The zero dynamics form for nonlinear differential-algebraic systems

Thomas Berger

***Abstract*—We show that any nonlinear differential-algebraic system can be locally transformed into zero dynamics form, which is a normal form with respect to the input-output behavior. Only mild assumptions on the maximal output zeroing submanifold are required and thus the zero dynamics form even generalizes the Byrnes-Isidori form for nonlinear systems with existing vector relative degree. Left- and right-invertibility of the system can be studied in terms of the solution properties of a subsystem in the zero dynamics form. This is the basis for the investigation of various classical control problems, such as output regulation and trajectory tracking.**

***Index Terms*—Differential-algebraic systems, nonlinear systems, descriptor systems, zero dynamics, output zeroing submanifold, system inversion.**

Nomenclature:

$\mathbb{N}$, $\mathbb{N}_0$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\mathbb{R}^{n\times m}$	the set of real $n \times m$ matrices
$\operatorname{rk} A$, $\operatorname{im} A$	rank and image of $A \in \mathbb{R}^{n\times m}$
$\mathbf{Gl}_n(\mathbb{R})$	the group of invertible matrices in $\mathbb{R}^{n\times n}$
$\mathscr{C}^k(X;Y)$	set of k-times continuously differentiable functions $f : X \to Y$, $k \in \mathbb{N}_0 \cup \{\infty\}$; $\mathscr{C}(X;Y) := \mathscr{C}^0(X;Y)$; if $k = \infty$ the function f is called *smooth*
$\operatorname{dom} f$	the domain of the function f
$f\|_I$	restriction of the function f to the set I

I. Introduction

We consider nonlinear descriptor systems governed by differential-algebraic equations (DAEs) of the form

$$\begin{aligned} E(x(t))\,\dot{x}(t) &= f(x(t)) + g(x(t))\,u(t), \\ y(t) &= h(x(t)), \end{aligned} \tag{1}$$

where $X \subseteq \mathbb{R}^n$ is open with $0 \in X$, $f \in \mathscr{C}(X;\mathbb{R}^l), h \in \mathscr{C}(X;\mathbb{R}^p)$ are vector-valued functions such that $f(0) = 0$, $h(0) = 0$, and $E \in \mathscr{C}(X;\mathbb{R}^{l\times n})$, $g \in \mathscr{C}(X;\mathbb{R}^{l\times m})$ are matrix-valued functions. The set of these systems is denoted by $\Sigma^X_{l,n,m,p}$; and we write $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$. Note that any system

$$F(x(t),\dot{x}(t)) = g(x(t))\,u(t)$$

with $F \in \mathscr{C}(X \times \mathbb{R}^n;\mathbb{R}^l)$ can be put into the form (1) by augmenting the state space, namely

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} w(t) \\ -F(x(t),w(t)) \end{pmatrix} + \begin{bmatrix} 0 \\ g(x(t)) \end{bmatrix} u(t).$$

In particular, the input-output behavior of the original system is not affected by this transformation. Therefore, the class $\Sigma^X_{l,n,m,p}$ encompasses any linear descriptor system (which may even be non-regular[1]) and various important classes of nonlinear descriptor systems (e.g. chemical process systems [1], mechanical systems [2], [3] and modified nodal analysis models of electrical circuits [4]). Nonlinear descriptor systems seem to have been first considered by Luenberger [5]; see also the recent textbooks [6], [7].

The functions $u : I \to \mathbb{R}^m$ and $y : I \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. Since solutions not necessarily exist globally (e.g. finite escape times may arise) we consider maximal solutions of (1).

Definition I.1 (Solutions)**.** For $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ a trajectory $(x,u,y) \in \mathscr{C}(I;X \times \mathbb{R}^m \times \mathbb{R}^p)$ is called a *solution* of (1), if $I = \operatorname{dom} x \subseteq \mathbb{R}$ is an open interval, $x \in \mathscr{C}^1(I;\mathbb{R}^l)$ and (x,u,y) solves (1) for all $t \in I$. A solution (x,u,y) of (1) is called *maximal*, if any other solution $(\tilde{x},\tilde{u},\tilde{y})$ of (1) satisfies

$$J := \operatorname{dom}\tilde{x} \cap \operatorname{dom} x \neq \emptyset \ \wedge\ (\tilde{x},\tilde{u},\tilde{y})|_J = (x,u,y)|_J \quad \Longrightarrow \quad \operatorname{dom}\tilde{x} \subseteq \operatorname{dom} x.$$

Note that the interval of definition I of a maximal solution of (1) depends on the choice of the input u. The *behavior* of (1) is defined as the set of maximal solution trajectories

$$\mathfrak{B}_{(1)} := \big\{ (x,u,y) \in \mathscr{C}(I;X \times \mathbb{R}^m \times \mathbb{R}^p) \ \big|\ I \subseteq \mathbb{R} \text{ open interval, } (x,u,y) \text{ is maximal solution of (1)} \big\}.$$

In the present paper, we derive the zero dynamics form for a system (1), which is a local input-output normal form. In the zero dynamics form, the zero dynamics of the system are decoupled. The zero dynamics are, loosely speaking, those dynamics that are not visible at the output when the output is identically zero, and they are defined as the set of trajectories

$$\mathscr{Z}\mathscr{D}_{(1)} := \big\{ \ (x,u,y) \in \mathfrak{B}_{(1)} \ \big|\ y = 0 \ \big\}.$$

If the system (1) is governed by an ordinary differential equation (ODE), i.e., $n = l$ and $E(x) = I$, then the concept of zero dynamics has been introduced by Byrnes and Isidori [8] and studied extensively since then, see e.g. the textbooks [9], [10]. The zero dynamics of DAE systems have been investigated in detail recently [11]–[13], see also [14]–[16] for some results on special classes.

For the derivation of the zero dynamics form we require a maximal output zeroing submanifold Z^* that satisfies

$$\dim\big(E(0)T_0Z^* + \operatorname{im} g(0)\big) = \dim Z^* + m,$$

where T_0Z^* is the tangent space of Z^* at 0 (see Section II for the precise definitions) and we require that h is a submersion at $x = 0$. These are very mild assumptions and they are always satisfied if the system is an ODE with $m \le p$ and has some vector relative degree. A special feature of the zero dynamics form is that, using a local state space transformation $\varphi : X_1 \to X_2$, $X_1, X_2 \subseteq X$ open, there exists a smooth vector field F_1 and a smooth function F_2 such that for all $(x,u,y) \in \mathfrak{B}_{(1)}$ with $x(t) \in X_2$ for all $t \in \operatorname{dom} x$ we have, with $\binom{z_1}{z_2} = \varphi^{-1}(x)$,

$$(x,u,0) \in \mathscr{Z}\mathscr{D}_{(1)} \iff \begin{cases} \dot{z}_1(t) = F_1(z_1(t)), \\ u(t) = F_2(z_1(t)), \quad \forall t \in \operatorname{dom} x \\ z_2(t) = 0. \end{cases}$$

This characterizes the zero dynamics as a set of solution trajectories

Thomas Berger is with the Fachbereich Mathematik, Universität Hamburg, Bundesstraße 55, 20146 Hamburg, Germany, thomas.berger@uni-hamburg.de

[1]A linear descriptor system $E\dot{x}(t) = Ax(t) + Bu(t)$, $y(t) = Cx(t)$ is called regular, if $E, A \in \mathbb{R}^{n\times n}$ and $\det(sE - A) \not\equiv 0$.

of an ODE. The zero dynamics form also allows to study invertibility properties in terms of the solution properties of one of its subsystems. This is the basis for the investigation of various classical control problems for nonlinear DAEs, such as output regulation and trajectory tracking as shown for linear DAEs in [12].

The present paper is organized as follows: In Section II we present some preliminary results, which are used for the derivation of the zero dynamics form in Section III; this is the main result of the present paper. System invertibility properties are studied in terms of the zero dynamics form in Section IV. An illustrative example is given in Section V and some conclusions in Section VI.

II. Preliminaries

We use the terminology of smooth manifolds and other differential geometric concepts as in [17]. Apart from that, by a *submanifold* we will always mean an embedded smooth k-submanifold of $\mathbb{R}^n$ for some $k \le n$. Furthermore, we define the *tangent space* to a submanifold M of $\mathbb{R}^n$ at $x \in M$ as the linear subspace

$$T_xM := \left\{ v \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists I \subseteq \mathbb{R} \text{ open interval } \exists \gamma \in \mathscr{C}^\infty(I;M): \\ \gamma(0) = x \;\wedge\; \dot\gamma(0) = v \end{array} \right\}.$$

We define the boundary ∂M of a submanifold M of $\mathbb{R}^n$ (if it exists) as the set

$$\partial M := \left\{ w \in \mathbb{R}^n \;\middle|\; \begin{array}{l} w \notin M,\ \exists \gamma \in \mathscr{C}^\infty((a,b);M), a<b: \\ \lim_{t\to b} \gamma(t) = w \end{array} \right\}.$$

Let $X \subseteq \mathbb{R}^n$ be an open set (which is a manifold) and $M \subseteq X$ be a submanifold. For any $x^0 \in M$ there exist $U \subseteq X$ open with $x^0 \in U$, $W \subseteq \mathbb{R}^k$ open for $k = \dim M \le n$ and a diffeomorphism $\varphi : M \cap U \to W$, i.e., $\varphi \in \mathscr{C}^\infty(M\cap U;W)$ and $\varphi^{-1} \in \mathscr{C}^\infty(W;M\cap U)$. Without loss of generality, W and φ can be chosen such that $0 \in W$ and $\varphi(x^0) = 0$. We call $\psi := \varphi^{-1}$ a *parametrization* of M at x^0 and record the following result which is important in due course.

Lemma II.1 (Parametrization and tangent space)**.** *Let M be a submanifold of an open set $X \subseteq \mathbb{R}^n$ and let $\psi : W \to M\cap U$ be a parametrization of M at $x^0 \in M$. Then*

$$\forall x \in M\cap U:\ T_xM = \operatorname{im} \psi'(\psi^{-1}(x)).$$

An important class of submanifolds in control theory are (locally) controlled invariant submanifolds, which have been introduced by Isidori and Moog [18], and before that by van der Schaft [19], [20] for Hamiltonian systems. A characterization of these submanifolds for DAE systems has been derived in [13].

Definition II.2 (Controlled invariant submanifolds)**.** Let $[E,f,g,h] \in \Sigma^X_{l,n,m,p}$ and M be a connected submanifold of X such that $0 \in M$. Then M is called *locally controlled invariant*, if there exists an open neighborhood U of $0 \in X$ such that

$$\begin{aligned} &\forall x^0 \in M\cap U\ \exists (x,u,y) \in \mathfrak{B}_{(1)}\ \exists t_0 \in \operatorname{dom} x,\ x(t_0) = x^0: \\ &\quad \big(\forall t \in \operatorname{dom} x, t \ge t_0:\ x(t) \in M\cap U\big) \\ &\quad \vee \big(\exists \hat t \in \operatorname{dom} x, \hat t > t_0\ \forall t \in [t_0,\hat t): \\ &\qquad\qquad x(t) \in M\cap U \;\wedge\; x(\hat t) \in \partial(M\cap U)\big). \end{aligned}$$

Roughly speaking, M is locally controlled invariant, if for any point x^0 in a neighborhood $M\cap U$ there exists a maximal solution (x,u,y) with x starting at x^0 and staying in $M\cap U$ or reaching its boundary in finite time. The latter is a special feature of local controlled invariance in the nonlinear case which cannot be prevented in general. Since the submanifold may be bounded, for instance it may be an open ball in $\mathbb{R}^n$, solutions starting in it may reach the boundary in finite time. However, until they do so they stay in $M\cap U$.

Of particular importance for the derivation of the zero dynamics form is the following concept of output zeroing submanifolds.

Definition II.3 (Output zeroing submanifold)**.** Let $[E,f,g,h] \in \Sigma^X_{l,n,m,p}$ and M be a connected submanifold of X such that $0 \in M$. Then M is called *output zeroing*, if M is locally controlled invariant and $h(x) = 0$ for all $x \in M$. An output zeroing submanifold M is called *locally maximal*, if there exists an open neighborhood U of $0 \in X$ such that any output zeroing submanifold $\tilde M$ satisfies $\tilde M \cap U \subseteq M \cap U$.

In the following we restate [13, Thm. 13], which provides a sequence of submanifolds which converges to a locally maximal output zeroing submanifold. This is called the zero dynamics algorithm and has been developed for ODE systems in [18], [21].

Theorem II.4 (Zero dynamics algorithm)**.** *Let $[E,f,g,h] \in \Sigma^X_{l,n,m,p}$ be such that E,f,g and h are smooth. Define $M_0 := h^{-1}(0)$ and for any $k \in \mathbb{N}$ the set M_k recursively as follows: Suppose that for some open neighborhood U_{k-1} of $0 \in X$, $M_{k-1}\cap U_{k-1}$ is a submanifold, define*

$$\tilde M_{k-1} := \bigcup \left\{ M_{k-1}\cap U \;\middle|\; \begin{array}{l} U \subseteq X \text{ open, } M_{k-1}\cap U \\ \text{is a submanifold} \end{array} \right\},$$

let M^c_{k-1} be the connected component of $\tilde M_{k-1}$ which contains $0 \in X$ and define

$$M_k := \left\{ x \in M^c_{k-1} \;\middle|\; f(x) \in E(x)T_xM^c_{k-1} + \operatorname{im} g(x) \right\}. \qquad (2)$$

Then we have the following:

(i) *The sequence (M_k) is nested, terminates and satisfies*
$$\exists k^* \in \mathbb{N}_0\ \forall j \in \mathbb{N}:\ M_0 \supsetneq M_1 \supsetneq \ldots \supsetneq M_{k^*} \supseteq M^c_{k^*} = M_{k^*+j} = M^c_{k^*+j}.$$

(ii) *If $Z^* := M^c_{k^*}$ satisfies, for some open neighborhood U of $0 \in X$, that $\dim E(x)T_xZ^*$ and $\dim\big(E(x)T_xZ^* + \operatorname{im} g(x)\big)$ are both constant for $x \in Z^*\cap U$, then Z^* is a locally maximal output zeroing submanifold.*

(iii) *There exists an open neighborhood U of $0 \in X$ such that for all open $O \subseteq U$ and all solutions $(x,u,y) \in \mathscr{C}(I;X\times\mathbb{R}^m\times\mathbb{R}^p)$ of (1) with $x(t) \in O$ for all $t \in I$, we have*
$$y = 0 \iff x(t) \in Z^*\cap O \quad \forall t \in I.$$

III. Zero dynamics form

In this section we derive the main result of the present paper, the zero dynamics form, and give some remarks on it. The zero dynamics form is derived using a locally maximal output zeroing submanifold which also allows for a decoupling of the zero dynamics in the new local coordinates. We need to assume that $E(0)T_0Z^* + \operatorname{im} g(0)$ has maximal dimension, which generalizes the assumption in the linear case (see [12, Thm. 3.6]) and guarantees so called locally autonomous zero dynamics (see [13, Thm. 17]). By further assuming that the output function h is a submersion at $x = 0$ it is possible to obtain the output variables as a part of these new coordinates.

Theorem III.1 (Zero dynamics form)**.** *Let $[E,f,g,h] \in \Sigma^X_{l,n,m,p}$ be such that E,f,g and h are smooth and assume, for the sets M_k as in (2), that for some open neighborhood U_k of $0 \in X$, $M_k\cap U_k$ is a submanifold, for all $k \in \mathbb{N}_0$. Use the notation from Theorem II.4 and assume furthermore that*

(1) $\dim\big(E(0)T_0Z^* + \operatorname{im} g(0)\big) = q+m$, where $q = \dim Z^*$, and
(2) h is a submersion at $x=0$, i.e., $h'(0)$ has full row rank p.

Then there exist open neighborhoods X_1, X_2 of $0 \in X$, a diffeomorphism $\varphi : X_1 \to X_2$ and $S \in \mathscr{C}^\infty(X_1;\mathbf{Gl}_l(\mathbb{R}))$ such that the local coordinate transformation $z(t) = \varphi^{-1}(x(t))$ and left-multiplication *by S put system* (1) *into the form*

$$\begin{aligned}\tilde{E}(z(t))\dot{z}(t) &= \tilde{f}(z(t)) + \tilde{g}(z(t))\,u(t),\\ y(t) &= \tilde{h}(z(t)),\end{aligned} \tag{3}$$

where

$$[\tilde{E},\tilde{f},\tilde{g},\tilde{h}] = [S(E\circ\varphi)\,\varphi', S(f\circ\varphi), S(g\circ\varphi), h\circ\varphi]. \tag{4}$$

Each of these functions is defined on X_1 and smooth and they satisfy, for all $z\in X_1$,

$$\tilde{E}(z) = \begin{bmatrix} I_q & E_{12}(z) & E_{13}(z)\\ 0 & E_{22}(z) & E_{23}(z)\\ 0 & E_{32}(z) & E_{33}(z)\end{bmatrix}, \quad \tilde{f}(z) = \begin{pmatrix} f_1(z)\\ f_2(z)\\ f_3(z)\end{pmatrix}, \quad \tilde{g}(z) = \begin{bmatrix} 0\\ I_m\\ 0\end{bmatrix},$$
$$\tilde{h}(z) = [0, I_p, 0]\,z,$$

where $E_{22}: X_1\to\mathbb{R}^{m\times p}$, $E_{33}: X_1\to\mathbb{R}^{l_3\times n_3}, n_3 = n-q-p, l_3 = l-q-m$, and all other entries are of appropriate sizes. Furthermore,

$$\forall z_1\in\mathbb{R}^q: \begin{pmatrix} z_1\\ 0\end{pmatrix}\in X_1 \implies f_3\left(\begin{pmatrix} z_1\\ 0\end{pmatrix}\right) = 0,$$

and any solution $(z_1,z_3)\in\mathscr{C}^1(I;\mathbb{R}^q\times\mathbb{R}^{n_3})$, $I\subseteq\mathbb{R}$ an open interval, of the subsystem

$$\begin{aligned}\dot{z}_1(t) + E_{13}(z_1(t),0,z_3(t))\,\dot{z}_3(t) &= f_1(z_1(t),0,z_3(t)),\\ E_{33}(z_1(t),0,z_3(t))\,\dot{z}_3(t) &= f_3(z_1(t),0,z_3(t))\end{aligned} \tag{5}$$

with $\begin{pmatrix} z_1(t)\\ 0\\ z_3(t)\end{pmatrix}\in X_1$ for all $t\in I$ satisfies $z_3 = 0$.

Proof. Step 1: By assumption (2) and the submersion theorem, there exist open sets $V_1, W_1\subseteq X$ with $0\in V_1$, $0\in W_1$, and a diffeomorphism $\varphi_1: V_1\to W_1$ with $\varphi_1(0) = 0$ (since $h(0) = 0$) such that

$$(h\circ\varphi_1)(z) = [0_{p\times q}, I_p, 0_{p\times n_3}]\,z$$

for all $z\in V_1$, where $n_3 = n-q-p$. Then the set $\mathscr{V}^* := \{\ v\in V_1 \mid \varphi_1(v)\in Z^*\ \}$ is a connected submanifold of X with $0\in\mathscr{V}^*$ and $\dim\mathscr{V}^* = \dim Z^* = q$.

Step 2: Let $\psi: V\to\mathscr{V}^*\cap U$ be a parametrization of $\mathscr{V}^*$ at $0\in\mathscr{V}^*$, where $V\subseteq\mathbb{R}^q$ open with $0\in V$. Since $Z^*\subseteq h^{-1}(0)$ we find that

$$\mathscr{V}^*\subseteq(h\circ\varphi_1)^{-1}(0) = V_1\cap\operatorname{im}\begin{bmatrix} I_q & 0\\ 0 & 0\\ 0 & I_{n_3}\end{bmatrix}.$$

By Lemma II.1,

$$\operatorname{im}\psi'(0) = T_0\mathscr{V}^*\subseteq\operatorname{im}\begin{bmatrix} I_q & 0\\ 0 & 0\\ 0 & I_{n_3}\end{bmatrix}$$

and hence $\left[\psi'(0), \begin{bmatrix} 0_{q\times p}\\ I_p\\ 0_{n_3\times p}\end{bmatrix}\right]$ has full column rank $q+p$. Let $W\in\mathbb{R}^{n\times n_3}$ be such that

$$\left[\psi'(0), \begin{bmatrix} 0_{q\times p}\\ I_p\\ 0_{n_3\times p}\end{bmatrix}, W\right]\in\mathbf{Gl}_n(\mathbb{R}).$$

Define

$$\varphi_2: V\times\mathbb{R}^p\times\mathbb{R}^{n_3}\to\mathbb{R}^n,\ \begin{pmatrix} z_1\\ y\\ z_3\end{pmatrix}\mapsto\psi(z_1) + \begin{bmatrix} 0_{q\times p}\\ I_p\\ 0_{n_3\times p}\end{bmatrix} y + Wz_3.$$

We calculate that

$$\varphi_2'(0) = \left[\psi'(0), \begin{bmatrix} 0_{q\times p}\\ I_p\\ 0_{n_3\times p}\end{bmatrix}, W\right]\in\mathbf{Gl}_n(\mathbb{R})$$

and hence, invoking the inverse function theorem, φ_2 is locally a diffeomorphism on a neighborhood of $0\in X$. Therefore, we may choose open $V_2\subseteq V\times\mathbb{R}^p\times\mathbb{R}^{n_3}$ sufficiently small with $0\in V_2$ such that $W_2 := \varphi_2(V_2)\subseteq V_1$ and $\varphi_2: V_2\to W_2$ is a diffeomorphism.

Step 3: Since $\varphi_1\circ\psi: V\to Z^*\cap\tilde{U}$ is a parametrization of Z^* at $0\in Z^*$ for some open set $\tilde{U}$ it follows from Lemma II.1 that

$$T_0 Z^* = \operatorname{im}(\varphi_1\circ\psi)'\big((\varphi_1\circ\psi)^{-1}(0)\big) = \operatorname{im}\varphi_1'(0)\psi'(0).$$

Then, by assumption (1) and $\varphi_2(0) = 0$ we have that

$$\begin{aligned}\operatorname{rk}[E(0)\varphi_1'(0)\psi'(0), g(0)] &= q+m\\ &= \operatorname{rk}[E(\varphi_1(\varphi_2(0)))\varphi_1'(\varphi_2(0))\psi'([I_q,0]\cdot 0), g(\varphi_1(\varphi_2(0)))]\end{aligned}$$

and from continuity we may infer existence of an open neighborhood $V_3\subseteq V_2$ of $0\in X$ such that, for all $z\in V_3$,

$$\operatorname{rk}[E(\varphi_1(\varphi_2(z)))\varphi_1'(\varphi_2(z))\psi'([I_q,0]z), g(\varphi_1(\varphi_2(z)))] = q+m.$$

Therefore, we may apply [13, Lem. 7] to its transpose and this gives existence of an open neighborhood $V_4\subseteq V_3$ of $0\in X$ and $S\in\mathscr{C}^\infty(V_4;\mathbf{Gl}_l(\mathbb{R}))$ such that

$$S(z)[E(\varphi_1(\varphi_2(z)))\varphi_1'(\varphi_2(z))\psi'([I_q,0]z), g(\varphi_1(\varphi_2(z)))] = \begin{bmatrix} I_q & 0\\ 0 & I_m\\ 0 & 0\end{bmatrix}.$$

for all $z\in V_4$. Let $W_4 := \varphi_1(\varphi_2(V_4))$.

Step 4: Define the diffeomorphism $\varphi := \varphi_1\circ\varphi_2: V_4\to W_4$ and calculate that, for all $z = \begin{pmatrix} z_1\\ y\\ z_3\end{pmatrix}\in V_4$,

$$\begin{aligned} S(z)E(\varphi(z))\varphi'(z) &= S(z)E(\varphi(z))\varphi_1'(\varphi_2(z))\varphi_2'(z)\\ &= S(z)E(\varphi(z))\varphi_1'(\varphi_2(z))\left[\psi'(z_1), \begin{bmatrix} 0_{q\times p}\\ I_p\\ 0_{n_3\times p}\end{bmatrix}, W\right]\\ &= \left[\begin{bmatrix} I_q\\ 0_{m\times q}\\ 0_{l_3\times q}\end{bmatrix}, S(z)\begin{bmatrix} 0_{q\times p}\\ I_p\\ 0_{n_3\times p}\end{bmatrix}, S(z)W\right] = \begin{bmatrix} I_q & E_{12}(z) & E_{13}(z)\\ 0 & E_{22}(z) & E_{23}(z)\\ 0 & E_{32}(z) & E_{33}(z)\end{bmatrix}\end{aligned}$$

where $E_{22}: V_4\to\mathbb{R}^{m\times p}$, $E_{33}: V_4\to\mathbb{R}^{l_3\times n_3}$ and all other entries are of appropriate sizes. Furthermore, we obtain

$$S(z)g(\varphi(z)) = \begin{bmatrix} 0_{q\times m}\\ I_m\\ 0_{l_3\times m}\end{bmatrix}, \quad h(\varphi(z)) = [0_{p\times q}, I_p, 0_{p\times n_3}]\,z$$

for all $z\in V_4$ and

$$S(z)f(\varphi(z)) = \begin{pmatrix} f_1(z)\\ f_2(z)\\ f_3(z)\end{pmatrix}$$

with $f_1: V_4\to\mathbb{R}^q, f_2: V_4\to\mathbb{R}^m, f_3: V_4\to\mathbb{R}^{l_3}$.

Step 5: Using assumption (1), we may infer from [13, Thm. 9] that there exists an open neighborhood O of $0\in X$ such that $f(x)\in E(x)T_x Z^* + \operatorname{im} g(x)$ for all $x\in Z^*\cap O$. Define

$$M := \{\ z\in V_4 \mid \varphi(z)\in Z^*\cap O\ \},$$

and observe that M is a connected submanifold of X with $0\in M$ and $\dim M = \dim Z^* = q$. Then we obtain that for all $z\in M$,

$$\begin{aligned} f(\varphi(z)) &\in E(\varphi(z))T_{\varphi(z)}Z^* + \operatorname{im} g(\varphi(z))\\ &= E(\varphi(z))\varphi'(z)T_z M + \operatorname{im} g(\varphi(z))\end{aligned}$$

since φ is a diffeomorphism, and therefore

$$S(z)f(\varphi(z)) \in S(z)E(\varphi(z))\varphi'(z)T_zM + \operatorname{im} S(z)g(\varphi(z))$$
$$\implies \begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \end{pmatrix} \in \begin{bmatrix} I_q & E_{12}(z) & E_{13}(z) \\ 0 & E_{22}(z) & E_{23}(z) \\ 0 & E_{32}(z) & E_{33}(z) \end{bmatrix} T_zM + \operatorname{im} \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}.$$

It is no loss of generality to assume that $Z^* \cap O \subseteq Z^* \cap \tilde{U}$. Then, for any $z \in M$ there exists $z_1 \in V$ with $\left(\begin{smallmatrix} z_1 \\ 0 \\ 0 \end{smallmatrix}\right) \in V_4$ such that

$$\varphi(z) = \varphi_1(\psi(z_1)).$$

Therefore,

$$\varphi(z) = \varphi_1\left(\varphi_2\left(\left(\begin{smallmatrix} z_1 \\ 0 \\ 0 \end{smallmatrix}\right)\right)\right) = \varphi\left(\left(\begin{smallmatrix} z_1 \\ 0 \\ 0 \end{smallmatrix}\right)\right)$$

and this gives $z = \left(\begin{smallmatrix} z_1 \\ 0 \\ 0 \end{smallmatrix}\right)$. On the other hand, for every such z we have $\varphi(z) = \varphi_1(\psi(z_1)) \in Z^* \cap O$ whenever z_1 is sufficiently small. It follows that there exists an open neighborhood $V_5 \subseteq V_4$ of $0 \in X$ such that

$$M = \operatorname{im} \begin{bmatrix} I_q \\ 0 \\ 0 \end{bmatrix} \cap V_5 \quad \text{and} \quad T_zM = \operatorname{im} \begin{bmatrix} I_q \\ 0 \\ 0 \end{bmatrix}$$

for all $z \in M$. Finally,

$$\begin{pmatrix} f_1(z) \\ f_2(z) \\ f_3(z) \end{pmatrix} \in \begin{bmatrix} I_q & 0 \\ 0 & I_m \\ 0 & 0 \end{bmatrix}$$

for all $z \in M$ and this gives

$$\forall z_1 \in \mathbb{R}^q : \begin{pmatrix} z_1 \\ 0 \end{pmatrix} \in V_5 \implies f_3\left(\begin{pmatrix} z_1 \\ 0 \end{pmatrix}\right) = 0.$$

Step 6: Let $\hat{U}$ be an open neighborhood of $0 \in X$ as given by Theorem II.4 (iii) and define $V_6 := \varphi^{-1}(\varphi(V_5) \cap \hat{U})$. Let $(z_1, z_3) \in \mathscr{C}^1(I; \mathbb{R}^q \times \mathbb{R}^{n_3})$, $I \subseteq \mathbb{R}$ an open interval, be a solution of (5) with $\left(\begin{smallmatrix} z_1(t) \\ 0 \\ z_3(t) \end{smallmatrix}\right) \in V_6$. Define

$$u(t) := E_{23}\left(\left(\begin{smallmatrix} z_1(t) \\ 0 \\ z_3(t) \end{smallmatrix}\right)\right) \dot{z}_3(t) - f_2\left(\left(\begin{smallmatrix} z_1(t) \\ 0 \\ z_3(t) \end{smallmatrix}\right)\right), \quad t \in I,$$

then $\left(\left(\begin{smallmatrix} z_1 \\ 0 \\ z_3 \end{smallmatrix}\right), u, y = 0\right)$ solves (3) and hence

$$\left(\varphi\left(\left(\begin{smallmatrix} z_1 \\ 0 \\ z_3 \end{smallmatrix}\right)\right), u, y = 0\right) \text{ solves (1) for all } t \in I.$$

Now, Theorem II.4 (iii) implies that

$$\varphi\left(\left(\begin{smallmatrix} z_1(t) \\ 0 \\ z_3(t) \end{smallmatrix}\right)\right) \in Z^* \cap \varphi(V_6) \quad \forall t \in I.$$

Without loss of generality we may assume that $Z^* \cap \varphi(V_6) \subseteq Z^* \cap O$ and hence $\left(\begin{smallmatrix} z_1(t) \\ 0 \\ z_3(t) \end{smallmatrix}\right) \in M$ for all $t \in I$ which implies $z_3 = 0$. Identifying $X_1 := V_6$, $X_2 := \varphi(V_6)$ finishes the proof of the theorem. □

Remark III.2.

(i) The zero dynamics form derived in Theorem III.1 is a generalization of those derived for linear DAEs in [12], [22]. For time-varying ODE systems, a zero dynamics form has been derived in [23]. The form (3) generalizes the Byrnes-Isidori form of ODE systems (see [9, Sec. 5]) since an existing vector relative degree is not required. However, if the system has some vector relative degree at $x = 0$, then the assumptions (i) and (ii) in Theorem III.1 are satisfied.

(ii) The name zero dynamics form for (3) may be justified by the fact that in this form the zero dynamics of the system are decoupled. With the notation from Theorem III.1, if $(x, u, y) \in \mathfrak{B}_{(1)}$ with $x(t) \in X_2$ for all $t \in \operatorname{dom} x$, then applying the local coordinate transformation $x(t) = \varphi(z(t))$ leads to the system (3) with a corresponding partitioning $z = (z_1^\top, y^\top, z_3^\top)^\top$. Invoking the last statement in Theorem III.1, we find that

$$\boxed{(x, u, y) \in \mathscr{Z}\mathscr{D}_{(1)} \iff \begin{cases} \dot{z}_1(t) = f_1(z_1(t), 0, 0), \\ u(t) = -f_2(z_1(t), 0, 0), \\ y(t) = 0, \; z_3(t) = 0. \end{cases}}$$

Therefore, z_1 is the solution of an ODE and it characterizes the "dynamics" within the zero dynamics and the input u is given by an algebraic equation depending on z_1.

(iii) The above property implies a local "zero output invertibility". That is, any two solution trajectories of (1) which generate a zero output and have the same initial value in a neighborhood of the origin must have the same input. More precisely, if $(x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathfrak{B}_{(1)}$ with $t_0 \in J := \operatorname{dom} x_1 \cap \operatorname{dom} x_2$, $x_1(t_0) = x_2(t_0) = x^0$ for some $x^0 \in X_2$ and additionally $y_1|_J = y_2|_J = 0$, then there exists $\varepsilon > 0$ such that $u_1|_{[t_0, t_0+\varepsilon)} = u_2|_{[t_0, t_0+\varepsilon)}$. Invertibility is discussed in more detail in Section IV.

(iv) The system (3) can be written in local coordinates as

$$\begin{aligned} \dot{z}_1(t) + E_{12}(z(t))\,\dot{y}(t) + E_{13}(z(t))\,\dot{z}_3(t) &= f_1(z(t)), \\ E_{22}(z(t))\,\dot{y}(t) + E_{23}(z(t))\,\dot{z}_3(t) &= f_2(z(t)) + u(t), \\ E_{32}(z(t))\,\dot{y}(t) + E_{33}(z(t))\,\dot{z}_3(t) &= f_3(z(t)), \end{aligned} \tag{6}$$

where $z(t) = (z_1(t)^\top, y(t)^\top, z_3(t)^\top)^\top$. From (6) we may obtain a realization of the inverse system of $[E, f, g, h]$ in the behavioral sense, i.e., where inputs and outputs have been interchanged (see e.g. [11]). To this end we introduce the variables $z_2(t) := y(t)$ and $z_4(t) := E_{22}(z(t))\dot{z}_2(t) + E_{23}(z(t))\dot{z}_3(t)$, where now $z(t) = (z_1(t)^\top, z_2(t)^\top, z_3(t)^\top)^\top$. The second equation in (6) can be solved explicitly for u, which is the output of the inverse system. The first and third equation in (6) together with the equations for the new variables determine the dynamics and constraints of the inverse system which reads

$$\begin{aligned} \dot{z}_1(t) + E_{12}(z(t))\,\dot{z}_2(t) + E_{13}(z(t))\,\dot{z}_3(t) &= f_1(z(t)), \\ E_{32}(z(t))\,\dot{z}_2(t) + E_{33}(z(t))\,\dot{z}_3(t) &= f_3(z(t)), \\ E_{22}(z(t))\,\dot{y}(t) + E_{23}(z(t))\,\dot{z}_3(t) &= z_4(t), \\ 0 &= -z_2(t) + y(t), \end{aligned}$$

with output equation

$$u(t) = -f_2(z(t)) + z_4(t).$$

In this sense the inverse system can be derived from the zero dynamics form. In contrast to classical approaches of system inversion for ODEs [24], [25], this procedure does not involve the application of the structure algorithm, but can be treated solely within the behavioral framework.

IV. System inversion

In this section we show that the zero dynamics form derived in Theorem III.1 allows to study invertibility properties of the system (1) by means of solution properties of a subsystem of the zero dynamics form.

The investigation of problems related to system inversion is known to be of high relevance since the important work by Brockett and Mesarović [26]. The basis for a systematic study of invertibility has been provided by Silverman [24] and Hirschorn [25], see also [27], [28]. We use the following definitions.

Definition IV.1 (System invertibility)**.** Let $[E, f, g, h] \in \Sigma^X_{l,n,m,p}$ and $x^0 \in X$. Then $[E, f, g, h]$ is called

(i) *left-invertible* at x^0, if

$$\forall (x_1,u_1,y_1),(x_2,u_2,y_2)\in\mathfrak{B}_{(1)}:\ \big[t_0\in J:=\operatorname{dom}x_1\cap\operatorname{dom}x_2 \\ \wedge\ x_1(t_0)=x_2(t_0)=x^0\ \wedge\ y_1|_J=y_2|_J\big] \\ \implies\ \exists\varepsilon>0:\ u_1|_{[t_0,t_0+\varepsilon)}=u_2|_{[t_0,t_0+\varepsilon)}.$$

(ii) *left-invertible* on an open set $U\subseteq X$, if it is left-invertible at every point $x^0\in U$.

(iii) *right-invertible* on an open set $O\subseteq\mathbb{R}^p$, if

$$\forall y\in\mathscr{C}^\infty(\mathbb{R};O)\ \exists I\subseteq\mathbb{R}\text{ open intvl. }\exists x\in\mathscr{C}^1(I;\mathbb{R}^n) \\ \exists u\in\mathscr{C}(I;\mathbb{R}^m):\ (x,u,y|_I)\text{ is a solution of (1).}$$

In the linear case, left-invertibility implies condition (1) in Theorem III.1 and right-invertibility implies condition (2), see [12]. For nonlinear DAEs this is not true in general, since it is possible to have rank drops at the origin which do not influence the invertibility properties.

Note that, whereas for ODE systems left-invertibility implies $m\le p$ and right-invertibility implies $p\le m$, this is not true for DAE systems in general. For example, the system

$$\dot{x}(t)=u_1(t),\quad 0=u_2(t),\quad y(t)=x(t)$$

is left-invertible (on $\mathbb{R}$), but satisfies $m=2>1=p$; the system

$$\dot{x}_1(t)=x_2(t)+u(t),\quad y_1(t)=x_1(t),\quad y_2(t)=x_2(t),$$

is right-invertible (on $\mathbb{R}^2$), but satisfies $m=1<2=p$. The reason for this is that the system may contain constrained input variables (u_2 in the first example) and free state variables appearing at the output (x_2 in the second example).

Under the assumptions of Theorem III.1, we may study left- and right-invertibility in terms of the zero dynamics form. We show that left-invertibility follows from a certain subsystem having unique solutions whenever a solution exists, and right-invertibility follows from existence of solutions for this subsystem.

Theorem IV.2 (Invertibility)**.** *Consider* $[E,f,g,h]\in\Sigma^X_{l,n,m,p}$ *and let the assumptions in Theorem III.1 be satisfied.*

(i) *If for all solutions* $w^i=(z_1^i,y^i,z_3^i)\in\mathscr{C}^1(I_i;(\mathbb{R}^q\times\mathbb{R}^p\times\mathbb{R}^{n_3})\cap X_1)$, $i=1,2$, *of the system*

$$\begin{aligned}\dot{z}_1(t)+E_{12}(z(t))\,\dot{y}(t)+E_{13}(z(t))\,\dot{z}_3(t)&=f_1(z(t)),\\ E_{32}(z(t))\,\dot{y}(t)+E_{33}(z(t))\,\dot{z}_3(t)&=f_3(z(t)),\end{aligned}\tag{7}$$

where $z(t)=(z_1(t)^\top,y(t)^\top,z_3(t)^\top)^\top$, *i.e.,* w^i *satisfies* (7) *for all* $t\in I_i$, *we have the implication*

$$t_0\in J:=I_1\cap I_2\ \wedge\ w^1(t_0)=w^2(t_0)\ \wedge\ y^1\Big|_J=y^2\Big|_J \\ \implies\ z_1^1\Big|_J=z_1^2\Big|_J\ \wedge\ z_3^1\Big|_J=z_3^2\Big|_J,$$

then $[E,f,g,h]$ *is left-invertible on* X_2.

(ii) *Let* $O\subseteq[0_{p\times q},I_p,0_{p\times n_3}]X_1$. *If for all* $y\in\mathscr{C}^\infty(\mathbb{R};O)$ *there exists an open interval* $I\subseteq\mathbb{R}$ *and* $(z_1,z_3)\in\mathscr{C}^1(I;\mathbb{R}^q\times\mathbb{R}^{n_3})$ *such that* $\begin{pmatrix}z_1(t)\\y(t)\\z_3(t)\end{pmatrix}\in X_1$ *and* (7) *holds for all* $t\in I$, *then* $[E,f,g,h]$ *is right-invertible on* O.

Proof. We show (i). Let $x^0\in X_2$ and let $(x_1,u_1,y_1),(x_2,u_2,y_2)\in\mathfrak{B}_{(1)}$ be such that $t_0\in J:=\operatorname{dom}x_1\cap\operatorname{dom}x_2$, $x_1(t_0)=x_2(t_0)=x^0$ and $y_1|_J=y_2|_J$. Choose an open interval $I\subseteq J$ with $t_0\in I$ such that $x_i(t)\in X_2$ for all $t\in I$. Define

$$\begin{pmatrix}z_1^i(t)\\y^i(t)\\z_3^i(t)\end{pmatrix}:=\varphi^{-1}(x_i(t)),\quad t\in I,$$

and observe that it solves (7) for all $t\in I$ by Theorem III.1. Therefore, the assumption gives that $z_1^1(t)=z_1^2(t)$ and $z_3^1(t)=z_3^2(t)$ for all $t\in I$ and hence

$$\begin{aligned}u_1(t)&=E_{22}(z_1^1(t),y^1(t),z_3^1(t))\,\dot{y}^1(t)\\&\quad+E_{23}(z_1^1(t),y^1(t),z_3^1(t))\,\dot{z}_3^1(t)-f_2(z_1^1(t),y^1(t),z_3^1(t))\\&=u_2(t)\end{aligned}$$

for $t\in I$ which proves the assertion.

We show (ii). Let $y\in\mathscr{C}^\infty(\mathbb{R};O)$. By the assumption we find an open interval $I\subseteq\mathbb{R}$ and $(z_1,z_3)\in\mathscr{C}^1(I;\mathbb{R}^q\times\mathbb{R}^{n_3})$ such that $\begin{pmatrix}z_1(t)\\y(t)\\z_3(t)\end{pmatrix}\in X_1$ and (7) holds for all $t\in I$. Defining

$$\begin{aligned}u(t)&=E_{22}(z_1(t),y(t),z_3(t))\,\dot{y}(t)\\&\quad+E_{23}(z_1(t),y(t),z_3(t))\,\dot{z}_3(t)-f_2(z_1(t),y(t),z_3(t))\end{aligned}$$

for $t\in I$ and

$$x(t)=\varphi\left(\begin{pmatrix}z_1(t)\\y(t)\\z_3(t)\end{pmatrix}\right),\quad t\in I,$$

leads to a trajectory $(x,u,y|_I)$ which solves (1) for all $t\in I$ by Theorem III.1. □

The conditions in Theorem IV.2 represent solution properties of the DAE subsystem (7) from the zero dynamics form which are dual in some sense. The output y is considered as an input to this system. For left-invertibility, we require that the DAE (7) has a unique solution (z_1,z_3) for any given y whenever such a solution exists. For right-invertibility, we require that the DAE (7) has a solution (z_1,z_3) for any given y. In this sense, left- and right-invertibility correspond to the "dual" properties of uniqueness and existence of solutions to (7), resp.

V. Example

We illustrate the results of the paper by an academic example. Consider the nonlinear DAE system

$$\begin{aligned}x_2(t)\dot{x}_1(t)+x_1(t)\dot{x}_2(t)&=u_1(t)-u_2(t),\\ 0&=u_1(t)+u_2(t),\\ \dot{x}_1(t)+\dot{x}_3(t)&=x_2(t)^2-x_1(t)^3+x_3(t),\\ y_1(t)&=x_1(t)^3-x_3(t),\\ y_2(t)&=x_2(t).\end{aligned}\tag{8}$$

We calculate Z^* using the zero dynamics algorithm in Theorem II.4:

$$M_0=h^{-1}(0)=\left\{\ x\in\mathbb{R}^3\ \middle|\ x_2=0,\ x_3=x_1^3\ \right\},\\ M_1=M_0=:Z^*.$$

The tangent space of Z^* at some $x\in Z^*$ is given by

$$T_xZ^*=\operatorname{im}\begin{bmatrix}1\\0\\3x_1^2\end{bmatrix}$$

and hence

$$E(0)T_0Z^*+\operatorname{im}g(0)=\operatorname{im}\begin{bmatrix}0&1&-1\\0&1&1\\1&0&0\end{bmatrix}=\mathbb{R}^3,$$

by which condition (1) in Theorem III.1 is satisfied. Clearly, condition (2) is satisfied as well. In order to compute the zero dynamics form (3) we need to find a diffeomorphism $\varphi:\mathbb{R}^3\to\mathbb{R}^3$ such that

$h(\varphi(z)) = \binom{z_2}{z_3}$. This leads to the choice

$$\varphi(z) = \varphi(z_1, z_2, z_3) = \begin{pmatrix} z_1 \\ z_3 \\ z_1^3 - z_2 \end{pmatrix},$$

which satisfies

$$\varphi'(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3z_1^2 & -1 & 0 \end{bmatrix}.$$

Then we obtain

$$E(\varphi(z))\, \varphi'(z) = \begin{bmatrix} z_3 & 0 & z_1 \\ 0 & 0 & 0 \\ 1 + 3z_1^2 & -1 & 0 \end{bmatrix}$$

and hence we need to find $S(z)$ such that

$$S(z) \begin{bmatrix} z_3 & 1 & -1 \\ 0 & 1 & 1 \\ 1 + 3z_1^2 & 0 & 0 \end{bmatrix} = I_3,$$

which gives

$$S(z) = \begin{bmatrix} 0 & 0 & \frac{1}{1+3z_1^2} \\ \frac{1}{2} & \frac{1}{2} & \frac{-z_3}{2+6z_1^2} \\ -\frac{1}{2} & \frac{1}{2} & \frac{z_3}{2+6z_1^2} \end{bmatrix}$$

and indeed $S : \mathbb{R}^3 \to \mathbb{R}^{3\times 3}$ is smooth. With this, the zero dynamics form (3) is now given by

$$\tilde{E}(z) = \begin{bmatrix} 1 & \frac{-1}{1+3z_1^2} & 0 \\ 0 & \frac{z_3}{2+6z_1^2} & \frac{z_1}{2} \\ 0 & \frac{-z_3}{2+6z_1^2} & -\frac{z_1}{2} \end{bmatrix}, \qquad \tilde{f}(z) = \begin{pmatrix} \frac{z_3^2 - z_2}{1+3z_1^2} \\ \frac{z_3(z_3^2 - z_2)}{2+6z_1^2} \\ \frac{-z_3(z_3^2 - z_2)}{2+6z_1^2} \end{pmatrix}$$

$$\tilde{g}(z) = \begin{bmatrix} 0 \\ I_2 \end{bmatrix}, \qquad \tilde{h}(z) = [0, I_2]\, z.$$

Using this form, the system (7) reads

$$\dot{z}_1(t) = \frac{\dot{y}_1(t) + y_2(t)^2 - y_1(t)}{1 + 3z_1^2(t)}$$

and obviously has unique solutions for all y_1 and y_2, i.e., the conditions in Theorem IV.2 (i) and (ii) are satisfied, which implies that system (8) is left-invertible (on $\mathbb{R}^3$) and right-invertible (on $\mathbb{R}^2$).

VI. Conclusion

In the present paper, we have derived the zero dynamics form for nonlinear DAE systems, which is a local input-output normal form. The zero dynamics form generalizes the Byrnes-Isidori form for nonlinear ODE systems, since existence of a vector relative degree is not required; only mild assumptions on the maximal output zeroing submanifold are needed for the derivation. In the zero dynamics form the zero dynamics are decoupled. Furthermore, left- and right-invertibility of the system can be studied in terms of the solution properties of a certain subsystem. These results are the basis for the investigation of various classical control problems for nonlinear DAEs, such as output regulation and trajectory tracking.

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