 \leq s \leq t} \left\| \left(\begin{smallmatrix} x_4(s) \\ u(s) \end{smallmatrix}\right) \right\|. \quad (11)$$

Step 3b: We show that x_4 and u can be chosen freely in a certain sense. Observe that the subsystem

$$\left[\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}, \begin{bmatrix} \tilde{B}_3 \\ \tilde{B}_4 \end{bmatrix}, [\tilde{C}_{31}, \tilde{C}_{32}], D \right] \in \Sigma_{r_3, q_3, m, p} \quad (12)$$

is completely controllable since $[E_{11}, A_{11}, B_1, C_1, D]$ is completely controllable. Choose $\hat{W} \in \mathbf{Gl}_{r_3-k}(\mathbb{R})$ such that

$$\hat{W}[J_{21}, J_{22}, \tilde{B}_4] = \begin{bmatrix} \tilde{J}_{21} & \tilde{J}_{22} & \tilde{B}_{41} \\ 0 & 0 & 0 \end{bmatrix}$$

with $\tilde{J}_{21} \in \mathbb{R}^{k_2 \times k}, \tilde{J}_{22} \in \mathbb{R}^{k_2 \times (q_3-k)}, \tilde{B}_{41} \in \mathbb{R}^{k_2 \times m}$ and $\text{rk}[\tilde{J}_{21}, \tilde{J}_{22}, \tilde{B}_{41}] = k_2$. Then complete controllability yields

$$\text{rk} \begin{bmatrix} I_k & \tilde{B}_3 \\ 0 & \tilde{B}_{41} \\ 0 & 0 \end{bmatrix} = \text{rk} \begin{bmatrix} I_k & J_{11} & J_{12} & \tilde{B}_3 \\ 0 & \tilde{J}_{21} & \tilde{J}_{22} & \tilde{B}_{41} \\ 0 & 0 & 0 & 0 \end{bmatrix} = k + k_2,$$

and hence $\operatorname{rk}\tilde{B}_{41} = k_2$. Then there exist $F_1 \in \mathbb{R}^{m\times k}, F_2 \in \mathbb{R}^{m\times(q_3-k)}$ such that

$$[\tilde{J}_{21}, \tilde{J}_{22}] = \tilde{B}_{41}[F_1, F_2].$$

Therefore, applying the feedback

$$u(t) = -F_1 x_3(t) - F_2(t) x_4(t) + v(t), \tag{13}$$

where $v \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^m)$, to the DAE associated with system (12), we obtain

$$\begin{aligned} \dot{x}_3(t) &= (J_{11} - \tilde{B}_3 F_1) x_3(t) + (J_{12} - \tilde{B}_3 F_2) x_4(t) + v(t) \\ 0 &= \tilde{B}_{41} v(t). \end{aligned} \tag{14}$$

This proves the following statement:
For all $x_4 \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^{q_3-k})$, $v \in \mathscr{C}^\infty(\mathbb{R};\mathbb{R}^m)$ with $x_4(0) = 0$, $v(0) = 0$, $\tilde{B}_{41} v = 0$ and the unique solution (x_1, x_2, x_3) of

$$\begin{bmatrix} N & 0 & \tilde{E}_{13} \\ 0 & I_{r_2} & \tilde{E}_{23} \\ 0 & 0 & I_k \end{bmatrix} \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} I_{r_1} & 0 & \tilde{A}_{13} \\ 0 & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & 0 & J_{11} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{bmatrix} \tilde{E}_{14} & \tilde{A}_{14} & \tilde{B}_1 \\ \tilde{E}_{24} & \tilde{A}_{24} & \tilde{B}_2 \\ 0 & J_{12} & \tilde{B}_3 \end{bmatrix} \begin{pmatrix} \dot{x}_4 \\ x_4 \\ u \end{pmatrix} \tag{15}$$

with $x_1(0) = 0$, $x_2(0) = 0$, $x_3(0) = 0$ and $u = -F_1 x_3 - F_2 x_4 + v$ as well as $y = \tilde{C}_{31} x_3 + \tilde{C}_{32} x_4 + Du$ we have, using the notation in (ii), $(x,u,y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$. Note that the condition $x_1(0) = 0$ may restrict the initial values of the derivatives of x_4 and u, which is only a slight restriction of their free choice.
Step 3c: In order to exploit the inequality (11) we consider arbitrary $(x,u,y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$. Using the partitioning in (ii) and equation (9) it follows that

$$\begin{aligned} N\dot{x}_1 + \tilde{E}_{13}\dot{x}_3 + \tilde{E}_{14}\dot{x}_4 &= x_1 + \tilde{A}_{13} x_3 + \tilde{A}_{14} x_4 + \tilde{B}_1 u, \\ \dot{x}_3 &= J_{11} x_3 + J_{12} x_4 + \tilde{B}_3 u, \\ 0 &= J_{21} x_3 + J_{22} x_4 + \tilde{B}_4 u. \end{aligned}$$

We ignore the equation for x_2 since it is always solvable provided the above equations are solvable. Since $x_3(0) = 0$ we calculate

$$x_3^{(j)}(t) = \int_0^t J_{11}^j e^{J_{11}(t-s)} \psi(s)\,\mathrm{d}s + \sum_{i=1}^{j} J_{11}^{i-1} \psi^{(j-i)}(t)$$

for all $j \in \mathbb{N}$ and all $t \in \mathbb{R}$, where $\psi(t) = J_{12} x_4(t) + \tilde{B}_3 u(t)$.

Therefore,

$$\begin{aligned} x_1(t) &= -\sum_{k=0}^{\nu-1} \left(N \tfrac{\mathrm{d}}{\mathrm{d}t}\right)^k \big((\tilde{A}_{13} - \tilde{E}_{13} J_{11}) x_3(t) + (\tilde{B}_1 - \tilde{E}_{13}\tilde{B}_3) u(t) + (\tilde{A}_{14} - \tilde{E}_{13} J_{12}) x_4(t) - \tilde{E}_{14}\dot{x}_4(t)\big) \\ &= -\sum_{k=0}^{\nu-1} N^k \Big((\tilde{A}_{14} - \tilde{E}_{13} J_{12}) x_4^{(k)}(t) - \tilde{E}_{14} x_4^{(k+1)}(t) + (\tilde{B}_1 - \tilde{E}_{13}\tilde{B}_3) u^{(k)}(t) \\ &\quad + (\tilde{A}_{13} - \tilde{E}_{13} J_{11}) \sum_{i=1}^{k} J_{11}^{i-1} \big(J_{12} x_4^{(k-i)}(t) + \tilde{B}_3 u^{(k-i)}(t) \big) \Big) \\ &\quad - \int_0^t \sum_{k=0}^{\nu-1} N^k (\tilde{A}_{13} - \tilde{E}_{13} J_{11}) J_{11}^k e^{J_{11}(t-s)} \big(J_{12} x_4(s) + \tilde{B}_3 u(s) \big)\,\mathrm{d}s \end{aligned}$$

for all $t \in \mathbb{R}$. We may now show that $\tilde{E}_{14} = 0$. If $\tilde{E}_{14} \neq 0$ (and $q_3 - k > 0$), then, due to the free choice of x_4 as proved in Step 3b, we may choose a sequence $(x^k, u^k, y^k) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$ such that $\sup_{t\geq 0} \|x_4^k(t)\| \leq 1$ and $\sup_{t\in[0,1]} \|\tilde{E}_{14}\dot{x}_4^k(t)\| \to \infty$ for $k \to \infty$. Furthermore, by (13) and (14) and choosing v^k such that $\sup_{t\geq 0} \|v^k(t)\| \leq 1$, we can guarantee that $\sup_{k\in\mathbb{N}} \sup_{t\in[0,1]} \|u^k(t)\| < \infty$. This contradicts (11) and hence $\tilde{E}_{14} = 0$.
Step 3d: Using a similar argument as above, we can show that x_1 cannot depend on derivatives of x_4. Furthermore, according to Step 3b, using the free choice of v^k under the condition $\tilde{B}_{41} v^k = 0$ in the sequence $(x^k, u^k, y^k) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$ we can show that x_1 cannot depend on derivatives of u as well. Note that if $\tilde{B}_{41}$ has

full column rank (i.e., $\nu = 0$), then $u = -F_1x_3 - F_2x_4$ and thus derivatives of u involve derivatives of x_4 which have already been excluded. Hence, x_1 must be of the form

$$\begin{aligned} x_1(t) = &\sum_{k=0}^{\nu-1} N^k(\tilde{E}_{13}J_{11} - \tilde{A}_{13})J_{11}^k x_3(t) + (\tilde{E}_{13}\tilde{B}_3 - \tilde{B}_1)u(t) + (\tilde{E}_{13}J_{12} - \tilde{A}_{14})x_4(t) \\ &+ \sum_{k=1}^{\nu-1} N^k(\tilde{E}_{13}J_{11} - \tilde{A}_{13})J_{11}^{k-1}\big(J_{12}x_4(t) + \tilde{B}_3u(t)\big). \end{aligned} \tag{16}$$

Step 3e: By complete controllability of $[E_{11}, A_{11}, B_1, C_1, D]$ we have

$$\left\{\ \tilde{x}(t)\ \middle|\ (x,u,y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}\ \right\} = \mathbb{R}^{n_1} \tag{17}$$

for all $t > 0$. Note that in $\overline{\mathfrak{B}}_{[E,A,B,C,D]}$ only inputs with $u(0) = 0$ are considered. However, reachability of every state is still guaranteed which can be seen as follows: For controllable ODE systems this is straightforward. For completely controllable DAE systems this can then be concluded from the feedback form (Berger and Reis, 2013, Thm. 3.3). Multiplying (16) by $N^{\nu-1}$ from the left it follows that for all $(x,u,y) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$ we have

$$M_1 \begin{pmatrix} x_1(t) \\ x_3(t) \\ x_4(t) \end{pmatrix} = M_2u(t)$$

for all $t \in \mathbb{R}$, where

$$\begin{aligned} M_1 &:= \left[N^{\nu-1}, N^{\nu-1}(\tilde{E}_{13}J_{11} - \tilde{A}_{13}), N^{\nu-1}(\tilde{E}_{13}J_{12} - \tilde{A}_{14})\right], \\ M_2 &:= N^{\nu-1}(\tilde{E}_{13}\tilde{B}_3 - \tilde{B}_1). \end{aligned}$$

Then it follows from (17) that $\operatorname{im} M_1 \subseteq \operatorname{im} M_2$, and it is a simple calculation that $\operatorname{im} N^{\nu-1} = \operatorname{im} M_1 \subseteq \operatorname{im} M_2 \subseteq \operatorname{im} N^{\nu-1}$, thus $\operatorname{im} M_2 = \operatorname{im} N^{\nu-1}$. Since $N^{\nu-1} \neq 0$ this implies $M_2 \neq 0$. By Step 3b, and invoking (13) and (14), it is then possible to find a sequence $(x^k, u^k, y^k) \in \overline{\mathfrak{B}}_{[E,A,B,C,D]}$ with $\sup_{k\in\mathbb{N}} \sup_{t\in[0,1]} \|u^k(t)\| < \infty$ and $\sup_{t\in[0,1]} \|N^{\nu-1}(\tilde{E}_{13}\tilde{B}_3 - \tilde{B}_1)(\frac{\mathrm{d}}{\mathrm{d}t})^{\nu-1}u^k(t)\| \to \infty$ for $k \to \infty$. If $u = -F_1x_3 - F_2x_4$, then we may choose x_4 accordingly. This contradicts (11) and proves that $N = 0$.

Step 4: We show that $[E_{11}, A_{11}, B_1, C_1, D]$ is impulse observable. Since impulse observability is invariant under equivalence transformations it is sufficient to show that the system in (9) is impulse observable. We calculate that

$$\ker(\tilde{S}E_{11}\tilde{T})^\top = \operatorname{im} \begin{bmatrix} I_{r_1} & 0 \\ 0 & 0 \\ -\tilde{E}_{13}^\top & 0 \\ 0 & I_{r_3-k} \end{bmatrix} =: \operatorname{im} Z.$$

Then

$$\begin{bmatrix} \tilde{S}E_{11}\tilde{T} \\ Z^\top(\tilde{S}A_{11}\tilde{T}) \\ C_1\tilde{T} \end{bmatrix} = \begin{bmatrix} 0 & 0 & \tilde{E}_{13} & 0 \\ 0 & I_{r_2} & \tilde{E}_{23} & \tilde{E}_{24} \\ 0 & 0 & I_k & 0 \\ 0 & 0 & 0 & 0 \\ I_{r_1} & 0 & \tilde{A}_{13} - \tilde{E}_{13}J_{11} & \tilde{A}_{14} - \tilde{E}_{13}J_{12} \\ 0 & 0 & J_{21} & J_{22} \\ 0 & 0 & \tilde{C}_{31} & \tilde{C}_{32} \end{bmatrix}$$

and the latter matrix has full column rank since $\tilde{C}_{32}$ has full column rank. Then impulse observability follows from (Berger et al., 2017, Cor. 6.2) and this finishes the proof of 2) $\Rightarrow$ 3).

3) $\Leftrightarrow$ 4): For T as in (2) we have, see Berger and Trenn (2014),

$$\operatorname{im} T \begin{bmatrix} I_{n_1} \\ 0 \\ 0 \end{bmatrix} = \mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]}.$$

Let $T_1 := T \begin{bmatrix} I_{n_1} \\ 0 \\ 0 \end{bmatrix}$ and observe that $C_1 = CT_1$. Then

$$\ker C \cap \operatorname{im} T_1 = \{ \, T_1 x \mid x \in \mathbb{R}^{n_1}, \, CT_1 x = 0 \, \} = T_1 \ker C_1,$$

$$\ker E \cap \operatorname{im} T_1 = \{ \, T_1 x \mid x \in \mathbb{R}^{n_1}, \, ET_1 x = 0 \, \} = T_1 \{ \, x \in \mathbb{R}^{n_1} \mid SET_1 x = 0 \, \} = T_1 \ker \begin{bmatrix} E_{11} \\ 0 \\ 0 \end{bmatrix} = T_1 \ker E_{11},$$

and

$$\begin{aligned} A^{-1}\big(E(\operatorname{im} T_1)\big) \cap \operatorname{im} T_1 &= \{ \, T_1 x \mid x \in \mathbb{R}^{n_1}, \, \exists y \in \mathbb{R}^{n_1} : \, AT_1 x = ET_1 y \, \} \\ &= T_1 \{ \, x \in \mathbb{R}^{n_1} \mid \exists y \in \mathbb{R}^{n_1} : \, SAT_1 x = SET_1 y \, \} \\ &= T_1 \{ \, x \in \mathbb{R}^{n_1} \mid \exists y \in \mathbb{R}^{n_1} : \, A_{11} x = E_{11} y \, \} = T_1 A_{11}^{-1}(\operatorname{im} E_{11}). \end{aligned}$$

Therefore, we find

$$\begin{aligned} \mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]} \cap \ker E \cap A^{-1}\left(E\left(\mathscr{V}^*_{[E,A,B]} \cap \mathscr{W}^*_{[E,A,B]}\right)\right) \cap \ker C &= T_1 \ker E_{11} \cap T_1 A_{11}^{-1}(\operatorname{im} E_{11}) \cap T_1 \ker C_1 \\ &= T_1 \big(\ker E_{11} \cap A_{11}^{-1}(\operatorname{im} E_{11}) \cap \ker C_1\big). \end{aligned}$$

Now, by Proposition 2.1 (ii) we have that $[E_{11}, A_{11}, B_1, C_1, D]$ is impulse observable if, and only if,

$$\ker E_{11} \cap A_{11}^{-1}(\operatorname{im} E_{11}) \cap \ker C_1 = \{0\}$$

and the statement follows from full column rank of T_1.

1') $\Leftrightarrow$ 2') follows from Proposition 2.7 and 3') $\Leftrightarrow$ 4') is a consequence of 3) $\Leftrightarrow$ 4). It remains to show 2') $\Leftrightarrow$ 3').

3') $\Rightarrow$ 2'): We modify the proof of "3) $\Rightarrow$ 2)". Since $[E,A,B,C,D]$ is behaviorally detectable, it follows that system $[\tilde{E},\tilde{A},\tilde{B},\tilde{C},D]$ defined in (6) is behaviorally detectable as well. Then, by (Berger and Reis, 2017, Thm. 3.8), the index-1 observer $[E_o^1, A_o^1, B_o^1, C_o^1, D_o^1]$ can be chosen to be asymptotic, and hence $[E_o, A_o, B_o, C_o, D_o]$ is an asymptotic index-1 observer for $[E,A,B,C,D]$.

2') $\Rightarrow$ 3'): By 2) $\Leftrightarrow$ 3) the completely controllable part $[E_{11}, A_{11}, B_1, C_1, D]$ is impulse observable. Furthermore, since there exists an asymptotic observer for $[E,A,B,C,D]$, it follows from (Berger and Reis, 2017, Thm. 3.5) that $[E,A,B,C,D]$ is behaviorally detectable. ☐

4 Observer design

In this section we present a procedure for the design of ODE observers for DAE systems satisfying the condition in Theorem 3.3. The presented method is essentially an extension of the method presented in Darouach and Boutayeb (1995) to systems which are not impulse observable. As a starting point we use the following result which was derived in (Hou and Müller, 1999a, Prop. 1).

Lemma 4.1. *For any $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ there exist orthogonal matrices $U \in \mathbb{R}^{l \times l}$, $V \in \mathbb{R}^{n \times n}$ such that*

$$\begin{aligned} U(sE - A)V &= \begin{bmatrix} s\tilde{E}_{11} - \tilde{A}_{11} & s\tilde{E}_{13} - \tilde{A}_{13} \\ 0 & sE_{33} - A_{33} \end{bmatrix}, \\ UB &= \begin{bmatrix} \tilde{B}_1 \\ 0 \end{bmatrix}, \quad CV = \begin{bmatrix} \tilde{C}_1 & C_3 \end{bmatrix}, \end{aligned} \tag{18}$$

where

(i) $[\tilde{E}_{11}, \tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, D] \in \Sigma_{\tilde{l}_1, \tilde{n}_1, m, p}$ *with* $\tilde{l}_1 = \operatorname{rk}[E_{11}, B_1] \le \tilde{n}_1 + m$,

(ii) $[E_{33}, A_{33}, 0, C_3, D] \in \Sigma_{l_3, n_3, m, p}$ *with* $l_3 \ge n_3$ *satisfies* $\operatorname{rk}_{\mathbb{C}}(\lambda E_{33} - A_{33}) = n_3$ *for all* $\lambda \in \mathbb{C}$.

As indicated by the notation used in (18) it is possible to transform it into KCD (2) by an additional transformation of the first block row and block column, hence we may view (18) as a precursor of the KCD. More precisely, there exist $\tilde{S} \in \mathbf{Gl}_{\tilde{l}_1}(\mathbb{R}), \tilde{T} \in \mathbf{Gl}_{\tilde{n}_1}(\mathbb{R})$ such that

$$\begin{aligned} \tilde{S}(s\tilde{E}_{11} - \tilde{A}_{11})\tilde{T} &= \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} \\ 0 & sE_{22} - A_{22} \end{bmatrix}, \\ \tilde{S}\tilde{B}_1 &= \begin{bmatrix} B_1 \\ 0 \end{bmatrix}, \quad \tilde{C}_1 \tilde{T} = \begin{bmatrix} C_1 & C_2 \end{bmatrix}, \end{aligned} \tag{19}$$

where the new block entries have the properties as in Theorem 2.2.

If $M \in \mathbb{R}^{l \times n}$ with $\operatorname{rk} M = r$, then, using QR factorization with pivoting (see Golub and van Loan (1996)), there exists an orthogonal matrix $T \in \mathbb{R}^{l \times l}$ such that

$$TM = \begin{bmatrix} \Sigma \\ 0 \end{bmatrix},$$

where $\Sigma \in \mathbb{R}^{r \times n}$ with $\operatorname{rk} \Sigma = r$, see also Beelen and Van Dooren (1988). We call T a *row compression* of the matrix M.

We are now in a position to state our observer design procedure. Feasibility of each step of the procedure will be proved in the subsequent Theorem 4.2.

Initialization. Let $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ be given such that Theorem 3.3 3) is satisfied. If an asymptotic ODE observer is sought, then assume that additionally $[E,A,B,C,D]$ is behaviorally detectable.

Step 1. Compute orthogonal matrices $U \in \mathbb{R}^{l \times l}$, $V \in \mathbb{R}^{n \times n}$ such that (18) holds, e.g. using the procedure presented in the proof of (Hou and Müller, 1999a, Prop. 1).

Step 2. Compute a row compression P of $\tilde{E}_{11}$ such that

$$P\tilde{E}_{11} = \begin{bmatrix} E_1 \\ 0 \end{bmatrix}, \quad P\tilde{A}_{11} = \begin{bmatrix} A_1 \\ A_2 \end{bmatrix}, \quad P\tilde{B}_1 = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix},$$

where $\operatorname{rk} \tilde{E}_{11} = r$, $E_1, A_1 \in \mathbb{R}^{r \times \tilde{n}_1}$, $B_{11} \in \mathbb{R}^{r \times m}$, $B_{12} \in \mathbb{R}^{(\tilde{l}_1 - r) \times m}$. Set

$$C_{11} := \begin{bmatrix} A_2 \\ \tilde{C}_1 \end{bmatrix} \in \mathbb{R}^{(p + \tilde{l}_1 - r) \times \tilde{n}_1}.$$

Step 3. Compute a row compression $Q = \left[\begin{smallmatrix} \tilde{Q}_1 & \tilde{Q}_2 \\ Q_3 & Q_4 \end{smallmatrix}\right]$ of $\left[\begin{smallmatrix} E_1 \\ C_{11} \end{smallmatrix}\right]$ such that

$$\begin{aligned} \tilde{Q}_1 E_1 + \tilde{Q}_2 C_{11} &= \Sigma, \\ Q_3 E_1 + Q_4 C_{11} &= 0, \end{aligned}$$

where $\Sigma \in \mathbf{Gl}_{\tilde{n}_1}(\mathbb{R})$. Solve

$$\Sigma Q_i = \tilde{Q}_i, \qquad i = 1, 2,$$

for Q_i (e.g. using Gaussian elimination with pivoting or QR factorization with pivoting), then $Q_1 E_1 + Q_2 C_{11} = I_{\tilde{n}_1}$.

Step 4. If $[E,A,B,C,D]$ is behaviorally detectable, then $\left[I_{\tilde{n}_1}, Q_1 A_1, 0, \left[\begin{smallmatrix} Q_3 A_1 \\ C_{11} \end{smallmatrix}\right], 0\right]$ is behaviorally detectable as well and hence, using classical methods (see e.g. (Datta, 2004, Sec. 10.2)), we may choose $K \in \mathbb{R}^{\tilde{n}_1 \times (p + \tilde{l}_1 - \tilde{n}_1)}$, $L_2 \in \mathbb{R}^{\tilde{n}_1 \times p}$ such that for

$$N := Q_1 A_1 + [K, -L_2] \begin{bmatrix} Q_3 A_1 \\ C_{11} \end{bmatrix} \in \mathbb{R}^{\tilde{n}_1 \times \tilde{n}_1}$$

we have $\sigma(N) \subseteq \mathbb{C}_-$. Otherwise, choose $K = 0$, $L_2 = 0$ and $N := Q_1 A_1$. Set

$$\begin{aligned} L_1 &:= N(Q_2 + K Q_4), \\ G &:= (Q_1 + K Q_3) B_{11}. \end{aligned}$$

Step 5. The (asymptotic) ODE observer is given by

$$[E_o, A_o, B_o, C_o, D_o] := \left[I_{\tilde{n}_1}, N, (L_1 + L_2) \left[\begin{smallmatrix} -B_{12} & 0 \\ -D & I_m \end{smallmatrix}\right] + [G, 0], V \begin{bmatrix} I_{\tilde{n}_1} \\ 0 \end{bmatrix}, V \begin{bmatrix} (Q_2 + K Q_4) \left[\begin{smallmatrix} -B_{12} & 0 \\ -D & I_m \end{smallmatrix}\right] \\ 0 \end{bmatrix} \right] \in \Sigma_{\tilde{n}_1, \tilde{n}_1, m+p, n}.$$

We prove feasibility of the above procedure and show that the result is indeed an (asymptotic) ODE observer for $[E,A,B,C,D]$.

Theorem 4.2. *Let $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$ be given such that Theorem 3.3 3) is satisfied. Then each step of the observer design procedure is feasible (Step 4 with $K = 0$ and $L_2 = 0$) and the resulting observer $[E_o, A_o, B_o, C_o, D_o]$ is an ODE observer for $[E,A,B,C,D]$. If $[E,A,B,C,D]$ is additionally behaviorally detectable, then Step 4 is feasible and the observer $[E_o, A_o, B_o, C_o, D_o]$ is asymptotic.*

Proof. Step 1 is feasible by Lemma 4.1 and Step 2 is always feasible. With the additional transformation as in (19) the form (18) can be put into KCD. By Theorem 3.3 3) the completely controllable part $[E_{11}, A_{11}, B_1, C_1, D]$ of $[E, A, B, C, D]$ is impulse observable and since E_{22} is invertible it follows that $[\tilde{E}_{11}, \tilde{A}_{11}, \tilde{B}_1, \tilde{C}_1, D]$ is impulse observable. As shown in Darouach and Boutayeb (1995) this implies that $\begin{bmatrix} E_1 \\ C_{11} \end{bmatrix}$ has full column rank and therefore Step 3 is feasible. It is further shown in Darouach and Boutayeb (1995) that if $[E, A, B, C, D]$ is behaviorally detectable, then $\left[I_{\tilde{n}_1}, Q_1 A_1, 0, \begin{bmatrix} Q_3 A_1 \\ C_{11} \end{bmatrix}, 0\right]$ is behaviorally detectable (i.e., detectable in the classical sense) which finally shows feasibility of Steps 4 and 5.

It remains to show that $[E_o, A_o, B_o, C_o, D_o]$ is an (asymptotic) ODE observer. This however is an immediate consequence of the fact that, since the DAE $\frac{\mathrm{d}}{\mathrm{d}t} E_{33} x_3 = A_{33} x_3$ does only have the trivial solution, the error $e = x - z$ between the system state and the observer output satisfies that $e = V \binom{\tilde{e}}{0}$ and, as shown in detail in Darouach and Boutayeb (1995), we have the dynamics $\frac{\mathrm{d}}{\mathrm{d}t} \tilde{e}(t) = N \tilde{e}(t)$. □

We like to note that if a reduced order observer is required, then the observer design method may be appropriately adjusted using the reduced order observer design for impulse observable systems presented in Darouach and Boutayeb (1995).

5 Conclusion

In the present paper we have considered the observer design approach to DAE systems introduced in Berger and Reis (2017). We have shown that a necessary and sufficient condition for the existence of an ODE observer is that the completely controllable part of the system is impulse observable; and that the observer is moreover asymptotic if, and only if, the system is additionally behaviorally detectable. Extending the observer design method for impulse observable systems given in Darouach and Boutayeb (1995) we presented a procedure for the construction of an (asymptotic) ODE observer.

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