A new bound for the distance to singularity of a regular matrix pencil

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For regular matrix pencils $\mathcal{A}(s) = sE - A$ the distance to the nearest singular pencil in the Frobenius norm of the coefficients is called the distance to singularity. We derive a new lower bound for this distance by using the spectral theory of tridiagonal Toeplitz matrices.

Introduction. A matrix pencil $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n\times n}$ is called *regular* if $\det \mathcal{A}(s)$ is not the zero polynomial, otherwise it is called *singular*. In the numerical treatment of matrix pencils it turns out that regular pencils which are close to singular pencils are difficult to handle [5]. In general, it is not even possible to compute canonical forms because rank decisions turn out to be impossible. This is an important issue in numerous applications, we mention only the solution theory of linear time-invariant differential-algebraic equations $\frac{d}{dt}Ex(t) = Ax(t)$, $t \in [0,\infty)$, $x(0) = x_0$. Here a unique solution exists for every consistent initial value if and only if the associated matrix pencil $\mathcal{A}(s) = sE - A$ is regular.

The distance to singularity. For a regular matrix pencil $sE - A$ the *distance to singularity* $\delta(E, A)$ was introduced in [2] as the Frobenius norm of the smallest perturbation that leads to a singular pencil

$$\delta(E,A) := \inf\left\{ \|[\Delta E, \Delta A]\|_F \;:\; \Delta E, \Delta A \in \mathbb{C}^{n\times n} \text{ are such that } s(E+\Delta E) - (A+\Delta A) \text{ is singular} \right\}, \quad (1)$$

where $\|M\|_F := \sqrt{\operatorname{tr}(M^*M)}$ is the Frobenius norm of a matrix $M \in \mathbb{C}^{m\times n}$, and M^* is the adjoint of M.

Recently, in [7] the number $\delta(E, A)$ was computed in the case that the perturbation $s\Delta E - \Delta A$ in (1) has rank one. In [2] upper and lower bounds for $\delta(E, A)$ were obtained, we mention here only

$$\frac{\sigma_{\min}(W_n(E,A))}{\sqrt{n}} \le \delta(E,A) \le \min\left\{ \sigma_{\min}\left(\begin{bmatrix} E \\ A \end{bmatrix}\right), \sigma_{\min}([E, A]) \right\}, \quad (2)$$

where $\sigma_{\min}(M)$ is the smallest singular value of a matrix $M \in \mathbb{C}^{m\times n}$ and for $k = 1, \ldots, n$, $W_k(E, A)$ is the bi-diagonal block matrix (see [3])

$$W_k(E,A) := \begin{bmatrix} E & & & \\ A & E & & \\ & \ddots & \ddots & \\ & & A & E \\ & & & A \end{bmatrix} \in \mathbb{C}^{(k+1)n\times kn}. \quad (3)$$

In what follows, we improve the lower bound in (2). The following characterization of the regularity of $\mathcal{A}(s)$ can be found in [3], see also [6, Thm. 3.1].

Theorem 1 *The matrix pencil $\mathcal{A}(s) = sE - A$ with $E, A \in \mathbb{C}^{n\times n}$ is regular if and only if* $\ker W_n(E, A) = \{0\}$.

We derive an upper bound for the spectral norm of $W_k(E, A)$ in (3) which is given by $\|M\| := \max_{\|x\|=1} \|Mx\|$ for $M \in \mathbb{C}^{m\times n}$.

Lemma 2 *For all $E, A, \in \mathbb{C}^{n\times n}$ and $k \ge 1$ we have* $\|W_k(E,A)\| \le \sqrt{1+\cos\left(\frac{\pi}{k+1}\right)}\sqrt{\|E\|^2+\|A\|^2}$.

P r o o f. Let $x_1, \ldots, x_k \in \mathbb{C}^n$ and set $x = (x_1^\top, \ldots, x_k^\top)^\top \in \mathbb{C}^{kn}$ with $\|x\| = 1$. We abbreviate $Z := E^*E + A^*A$ and

$$\|W_k(E,A)^*W_k(E,A)x\|^2 = \left\| \begin{bmatrix} Z & A^*E & & \\ E^*A & Z & \ddots & \\ & \ddots & \ddots & A^*E \\ & & E^*A & Z \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ \\ x_k \end{bmatrix} \right\|^2$$

$$= \|Zx_1 + A^*Ex_2\|^2 + \sum_{i=2}^{k-1} \|E^*Ax_{i-1} + Zx_i + A^*Ex_{i+1}\|^2 + \|E^*Ax_{k-1} + Zx_k\|^2$$

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** Supported by the NCN (National Science Center), decision No. DEC-2013/11/B/ST1/03613

$$\leq (\|Z\|\|x_1\| + \|A^*E\|\|x_2\|)^2 + \sum_{i=2}^{k-1}(\|E^*A\|\|x_{i-1}\| + \|Z\|\|x_i\| + \|A^*E\|\|x_{i+1}\|)^2 + (\|E^*A\|\|x_{k-1}\| + \|Z\|\|x_k\|)^2$$

$$= \left\| \underbrace{\begin{bmatrix} \|Z\| & \|A^*E\| & & \\ \|E^*A\| & \|Z\| & \ddots & \\ & \ddots & \ddots & \|A^*E\| \\ & & \|E^*A\| & \|Z\| \end{bmatrix}}_{=:M} \begin{bmatrix} \|x_1\| \\ \vdots \\ \vdots \\ \|x_k\| \end{bmatrix} \right\|^2 .$$

Since $\|x\| = 1$ implies $\|(\|x_1\|, \dots, \|x_k\|)\| = 1$ we see that

$$\|W_k(E, A)\|^2 = \|W_k(E, A)^* W_k(E, A)\| \leq \sigma_{\max}(M). \tag{4}$$

As $\|A^*E\| = \|E^*A\|$, M is a symmetric tridiagonal Toeplitz matrix and by [1, Theorem 2.4] its eigenvalues are given by

$$\lambda_j = \|Z\| + 2\|E^*A\| \cos\left(\frac{j\pi}{k+1}\right), \quad j = 1, \dots, k$$

and therefore

$$\sigma_{\max}(M) = \|Z\| + 2\|E^*A\| \cos\left(\frac{\pi}{k+1}\right) \leq \|Z\| + 2\|E\|\|A\| \cos\left(\frac{\pi}{k+1}\right) \leq \left(1 + \cos\left(\frac{\pi}{k+1}\right)\right)\left(\|E\|^2 + \|A\|^2\right).$$

Together with (4) this completes the proof. □

A new lower bound. The next theorem contains an improvement of the lower bound in (2).

Theorem 3 *Let $\mathcal{A}(s) = sE - A$ be a regular matrix pencil with $E, A \in \mathbb{C}^{n\times n}$. Then*

$$\frac{\sigma_{\min}(W_n(E, A))}{\sqrt{1 + \cos\left(\frac{\pi}{n+1}\right)}} \leq \delta(E, A). \tag{5}$$

P r o o f. Assume that $\widetilde{A}(s) = s\widetilde{E} - \widetilde{A}$ satisfies $\|[E - \widetilde{E}, A - \widetilde{A}]\|_F < \sigma_{\min}(W_n(E, A))\sqrt{1 + \cos\left(\frac{\pi}{n+1}\right)}^{-1}$. Using the norm inequality $\|\cdot\| \leq \|\cdot\|_F$ (cf. [4, Section 2.3.2]), a simple calculation yields

$$\sqrt{\|E - \widetilde{E}\|^2 + \|A - \widetilde{A}\|^2} \leq \sqrt{\|E - \widetilde{E}\|_F^2 + \|A - \widetilde{A}\|_F^2} = \|[E - \widetilde{E}, A - \widetilde{A}]\|_F$$

and by Lemma 2 we find that

$$\|W_n(E, A) - W_n(\widetilde{E}, \widetilde{A})\| \leq \sqrt{1 + \cos\left(\frac{\pi}{n+1}\right)}\sqrt{\|E - \widetilde{E}\|^2 + \|A - \widetilde{A}\|^2} < \sigma_{\min}(W_n(E, A)).$$

From [4, Thm. 2.5.3] we conclude $\sigma_{\min}(W_n(\widetilde{E}, \widetilde{A})) > 0$ and thus $\ker W_n(\widetilde{E}, \widetilde{A}) = \{0\}$. Now Theorem 1 shows that $\widetilde{\mathcal{A}}(s)$ is regular. □

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