Thomas Berger

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Zusammenfassung

In der vorliegenden Dissertation werden differential-algebraische Gleichungen (differential-algebraic equations, DAEs) der Form $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + f$ betrachtet, wobei E und A beliebige Matrizen sind. Falls E nichtverschwindende Einträge hat, dann kommen in der Gleichung Ableitungen der entsprechenden Komponenten von x vor. Falls E eine Nullzeile hat, dann kommen in der entsprechenden Gleichung keine Ableitungen vor und sie ist rein algebraisch. Daher werden Gleichungen vom Typ $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + f$ differential-algebraische Gleichungen genannt.

Ein Ziel dieser Dissertation ist es, eine strukturelle Zerlegung einer DAE in vier Teile herzuleiten: einen ODE-Anteil, einen nilpotenten Anteil, einen unterbestimmten Anteil und einen überbestimmten Anteil. Jeder Anteil beschreibt ein anderes Lösungsverhalten in Hinblick auf Existenz und Eindeutigkeit von Lösungen für eine vorgegebene Inhomogenität f und Konsistenzbedingungen an f. Die Zerlegung, namentlich die quasi-Kronecker Form (QKF), verallgemeinert die wohlbekannte Kronecker-Normalform und behebt einige ihrer Nachteile.

Die QKF wird ausgenutzt, um verschiedene Konzepte der Kontrollierbarkeit und Stabilisierbarkeit für DAEs mit $f = Bu$ zu studieren. Hier bezeichnet u den Eingang des differential-algebraischen Systems. Es werden Zerlegungen unter System- und Feedback-Äquivalenz, sowie die Folgen einer Behavioral-Steuerung $K_x x + K_u u = 0$ für die Stabilisierung des Systems untersucht.

Falls für das DAE-System zusätzlich eine Ausgangs-Gleichung $y = Cx$ gegeben ist, dann lässt sich das Konzept der Nulldynamik wie folgt definieren: die Nulldynamik ist, grob gesagt, die Dynamik, die am Ausgang nicht sichtbar ist, d.h. die Menge aller Lösungs-Trajektorien (x, u, y) mit $y = 0$. Für rechts-invertierbare Systeme mit autonomer Nulldynamik wird eine Zerlegung hergeleitet, welche die Nulldynamik entkoppelt. Diese versetzt uns in die Lage, eine Behavior-Steuerung zu entwickeln, die das System stabilisiert, vorausgesetzt die Nulldynamik selbst ist stabil.

Wir betrachten auch zwei Regelungs-Strategien, die von den Eigenschaften der oben genannten System-Klasse profitieren: Hochverstärkungs- und Funnel-Regelung. Ein System $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu$, $y = Cx$, hat die Hochverstärkungseigenschaft, wenn es durch die Anwen-

dung der proportionalen Ausgangsrückführung $u = -ky$, mit $k > 0$ hinreichend groß, stabilisiert werden kann. Wir beweisen, dass rechtsinvertierbare Systeme mit asymptotisch stabiler Nulldynamik, die eine bestimmte Relativgrad-Annahme erfüllen, die Hochverstärkungseigenschaft haben. Während der Hochverstärkungs-Regler recht einfach ist, ist es jedoch a priori nicht bekannt, wie groß die Verstärkungskonstante k gewählt werden muss. Dieses Problem wird durch den Funnel-Regler gelöst: durch die adaptive Justierung der Verstärkung über eine zeitabhängige Funktion $k(\cdot)$ und die Ausnutzung der Hochverstärkungseigenschaft wird erreicht, dass große Werte $k(t)$ nur dann angenommen werden, wenn sie nötig sind. Eine weitere wesentliche Eigenschaft ist, dass der Funnel-Regler das transiente Verhalten des Fehlers $e = y - y_{\rm ref}$ der Bahnverfolgung, wobei $y_{\rm ref}$ die Referenztrajektorie ist, beachtet. Für einen vordefinierten Performanz-Trichter (funnel) ψ wird erreicht, dass $\|e(t)\| < \psi(t)$.

Schließlich wird der Funnel-Regler auf die Klasse von MNA-Modellen von passiven elektrischen Schaltkreisen mit asymptotisch stabilen invarianten Nullstellen angewendet. Dies erfordert die Einschränkung der Menge der zulässigen Referenztrajektorien auf solche die, in gewisser Weise, die Kirchhoffschen Gesetze punktweise erfüllen.

Abstract

In the present thesis we consider differential-algebraic equations (DAEs) of the form $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + f$, where E and A are arbitrary matrices. If E has nonzero entries, then derivatives of the respective components of x are involved in the equation. If E has a zero row, then the respective equation involves no derivatives and is purely algebraic. This justifies to call $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + f$ a differential-algebraic equation.

One aim of the thesis is to derive a structural decomposition of the DAE into four parts: the ODE part, nilpotent part, underdetermined part and overdetermined part. Each part describes a different solution behavior regarding existence and uniqueness of solutions for given inhomogeneities f and consistency conditions on f. The decomposition, namely the quasi-Kronecker form (QKF), generalizes the well-known Kronecker canonical form and resolves some of its disadvantages.

The QKF is exploited to study the different controllability and stabilizability concepts for DAEs with $f = Bu$. Here u denotes the input of the differential-algebraic system. Decompositions under system and feedback equivalence and the consequence of behavioral control $K_x x + K_u u = 0$ for the stabilization of the system is investigated.

If the DAE system is accompanied by an output equation $y = Cx$, we may define the concept of zero dynamics: roughly speaking, the zero dynamics are those dynamics which are not visible at the output, i.e., the set of all solution trajectories (x, u, y) with $y = 0$. For right-invertible systems with autonomous zero dynamics a decomposition is derived, which decouples the zero dynamics of the system and enables us to derive a behavioral control which stabilizes the system, provided that the zero dynamics are stable as well.

We will also consider two control strategies which benefit from the properties of the above mentioned system class: high-gain and funnel control. We say that a system $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + Bu$, $y = Cx$, has the high-gain property if it is stabilizable by the application of the feedback $u = -ky$ for sufficiently large $k > 0$. It is proved that right-invertible systems with asymptotically stable zero dynamics which satisfy a certain relative degree assumption possess the high-gain property. While the high-gain controller is quite simple, it is, however, not known a priori how large the gain constant k must be chosen. This problem is resolved by the funnel controller: by adaptively adjusting the gain via

a time-varying function $k(\cdot)$ and exploiting the high-gain property, it is achieved that high values of $k(t)$ are used only when required. Moreover, and most important, the funnel controller takes the transient behavior of the tracking error $e = y - y_{\text{ref}}$, where y_{ref} is a reference signal, into account. For a prespecified performance funnel ψ it can be guaranteed that $\|e(t)\| < \psi(t)$.

Finally, the funnel controller is applied to the class of MNA models of passive electrical circuits with asymptotically stable invariant zeros. This requires to restrict the set of allowable reference trajectories such that any trajectory, in a sense, satisfies Kirchhoff's laws pointwise.

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Contents

1 Introduction

About fifty years ago, Gantmacher published his famous book [100] and therewith laid the foundations for the rediscovery of differential-algebraic equations (DAEs), the first main theories of which were developed by Weierstrass [243] and Kronecker [149] in terms of matrix pencils. DAEs have then been discovered to be an appropriate tool for modeling many problems in economics [173], demography [62], mechanical systems [8, 46, 99, 111, 197], multibody dynamics [90,111,216,221], electrical networks [8,30,61,89,168,183,207], fluid mechanics [8, 106, 168] and chemical engineering [79, 84, 85, 194], which often cannot be modeled by standard ordinary differential equations (ODEs). These problems indeed have in common that the dynamics are algebraically constrained, for instance by tracks, Kirchhoff laws, or conservation laws. As a result of the power in application, DAEs are nowadays an established field in applied mathematics and subject of various monographs [46, 62, 63, 80, 107] and textbooks [152, 154].

In general, DAEs are implicit differential equations, and in the simplest case just a combination of differential equations along with algebraic constraints (from which the name DAE comes from). These algebraic constraints however may cause that the solutions of initial value problems are no longer unique, or that there do not exist solutions at all. Furthermore, when considering inhomogeneous problems, the inhomogeneity has to be 'consistent' with the DAE in order for solutions to exist. Dealing with these problems, a broad solution theory for DAEs has been developed, starting with the pioneer contribution by Wilkinson [244]. Nowadays, the whole theory can be looked up in the aforementioned textbooks and monographs; for a comprehensive representation see the survey [229]. A good overview of DAE theory and a historical background can also be found in [158].

A lot of the aforementioned contributions deal with regular DAEs, i.e., equations of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + f(t), \quad x(0) = x^0,$$

where for any continuous inhomogeneity f there exist initial values x^0 for which the corresponding initial value problem has a solution (i.e., a differentiable function x satisfying the equation for all $t \in \mathbb{R}$) and this solution is unique. This has been proved to be equivalent to the condition that E, A are square matrices and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.

The aim of the present thesis is to present a control theory for differential-algebraic systems. Most results are on nonregular systems, in particular E and A may be rectangular. Applications with the need for nonregular DAEs appear in the modeling of electrical circuits [89, 207] for instance. Furthermore, systems arising from modern automatic modeling tools are usually nonregular DAEs. We also like to stress that general, possibly nonregular, DAE systems are a sub-class of the class of so-called differential behaviors, introduced by Willems in [245], see also [198, 246] and the survey [248]. In the present thesis we will pay a special attention to the behavioral setting, formulating most of the results and the concepts by using the underlying set of trajectories (behavior) of the system.

The guiding research idea is funnel control. In Chapters 2–4 we develop a theory which is also the basis for the application of the funnel controller to nonregular DAE systems in Chapter 5 and to passive electrical circuits in Chapter 6. For the application of funnel control it is necessary that the inputs and outputs of the DAE system (see (1.3.1)) are fixed a priori by the designer in order to establish the control law. This is different from other approaches based on the behavioral setting, see [65], where only the free variables in the system are viewed as inputs; this may require a reinterpretation of states as inputs and of inputs as states. In the present thesis we will assume that such a reinterpretation of variables has already been done or is not feasible, and the given DAE system is fix.

1.1 The Weierstraß and Kronecker canonical form

Weierstrass [243] and Kronecker [149] have independently developed the fundamental decompositions of regular and nonregular matrix pencils, resp., which are nowadays the basis for nearly any theory on time-invariant DAEs. Let us consider regular matrix pencils $sE - A \in \mathbb{R}[s]^{n\times n}$ first, i.e., $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$. We may then observe that via a suitable choice of new variables and appropriate manipulations of the equations we may equivalently express the system

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) \qquad \text{as} \qquad \begin{array}{rcl} \tfrac{\mathrm{d}}{\mathrm{d}t}x_1(t) &=& Jx_1(t) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}Nx_2(t) &=& x_2(t), \end{array}$$

where N is a nilpotent matrix. Here, x_1 are called the differential variables and x_2 the algebraic variables; the latter notion arising from the fact that the equation $\frac{\mathrm{d}}{\mathrm{d}t}Nx_2(t) = x_2(t)$ has only the trivial solution (this equation gets interesting when inhomogeneities are involved). Weierstrass observed that this transformation can be obtained by only applying an equivalence transformation to the matrix pencil $sE - A$, that is there exist $S, T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$S(sE - A)T = \begin{bmatrix} sI - J & 0 \\ 0 & sN - I \end{bmatrix},$$

and J, N are in Jordan canonical form and N is nilpotent. This form is called the Weierstraß canonical form (WCF).

If $sE - A \in \mathbb{R}[s]^{m\times n}$ is an arbitrary matrix pencil, then, compared to the ODE and nilpotent part in the WCF, two additional parts arise in its Kronecker canonical form (KCF): an underdetermined part and an overdetermined part. They consist of blocks of the type

$$sK_i - L_i \qquad \text{or} \qquad sK_i^\top - L_i^\top, \quad \text{resp.,}$$

where

$$K_i = \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix}, \quad L_i = \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(i-1)\times i}, \quad i \in \mathbb{N}.$$

Clearly, in equations of the type $\left(\frac{\mathrm{d}}{\mathrm{d}t}K_i - L_i\right)x(t) = f(t)$ the component x_i can be chosen freely and solutions exist for any $x(0) \in \mathbb{R}^i$ and for any $f \in \mathcal{C}(\mathbb{R}; \mathbb{R}^i)$, but it is far from being unique; in equations of the type $\left(\frac{\mathrm{d}}{\mathrm{d}t}K_i^\top - L_i^\top\right)x(t) = f(t)$ a solution x does only exist if the initial value $x(0)$ and the inhomogeneity f satisfy a certain consistency condition, but then the solution is unique - we omit the details here.

A major drawback of the WCF and the KCF is that, due to possible complex eigenvalues of the matrix J, the Jordan form and accompanying transformation matrices may be complex-valued. Hence, even in the case of a real-valued matrix pencil $sE - A$, its WCF or KCF are in general complex-valued. Furthermore, it is often not necessary that J and N are in Jordan form, but the knowledge that the pencil $sE - A$ can be decomposed into 2 (4, resp.) parts of the following types suffices: ODE part, nilpotent part, (underdetermined part, overdetermined part). These blocks can be described via simple rank conditions. This leads to the quasi-Weierstraß form (QWF) and the quasi-Kronecker form (QKF) discussed in Chapter 2.

1.2 Controllability

Controllability is, roughly speaking, the property of a system that any two trajectories can be concatenated by another admissible trajectory. The precise concept however depends on the specific framework, as quite a number of different concepts of controllability are present today.

Since the famous work by Kalman [137–139], who introduced the notion of controllability about fifty years ago, the field of mathematical control theory has been revived and rapidly growing ever since, emerging into an important area in applied mathematics, mainly due to its contributions to fields such as mechanical, electrical and chemical engineering (see e.g. [2, 78, 232]). For a good overview of standard mathematical control theory, i.e., involving ODEs, and its history see e.g. [115, 131, 132, 136, 213, 223].

DAEs found its way into control theory ever since the famous book by Rosenbrock [210], in which he developed his ideas of the description of linear systems by polynomial system matrices. Then a rapid development followed with important contributions of Rosenbrock himself [211] and Luenberger [169–172], not to forget the work by Pugh et al. [201], Verghese et al. [237, 239–241], Pandolfi [192, 193], Cobb [73, 74, 76, 77], Yip et al. [254] and Bernard [42]. Pioneer contributions for the development of the concepts of controllability are certainly [77, 241, 254]. Further developments were made by Lewis and Özçaldiran [161, 162] and by Bender and Laub [21, 22]. The first monograph which summarizes the development of control theory for

DAEs so far was the one by Dai [80].

All of the above mentioned contributions deal with regular systems. However, a major drawback in the consideration of regular systems arises when it comes to feedback: the class of regular DAE systems is not closed under the action of a feedback group [12]. This also justifies the investigation of nonregular DAE systems. In Chapter 3, we introduce and investigate the different controllability concepts for DAEs (which are not consistently treated in the literature; we clarify this) as well as feedback and important system decompositions.

1.3 Zero dynamics

Consider a differential-algebraic input-output system of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t)\,, \tag{1.3.1}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$ and the functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. The zero dynamics of the system is the set of those trajectories $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ which solve (1.3.1) and satisfy $y = 0$, i.e., the zero dynamics are, loosely speaking, the vector space of those dynamics of the system which are not visible at the output. The concept of zero dynamics has been introduced by Byrnes and Isidori [54]. For ODEs, exploiting the zero dynamics proved fruitful in a lot of control theoretic topics such as output regulation [56, 58], stabilization [131, Sec. 7.1], adaptive control [215] and distributed parameter systems [59]. For DAEs, the zero dynamics have been used in [28, 34, 35] to prove high-gain stabilizability and feasibility of funnel control.

The concepts of autonomous and asymptotically stable zero dynamics are introduced and investigated in Chapter 4. Particular emphasis is placed on algebraic and geometric characterizations (via invariant subspaces) and system decomposition such as the zero dynamics form. The transmission zeros of the system are related to asymptotic stability of the zero dynamics. Furthermore, system inversion is studied for systems with autonomous zero dynamics.

1.4 High-gain and funnel control

Consider a system (1.3.1) with the same number of inputs and outputs. A classical control strategy in order to achieve stabilization is constant high-gain output feedback, that is the application of the controller

$$u(t) = -k\, y(t) \tag{1.4.1}$$

to the system (1.3.1). Stabilization is achieved if any solution x of the closed-loop system

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = (A - kBC)x(t)$$

satisfies $\lim_{t\to\infty} x(t) = 0$. We show in Chapter 5 that stabilization can be achieved for right-invertible systems with asymptotically stable zero dynamics which satisfy a certain relative degree assumption (the matrix Γ in (5.1.2) exists and satisfies $\Gamma = \Gamma^\top \geq 0$). Regular systems with arbitrary known positive strict relative degree are also treated, but a derivative output feedback has to be used in this case. A drawback of high-gain control is that it is not known a priori how large the high-gain constant must be.

Another strategy is the 'classical' adaptive high-gain controller

$$u(t) = -k(t)\, y(t), \quad \dot{k}(t) = \|y(t)\|^2, \quad k(0) = k^0, \tag{1.4.2}$$

which resolves the above mentioned problem by adaptively increasing the high gain. The drawback of the control strategy (1.4.2) is that, albeit k is bounded, it is monotonically increasing and potentially so large that the input is very sensitive to noise corrupting the output measurement. Further drawbacks are that (1.4.2) does not tolerate mild output perturbations, tracking would require an internal model and, most importantly, transient behaviour is not taken into account. These issues are discussed for ODE systems (with strictly proper transfer function of strict relative degree one and asymptotically stable zero dynamics) in the survey [123].

To overcome these drawbacks, we introduce, for $\hat{k} > 0$, the funnel controller (introduced by Ilchmann, Ryan and Sangwin [125] for ODEs; see also [123] and the references therein)

$$u(t) = -k(t)\, y(t), \qquad k(t) = \frac{\hat{k}}{1 - \varphi(t)^2 \|y(t)\|^2}\,, \tag{1.4.3}$$

where $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is a suitable function. The control objective is that the application of the funnel controller (1.4.3), which is a proportional nonlinear time-varying high-gain output feedback, to (1.3.1) yields a closed-loop system where the output evolves within the funnel, i.e., $\|y(t)\| < \varphi(t)^{-1}$ for all $t > 0$. In particular, this prescribes the transient behavior of the output. The intuition of the control law (1.4.3) is as follows: if $\|y(t)\|$ gets close to the funnel boundary $\varphi(t)^{-1}$, then k increases and the high-gain property of the system is exploited and forces $\|y(t)\|$ away from the funnel boundary; k decreases if a high gain is not necessary. The control design (1.4.3) has two advantages: k is nonmonotone and (1.4.3) is a static time-varying proportional output feedback of striking simplicity.

In Chapter 5 we consider the output regulation problem by funnel control: given any reference signal y_{ref}, the funnel controller achieves tracking of a reference signal by the output signal within a prespecified performance funnel. We show that funnel control is feasible for all systems which have the high-gain property: right-invertible DAE systems with asymptotically stable zero dynamics for which the matrix Γ in (5.1.2) exists and satisfies $\Gamma = \Gamma^\top \geq 0$ (this class includes regular systems with strict relative degree one or proper inverse transfer function). Regular systems with arbitrary known positive strict relative degree are also treated, where the funnel controller is combined with a filter; this filter 'adjusts' the higher relative degree.

1.5 Electrical circuits

Consider a system (1.3.1) with $l = n$ and $p = m$, which arises from modified nodal analysis (MNA) models of electrical circuits, i.e.,

$$sE - A = \begin{bmatrix} s\mathrm{A}_\mathcal{C}\mathcal{C}\mathrm{A}_\mathcal{C}^\top + \mathrm{A}_\mathcal{R}\mathcal{G}\mathrm{A}_\mathcal{R}^\top & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{V} \\ -\mathrm{A}_\mathcal{L}^\top & s\mathcal{L} & 0 \\ -\mathrm{A}_\mathcal{V}^\top & 0 & 0 \end{bmatrix}, \quad B = C^\top = \begin{bmatrix} -\mathrm{A}_\mathcal{I} & 0 \\ 0 & 0 \\ 0 & -I_{n_\mathcal{V}} \end{bmatrix},$$

$$x = (\eta^\top, i_\mathcal{L}^\top, i_\mathcal{V}^\top)^\top, \quad u = (i_\mathcal{I}^\top, v_\mathcal{V}^\top)^\top, \quad y = (-v_\mathcal{I}^\top, -i_\mathcal{V}^\top)^\top,$$

where

$$\mathcal{C} \in \mathbb{R}^{n_\mathcal{C} \times n_\mathcal{C}}, \mathcal{G} \in \mathbb{R}^{n_\mathcal{G} \times n_\mathcal{G}}, \mathcal{L} \in \mathbb{R}^{n_\mathcal{L} \times n_\mathcal{L}},$$
$$\mathrm{A}_\mathcal{C} \in \mathbb{R}^{n_e \times n_\mathcal{C}}, \mathrm{A}_\mathcal{R} \in \mathbb{R}^{n_e \times n_\mathcal{G}}, \mathrm{A}_\mathcal{L} \in \mathbb{R}^{n_e \times n_\mathcal{L}}, \mathrm{A}_\mathcal{V} \in \mathbb{R}^{n_e \times n_\mathcal{V}}, \mathrm{A}_\mathcal{I} \in \mathbb{R}^{n_e \times n_\mathcal{I}},$$
$$n = n_e + n_\mathcal{L} + n_\mathcal{V}, \quad m = n_\mathcal{I} + n_\mathcal{V}.$$

Here $\mathrm{A}_\mathcal{C}$, $\mathrm{A}_\mathcal{R}$, $\mathrm{A}_\mathcal{L}$, $\mathrm{A}_\mathcal{V}$ and $\mathrm{A}_\mathcal{I}$ denote the element-related incidence matrices, $\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the consecutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials, $i_\mathcal{L}(t)$, $i_\mathcal{V}(t)$, $i_\mathcal{I}(t)$ are the vectors of currents through inductances, voltage and current sources, and $v_\mathcal{V}(t)$, $v_\mathcal{I}(t)$ are the voltages of voltage and current sources, resp.

In Chapter 6 we generalize a characterization of asymptotic stability of the circuit and give sufficient topological criteria for its invariant zeros being located in the open left half-plane. We show that asymptotic stability of the zero dynamics can be characterized by means of the interconnectivity of the circuit and that it implies that the circuit is high-gain stabilizable with any positive high-gain factor. Thereafter we consider the output regulation problem for electrical circuits by funnel control. We show that for circuits with asymptotically stable zero dynamics, the funnel controller achieves tracking of a class of reference signals within a prespecified funnel; this means in particular that the transient behaviour of the output error can be prescribed and the funnel controller does neither incorporate any internal model for the reference signals nor any identification mechanism, it is simple in its design. The results are illustrated by a simulation of a discretized transmission line.

1.6 Previously published results and joint work

Parts of the present thesis have already been published or submitted for publication as indicated in the following table.

Section	contained in
Section 2.1	new
Section 2.2	Berger, Ilchmann and Trenn [36]

Section 2.3	Berger and Trenn [40, 41]
Subsection 2.4.1	Berger and Trenn [40]
Subsection 2.4.2	new
Chapter 3	Berger and Reis [38]
Sections 4.1 and 4.2	Berger [28]; Remarks 4.1.13–4.1.15 are new
Section 4.3	Berger [28, 29]; Lemma 4.3.11 is already published in Berger, Ilchmann and Reis [35]
Section 4.4	Berger [29]; Proposition 4.4.6 about the stabilizing state feedback for regular systems is new
Section 5.1	Berger [27, 28]; the high-gain stabilization Theorem 5.1.4 is new
Section 5.2	Berger [27, 28]; the simulation of the mechanical system in Subsection 5.2.2 is contained in Berger, Ilchmann and Reis [34]
Subsection 5.3.1	Berger [28]; Proposition 5.3.1 and Corollary 5.3.3 contain new results
Subsections 5.3.2 and 5.3.3	Berger, Ilchmann and Reis [34]
Sections 6.1–6.7	Berger and Reis [39]

The solution theory developed in Section 2.4 defines a solution as a $\mathcal{L}^1_{\text{loc}}$-function with some additional properties, see Definition 2.4.1. This allows for a uniform solution theory in the present thesis without the need for distributional solutions as in [40].

2 Decomposition of matrix pencils

In this chapter we study (singular) linear matrix pencils

$$sE - A \in \mathbb{K}[s]^{m\times n}, \qquad \text{where } \mathbb{K} \text{ is } \mathbb{Q}, \mathbb{R} \text{ or } \mathbb{C}.$$

Two matrix pencils $sE_1 - A_1$ and $sE_2 - A_2$ are called *equivalent* if, and only if, there exist invertible matrices S and T such that

$$\big(SE_1T, SA_1T\big) = \big(E_2, A_2\big);$$

we write $(E_1, A_1) \cong (E_2, A_2)$. Indeed, this is an equivalence relation on $\mathbb{K}^{m\times n} \times \mathbb{K}^{m\times n}$. In the literature this is also sometimes called strict or strong equivalence, see e.g. [100, Ch. XII, § 1] and [152, Def. 2.1]. Based on this notion of equivalence it is of interest to find the 'simplest' matrix pencil within an equivalence class. As discussed in Section 1.1, for regular matrix pencils this problem was solved by Weierstrass [243] and for general matrix pencils later by Kronecker [149] (see also [100,152]). Nevertheless, the analysis of matrix pencils is still an active research area, mainly because of numerical issues or to find ways to obtain the WCF and KCF efficiently (see e.g. [17,81,82,234–236]).

A main goal in this chapter is to highlight the importance of the *Wong sequences* [250] for the analysis of matrix pencils. The Wong sequences for the matrix pencil $sE - A$ are given by the following sequences of subspaces (see the List of Symbols for the definition of the preimage)

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{K}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i) \subseteq \mathbb{K}^n, \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i) \subseteq \mathbb{K}^n. \end{aligned}$$

As a motivation for the Wong sequences we may consider a classical (i.e., continuously differentiable) solution $x : \mathbb{R} \to \mathbb{K}^n$ of $E\dot{x}(t) = Ax(t)$. Using that the linear spaces $\mathcal{V}_i$ are closed and thus invariant under differentiation, the following implications hold true:

$$\begin{aligned}
&\forall\, t \in \mathbb{R}:\ x(t) \in \mathbb{K}^n = \mathcal{V}_0\\
\Longrightarrow\quad &\forall\, t \in \mathbb{R}:\ \dot{x}(t) \in \mathcal{V}_0 \quad \overset{E\dot{x}=Ax}{\Longrightarrow} \quad \forall\, t \in \mathbb{R}:\ x(t) \in A^{-1}(E\mathcal{V}_0) = \mathcal{V}_1\\
\Longrightarrow\quad &\forall\, t \in \mathbb{R}:\ \dot{x}(t) \in \mathcal{V}_1 \quad \overset{E\dot{x}=Ax}{\Longrightarrow} \quad \forall\, t \in \mathbb{R}:\ x(t) \in A^{-1}(E\mathcal{V}_1) = \mathcal{V}_2\\
\Longrightarrow\quad &\text{etc.}
\end{aligned}$$

Therefore, after finitely many iterations it is established that the solution x must evolve in $\mathcal{V}^* := \bigcap_{i\in\mathbb{N}_0} \mathcal{V}_i$, i.e., $x(t) \in \mathcal{V}^*$ for all $t \in \mathbb{R}$. In fact, it is shown in Section 2.3 that $\mathcal{V}^*$ consists of the ODE part and the underdetermined part of the DAE and thus contains all solutions. $\mathcal{W}^* := \bigcup_{i\in\mathbb{N}_0} \mathcal{W}_i$ in turn constitutes the nilpotent part and the overdetermined part of the DAE - which lead to trivial solutions of the homogeneous DAE $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$. All four parts together constitute the quasi-Kronecker form of the matrix pencil $sE - A$, see Theorem 2.3.3.

In Section 2.2 we first consider regular matrix pencils $sE - A$ and show that $\mathcal{V}^* \cap \mathcal{W}^* = \{0\}$, which means that the underdetermined part is not present, and that $\mathcal{V}^* + \mathcal{W}^* = \mathbb{K}^n$, which means that the overdetermined part is not present. Therefore, the Wong sequences directly lead to a decomposition of the pencil into an ODE part and a nilpotent part - the quasi-Weierstraß form, see Theorem 2.2.5.

The QKF is derived step by step via several interim decompositions, which are interesting in their own right. The WCF and the KCF can be obtained as a corollary from the QWF and the QKF, resp. An overview of all decompositions used in this chapter and their relations is provided in Section 2.1.

The consequences of the quasi-Kronecker form for the characterization of solutions of the DAE $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + f(t)$ are discussed in Section 2.4. The solutions to each part are determined separately in Theorem 2.4.8 which allow to characterize consistency of the inhomogeneity and initial values. In particular, it is shown that the Wong sequences completely characterize the solution behavior of the DAE.

An important feature of the Wong sequences is that they (and some modifications of them) fully determine the KCF of the underlying matrix pencil (without the corresponding transformation matrices). More precisely, the row and column minimal indices, the degrees of the finite and infinite elementary divisors and the finite eigenvalues can be calculated using only the Wong sequences, see Subsection 2.3.4.

An advantage of the Wong sequence approach is that we respect the domain of the entries in the matrix pencil, e.g. if our matrices are real-valued, then all transformations remain real-valued. We formulated our results in such a way that they are valid for $\mathbb{K} = \mathbb{Q}$, $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{C}$. Especially for $\mathbb{K} = \mathbb{Q}$ it was also necessary to re-check known results, whether their proofs are also valid in $\mathbb{Q}$. We believe that the case $\mathbb{K} = \mathbb{Q}$ is of special importance because this allows for the implementation of our approach in exact arithmetic which might be feasible if the matrices are sparse and not too big. In fact, we believe that the construction of the QWF and QKF is also possible if the matrix pencil $sE - A$ contains symbolic entries as it is common for the analysis of electrical circuits, where one might just add the symbol R into the matrix instead of a specific value of the corresponding resistor; however, we have not formalized this. This is a major difference of our approach to the ones available in literature which often aim for unitary transformation matrices (due to numerical stability) and are therefore not suitable for symbolic calculations.

We like to stress that indeed most of the results in Section 2.2 can be derived using more general results from Section 2.3. However, we like to point out the peculiarities of the regular case considered in Section 2.2. In particular, the Wong sequences constitute a direct sum $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{K}^n$ and the transformation matrices for the QWF can be calculated easily in the regular case compared to the more general setting. Furthermore, the transformation to WCF is calculated explicitly.

2.1 Definitions

In this section we define different decompositions of matrix pencils which will be derived in Sections 2.2 and 2.3. First we consider decompositions for the class of regular matrix pencils: $sE - A \in \mathbb{K}[s]^{m \times n}$ is called *regular* if, and only if, $m = n$ and $\det(sE - A) \in \mathbb{K}[s] \setminus \{0\}$.

Definition 2.1.1 (Quasi-Weierstraß form).
A regular matrix pencil $sE - A \in \mathbb{K}[s]^{n \times n}$ is said to be in *quasi-Weierstraß form (QWF)* if, and only if,

$$sE - A = s \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix} \tag{2.1.1}$$

for some $n_1, n_2 \in \mathbb{N}_0$, $J \in \mathbb{K}^{n_1 \times n_1}$, $N \in \mathbb{K}^{n_2 \times n_2}$ such that N is nilpotent.

It is shown in Theorem 2.2.5 that any regular matrix pencil can be transformed into QWF and the transformation matrices can be calculated via the Wong sequences.

If the entries J and N in (2.1.1) are in Jordan canonical form, then (2.1.1) is called the Weierstraß canonical form. This form is derived in Corollary 2.2.19, again using the Wong sequences.

Definition 2.1.2 (Weierstraß canonical form)**.**
A regular matrix pencil $sE - A \in \mathbb{K}[s]^{n \times n}$ is said to be in *Weierstraß canonical form (WCF)* if, and only if, $sE-A$ satisfies (2.1.1) such that J and N are in Jordan canonical form and N is nilpotent.

If regularity of $sE - A$ is not required, then additional blocks may appear in the decomposition of the pencil. As a first step towards the quasi-Kronecker form, which is derived in Section 2.3, we derive the interim quasi-Kronecker triangular form in Theorem 2.3.19, which decouples the regular part, the underdetermined part and the overdetermined part of the pencil. This form is defined as follows.

Definition 2.1.3 (Interim quasi-Kronecker triangular form)**.**
A pencil $sE - A \in \mathbb{K}[s]^{m \times n}$ is said to be in *interim quasi-Kronecker triangular form (IQKTF)* if, and only if,

$$sE - A = s \begin{bmatrix} E_P & E_{PR} & E_{PQ} \\ 0 & E_R & E_{RQ} \\ 0 & 0 & E_Q \end{bmatrix} - \begin{bmatrix} A_P & A_{PR} & A_{PQ} \\ 0 & A_R & A_{RQ} \\ 0 & 0 & A_Q \end{bmatrix}, \tag{2.1.2}$$

where

(i) $E_P, A_P \in \mathbb{K}^{m_P \times n_P}$, $m_P < n_P$, are such that $\operatorname{rk}_{\mathbb{C}}(\lambda E_P - A_P) = m_P$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$ (for $\lambda = \infty$ see the List of Symbols),

(ii) $E_R, A_R \in \mathbb{K}^{m_R \times n_R}$, $m_R = n_R$, with $sE_R - A_R$ *regular*, i.e., $\det(sE_R - A_R) \not\equiv 0$,

(iii) $E_Q, A_Q \in \mathbb{K}^{m_Q \times n_Q}$, $m_Q > n_Q$, are such that $\operatorname{rk}_{\mathbb{C}}(\lambda E_Q - A_Q) = n_Q$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$.

If the off-diagonal block entries in the IQKTF (2.1.2) are zero, then the decomposition is called the interim quasi-Kronecker form.

Definition 2.1.4 (Interim quasi-Kronecker form).
A pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ is said to be in *interim quasi-Kronecker form (IQKF)* if, and only if,

$$sE - A = s\begin{bmatrix} E_P & 0 & 0 \\ 0 & E_R & 0 \\ 0 & 0 & E_Q \end{bmatrix} - \begin{bmatrix} A_P & 0 & 0 \\ 0 & A_R & 0 \\ 0 & 0 & A_Q \end{bmatrix}, \tag{2.1.3}$$

such that (i)–(iii) from Definition 2.1.3 are satisfied.

The IQKF is derived in Corollary 2.3.20 along with the transformation matrices, which can be calculated with the help of the Wong sequences. The IQK(T)F is interesting in its own right and provides an intuitive decoupling of the matrix pencil into three parts which have the solution properties (cf. also Section 2.4) 'existence, but non-uniquess' (underdetermined part), 'existence and uniqueness' (regular part) and 'uniqueness, but possible non-existence' (overdetermined part).

If a decoupling of the regular part is desired as well, this can be achieved by the quasi-Kronecker (triangular) form, where again the Wong sequences suffice for the transformation. The quasi-Kronecker triangular form, defined as follows, is derived in Theorem 2.3.1.

Definition 2.1.5 (Quasi-Kronecker triangular form).
A pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ is said to be in *quasi-Kronecker triangular form (QKTF)* if, and only if,

$$sE - A = s\begin{bmatrix} E_P & E_{PJ} & E_{PN} & E_{PQ} \\ 0 & E_J & E_{JN} & E_{JQ} \\ 0 & 0 & E_N & E_{NQ} \\ 0 & 0 & 0 & E_Q \end{bmatrix} - \begin{bmatrix} A_P & A_{PJ} & A_{PN} & A_{PQ} \\ 0 & A_J & A_{JN} & A_{JQ} \\ 0 & 0 & A_N & A_{NQ} \\ 0 & 0 & 0 & A_Q \end{bmatrix}, \tag{2.1.4}$$

where

(i) $E_P, A_P \in \mathbb{K}^{m_P\times n_P}$, $m_P < n_P$, are such that $\operatorname{rk}_{\mathbb{C}}(\lambda E_P - A_P) = m_P$ for all $\lambda \in \mathbb{C}\cup\{\infty\}$,

(ii) $E_J, A_J \in \mathbb{K}^{m_J\times n_J}$, $m_J = n_J$, and $\operatorname{rk}_{\mathbb{C}}(\lambda E_J - A_J) = n_J$ for $\lambda = \infty$, i.e., E_J is invertible,

(iii) $E_N, A_N \in \mathbb{K}^{m_N\times n_N}$, $m_N = n_N$, and $\operatorname{rk}_{\mathbb{C}}(\lambda E_N - A_N) = n_N$ for all $\lambda \in \mathbb{C}$, i.e., A_N is invertible and $A_N^{-1}E_N$ is nilpotent,

(iv) $E_Q, A_Q \in \mathbb{K}^{m_Q \times n_Q}$, $m_Q > n_Q$, are such that $\mathrm{rk}_{\mathbb{C}}(\lambda E_Q - A_Q) = n_Q$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$.

If the off-diagonal block entries in the QKTF (2.1.4) are zero, then the decomposition is called the quasi-Kronecker form.

Definition 2.1.6 (Quasi-Kronecker form)**.**
A pencil $sE - A \in \mathbb{K}[s]^{m \times n}$ is said to be in *quasi-Kronecker form (QKF)* if, and only if,

$$sE - A = s \begin{bmatrix} E_P & 0 & 0 & 0 \\ 0 & E_J & 0 & 0 \\ 0 & 0 & E_N & 0 \\ 0 & 0 & 0 & E_Q \end{bmatrix} - \begin{bmatrix} A_P & 0 & 0 & 0 \\ 0 & A_J & 0 & 0 \\ 0 & 0 & A_N & 0 \\ 0 & 0 & 0 & A_Q \end{bmatrix}, \qquad (2.1.5)$$

such that (i)–(iv) from Definition 2.1.5 are satisfied.

The QKF is derived in Theorem 2.3.3. If more structure of the block entries is desired, it is possible to refine the QKF to the well-known Kronecker canonical form, which is derived in Corollary 2.3.21. We use the notation N_k, L_k, K_k for $k \in \mathbb{N}$ which is defined in the List of Symbols.

Definition 2.1.7 (Kronecker canonical form)**.**
A pencil $sE - A \in \mathbb{K}[s]^{m \times n}$ is said to be in *Kronecker canonical form (KCF)* if, and only if, there exist $a, b, c, d \in \mathbb{N}_0$ and $\varepsilon_1, \ldots, \varepsilon_a$, $\rho_1, \ldots, \rho_b$, $\sigma_1, \ldots, \sigma_c$, $\eta_1, \ldots, \eta_d \in \mathbb{N}_0$, $\lambda_1, \ldots, \lambda_b \in \mathbb{K}$, such that

$$\begin{aligned} sE - A = \operatorname{diag}\big(&\mathcal{P}_{\varepsilon_1}(s), \ldots, \mathcal{P}_{\varepsilon_a}(s), \mathcal{J}_{\rho_1}^{\lambda_1}(s), \ldots, \mathcal{J}_{\rho_b}^{\lambda_b}(s), \\ &\mathcal{N}_{\sigma_1}(s), \ldots, \mathcal{N}_{\sigma_c}(s), \mathcal{Q}_{\eta_1}(s), \ldots, \mathcal{Q}_{\eta_d}(s)\big), \end{aligned} \qquad (2.1.6)$$

where

$$\begin{aligned}
\mathcal{P}_\varepsilon(s) &= s\begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} &&= sL_{\varepsilon+1} - K_{\varepsilon+1} \in \mathbb{K}[s]^{\varepsilon \times (\varepsilon+1)}, \\
\mathcal{J}_\rho^\lambda(s) &= (s-\lambda)I_\rho - \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \quad 0 \end{bmatrix} &&= (s-\lambda)I_\rho - N_\rho \in \mathbb{K}[s]^{\rho \times \rho}, \\
\mathcal{N}_\sigma(s) &= s\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \quad 0 \end{bmatrix} - I_\sigma &&= sN_\sigma - I_\sigma \in \mathbb{K}[s]^{\sigma \times \sigma}, \\
\mathcal{Q}_\eta(s) &= s\begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & 0 \\ & & 1 \end{bmatrix} - \begin{bmatrix} 1 & & \\ 0 & \ddots & \\ & \ddots & 1 \\ & & 0 \end{bmatrix} &&= sL_{\eta+1}^\top - K_{\eta+1}^\top \in \mathbb{K}[s]^{(\eta+1) \times \eta},
\end{aligned}$$

and $\varepsilon \in \mathbb{N}_0$, $\rho \in \mathbb{N}$, $\lambda \in \mathbb{K}$, $\sigma \in \mathbb{N}$, $\eta \in \mathbb{N}_0$.

2.2 Quasi-Weierstraß form

In this section we derive the quasi-Weierstraß form for regular matrix pencils $sE - A \in \mathbb{K}[s]^{n\times n}$. In Subsection 2.2.1 we derive some properties of the Wong sequences and exploit them to derive the QWF. In Subsection 2.2.2 we consider chains of generalized eigenvectors and show how to exploit them to re-derive the Weierstraß canonical form.

The results of this section have already been published in a joint work with Achim Ilchmann and Stephan Trenn [36].

2.2.1 Wong sequences and QWF

As pointed out earlier, our approach is based on the Wong sequences. They can be calculated via a recursive subspace iteration. Since they are used in later sections as well, we define the Wong sequences for general matrix pencils $sE - A \in \mathbb{K}[s]^{m\times n}$.

Definition 2.2.1 (Wong sequences [250])**.**
Let $sE - A \in \mathbb{K}[s]^{m\times n}$. Then the sequences of subspaces $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ and $(\mathcal{W}_i)_{i\in\mathbb{N}_0}$ defined by

$$\mathcal{V}_0 := \mathbb{K}^n, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i) \quad \forall\, i \in \mathbb{N}_0 \tag{2.2.1}$$

$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i) \quad \forall\, i \in \mathbb{N}_0 \tag{2.2.2}$$

are called *Wong sequences*. Let $\mathcal{V}^* := \bigcap_{i\in\mathbb{N}_0} \mathcal{V}_i$ and $\mathcal{W}^* := \bigcup_{i\in\mathbb{N}_0} \mathcal{W}_i$ be the *limits of the Wong sequences*.

It is easy to see that the Wong sequences are nested, terminate and satisfy

$$\left.\begin{array}{l}\exists\, k^*\in\mathbb{N}_0\ \forall\, j\in\mathbb{N}_0 : \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_{k^*} = \mathcal{V}_{k^*+j} = \mathcal{V}^* \\ \hfill = A^{-1}(E\mathcal{V}^*) \supseteq \ker A, \\ \exists\, \ell^*\in\mathbb{N}_0\ \forall\, j\in\mathbb{N}_0 : \mathcal{W}_0 \subseteq \ker E = \mathcal{W}_1 \subsetneq \cdots \subsetneq \mathcal{W}_{\ell^*} = \mathcal{W}_{\ell^*+j} \\ \hfill = \mathcal{W}^* = E^{-1}(A\mathcal{W}^*),\end{array}\right\} \tag{2.2.3}$$

as well as

$$A\mathcal{V}^* \subseteq E\mathcal{V}^* \qquad \text{and} \qquad E\mathcal{W}^* \subseteq A\mathcal{W}^*\,. \tag{2.2.4}$$

In the following Lemma 2.2.2 some elementary properties of the Wong sequences are derived, they are essential for proving basic properties of the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$ in Proposition 2.2.3. These results are inspired by the observation of Campbell [62, p. 37] who proves, for $\mathbb{K} = \mathbb{C}$, that the space of consistent initial values is given by $\operatorname{im}\big((A - \lambda E)^{-1}E\big)^{\nu}$ for any $\lambda \in \mathbb{C} \setminus \operatorname{spec}(sE - A)$ and $\nu \in \mathbb{N}_0$ the index of the matrix $(A-\lambda E)^{-1}E$, [62, p. 7]. However, Campbell did not consider the Wong sequences explicitly.

Lemma 2.2.2 (Spectrum and properties of $\mathcal{V}_i$ and $\mathcal{W}_i$)**.**
If $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular, then the Wong sequences (2.2.1) *and* (2.2.2) *satisfy*

$$\forall\, \lambda \in \mathbb{K}\setminus \operatorname{spec}(sE - A)\ \forall\, i \in \mathbb{N}_0 :\ \mathcal{V}_i = \operatorname{im}\big((A - \lambda E)^{-1}E\big)^{i} \quad \text{and} \quad \mathcal{W}_i = \ker\big((A - \lambda E)^{-1}E\big)^{i}.$$

In particular,

$$\forall\, i \in \mathbb{N}_0 :\ \dim \mathcal{V}_i + \dim \mathcal{W}_i = n. \tag{2.2.5}$$

Proof: Since $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular, let

$$\widehat{E} := (A - \lambda E)^{-1}E, \quad \text{for arbitrary but fixed } \lambda \in \mathbb{K}\setminus \operatorname{spec}(sE - A). \tag{2.2.6}$$

Step 1: We prove by induction: $\mathcal{V}_i = \operatorname{im} \widehat{E}^i$ for all $i \in \mathbb{N}_0$. Clearly, $\mathcal{V}_0 = \mathbb{K}^n = \operatorname{im} \widehat{E}^0$. Suppose that $\operatorname{im} \widehat{E}^i = \mathcal{V}_i$ holds for some $i \in \mathbb{N}_0$.

Step 1a: We show: $\mathcal{V}_{i+1} \supseteq \operatorname{im} \widehat{E}^{i+1}$. Let $x \in \operatorname{im} \widehat{E}^{i+1} \subseteq \operatorname{im} \widehat{E}^i$. Then there exists $y \in \operatorname{im} \widehat{E}^i$ such that $x = \big(A - \lambda E\big)^{-1}Ey$. Therefore, $(A-\lambda E)x = Ey = E(y+\lambda x-\lambda x)$ and so, for $\widehat{y} := y+\lambda x \in \operatorname{im} \widehat{E}^i = \mathcal{V}_i$, we have $Ax = E\widehat{y}$. This implies $x \in \mathcal{V}^{i+1}$.

Step 1b: We show: $\mathcal{V}_{i+1} \subseteq \operatorname{im} \widehat{E}^{i+1}$. Let $x \in \mathcal{V}_{i+1}$ and choose $y \in \mathcal{V}_i$ such that $Ax = Ey$. Then $(A - \lambda E)x = E(y - \lambda x)$ or, equivalently, $x = (A-\lambda E)^{-1}E(y-\lambda x)$. From $x \in \mathcal{V}_{i+1} \subseteq \mathcal{V}_i$ it follows that $y - \lambda x \in \mathcal{V}_i = \operatorname{im} \widehat{E}^i$ and therefore $x \in \operatorname{im} \widehat{E}^{i+1}$.

Step 2: We prove by induction: $\mathcal{W}_i = \ker \widehat{E}^i$ for all $i \in \mathbb{N}_0$. Clearly, $\mathcal{W}_0 = \{0\} = \ker \widehat{E}^0$. Suppose that $\ker \widehat{E}^i = \mathcal{W}_i$ for some $i \in \mathbb{N}_0$.
First observe that $(I + \lambda\widehat{E})$ restricted to $\ker \widehat{E}^i$ is an operator $(I + \lambda\widehat{E}) : \ker \widehat{E}^i \to \ker \widehat{E}^i$ with inverse $\sum_{j=0}^{i-1}(-\lambda)^j \widehat{E}^j$. Thus the following

equivalences hold

$$\begin{aligned} x \in \mathcal{W}_{i+1} &\iff \exists\, y \in \mathcal{W}_i : \; Ex = Ay = (A - \lambda E)y + \lambda E y \\ &\iff \exists\, y \in \mathcal{W}_i = \ker \widehat{E}^i : \; \widehat{E}x = (I + \lambda \widehat{E})y =: \widehat{y} \\ &\iff \exists\, \widehat{y} \in \ker \widehat{E}^i : \; \widehat{E}x = \widehat{y} \\ &\iff x \in \ker \widehat{E}^{i+1}. \end{aligned}$$ □

Next we prove important properties of the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$, some of which can be found in [250], but the present presentation is more straightforward.

Proposition 2.2.3 (Properties of $\mathcal{V}^*$ and $\mathcal{W}^*$)**.**
If $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular, then $\mathcal{V}^*$ and $\mathcal{W}^*$ as in (2.2.3) *satisfy:*

(i) $k^* = l^*$, *where k^*, l^* are given in* (2.2.3)*,*

(ii) $\mathcal{V}^* \oplus \mathcal{W}^* = \mathbb{K}^n$*,*

(iii) $\boxed{\ker E \cap \mathcal{V}^* = \{0\}}$, $\boxed{\ker A \cap \mathcal{W}^* = \{0\}}$, $\boxed{\ker E \cap \ker A = \{0\}}$.

Proof: (i): This is a consequence of (2.2.5).

(ii): In view of (2.2.5), it suffices to show that $\mathcal{V}^* \cap \mathcal{W}^* = \{0\}$. Using the notation as in (2.2.6), we may conclude: If $x \in \mathcal{V}^* \cap \mathcal{W}^* = \operatorname{im} \widehat{E}^{k^*} \cap \ker \widehat{E}^{k^*}$, then there exists $y \in \mathbb{K}^n$ such that $x = \widehat{E}^{k^*} y$ and so $0 = \widehat{E}^{k^*} x = \left(\widehat{E}^{k^*}\right)^2 y = \widehat{E}^{2k^*} y$, whence, in view of $y \in \ker \widehat{E}^{2k^*} = \ker \widehat{E}^{k^*}$, $0 = \widehat{E}^{k^*} y = x$.

(iii): This is a direct consequence from (2.2.3) and (ii). □

Example 2.2.4 (Regular pencil)**.**
Consider the linear pencil $sE - A \in \mathbb{K}[s]^{4\times 4}$ given by

$$A := \begin{bmatrix} 3 & 0 & 1 & 0 \\ 0 & 2 & 2 & -1 \\ 1 & 2 & 3 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \qquad E := \begin{bmatrix} 1 & -1 & -3 & 0 \\ 0 & 2 & 0 & -1 \\ -3 & -1 & 1 & 2 \\ -2 & -2 & 0 & 2 \end{bmatrix}.$$

Since $\det(sE - A) = 36s(s-1)$, the pencil is regular and not equivalent to a pencil $sI_4 - J$, $J \in \mathbb{K}^{4\times 4}$, i.e., it is not an ODE. A straightforward calculation gives

$$\mathcal{V}_1 = \operatorname{im} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathcal{W}_1 = \ker E = \operatorname{im} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$$

and

$$\mathcal{V}_2 = \operatorname{im} V, \qquad \text{where } V := \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ -1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$\mathcal{W}_2 = \operatorname{im} W, \qquad \text{where } W := \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}.$$

Both sequences terminate after these two iterations and therefore $\mathcal{V}^* = \mathcal{V}_2$, $\mathcal{W}^* = \mathcal{W}_2$ and $k^* = \ell^* = 2$. The statements of Proposition 2.2.3 and (2.2.4) are readily verified. Finally, we stress, in view of (2.2.5), that for this example

$$\mathcal{V}_1 \cap \mathcal{W}_1 = \mathcal{W}_1 \supsetneq \{0\}.$$

We are now in a position to state the main result of this section: The Wong sequences $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ and $(\mathcal{W}_i)_{i\in\mathbb{N}_0}$, converge in finitely many steps to the subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$, and the latter constitute a transformation of the original pencil $sE - A$ into two decoupled pencils.

Theorem 2.2.5 (The quasi-Weierstraß form).
Consider a regular matrix pencil $sE - A \in \mathbb{K}[s]^{n\times n}$ *and corresponding spaces* $\mathcal{V}^*$ *and* $\mathcal{W}^*$ *as in* (2.2.3). *Let*

$$\begin{aligned} n_1 &:= \dim \mathcal{V}^*, & V &\in \mathbb{K}^{n\times n_1} : \operatorname{im} V = \mathcal{V}^* \\ \text{and } n_2 &:= n - n_1 = \dim \mathcal{W}^*, & W &\in \mathbb{K}^{n\times n_2} : \operatorname{im} W = \mathcal{W}^*. \end{aligned}$$

Then $[V, W]$ *and* $[EV, AW]$ *are invertible and* $[EV, AW]^{-1}(sE - A)[V, W]$ *is in QWF* (2.1.1) *such that* $N^{k^*} = 0$ *for* k^* *as given in* (2.2.3).

Before we prove Theorem 2.2.5, some comments may be warranted.

Remark 2.2.6 (The quasi-Weierstraß form).
Let $sE - A \in \mathbb{K}[s]^{n\times n}$ be a regular matrix pencil and use the notation from Theorem 2.2.5.

(i) It is immediate, and will be used in later analysis, that $[EV, AW]^{-1}(sE - A)[V, W]$ is in QWF (2.1.1) if, and only if,

$$AV = EVJ \quad \text{and} \quad EW = AWN \tag{2.2.7}$$

or, equivalently, if

$$E = [EV, AW]\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}[V, W]^{-1}$$
$$\text{and} \quad A = [EV, AW]\begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}[V, W]^{-1}. \tag{2.2.8}$$

(ii) If (2.2.7) is solvable and if $[EV, AW]$ is invertible, then it is straightforward to see that J and N in (2.2.7) are uniquely given by

$$J := (EV)^{+}AV \quad \text{and} \quad N := (AW)^{+}EW, \quad \text{resp.,} \tag{2.2.9}$$

where $M^{+} := (M^{*}M)^{-1}M^{*}$ for $M \in \mathbb{K}^{p\times q}$ with $\operatorname{rk}_{\mathbb{K}} M = q$.

(iii) The spaces $\mathcal{V}^*$ and $\mathcal{W}^*$ determine uniquely – up to similarity – the solutions J and N of (2.2.7), resp. More precisely, let

$$\widehat{V} \in \mathbb{K}^{n\times n_1}\colon \ \operatorname{im}\widehat{V} = \mathcal{V}^* \quad \text{and} \quad \widehat{W} \in \mathbb{K}^{n\times n_2}\colon \ \operatorname{im}\widehat{W} = \mathcal{W}^*.$$

Then

$$\exists\, S \in \mathbf{Gl}_{n_1}(\mathbb{K})\colon \ VS = \widehat{V} \quad \text{and} \quad \exists\, T \in \mathbf{Gl}_{n_2}(\mathbb{K})\colon \ WT = \widehat{W},$$

and a simple calculation yields that J and N are similar to

$$(E\widehat{V})^{+}A\widehat{V} = S^{-1}JS \quad \text{and} \quad (A\widehat{W})^{+}E\widehat{W} = T^{-1}NT, \quad \text{resp.}$$

(iv) If $\det E \neq 0$, then $\mathcal{V}^* = \mathcal{V}_i = \mathbb{K}^n$ and $\mathcal{W}^* = \mathcal{W}_i = \{0\}$ for all $i \in \mathbb{N}_0$. Therefore

$$E^{-1}(sE - A) = sI - E^{-1}A$$

is in QWF.

(v) Let $\mathbb{K} = \mathbb{C}$. In view of (iii), the matrices V and W may always be chosen so that J and N in (2.1.1) are in Jordan canonical form, in this case (2.1.1) is in WCF.

(vi) For $\mathbb{K} = \mathbb{C}$, there are various numerical methods available to calculate the WCF, see e.g. [17, 81, 82]. However, since the QWF does not invoke any eigenvalues and eigenvectors (here only the decoupling (2.1.1) and J and N without any special structure is important), it is possible that the above mentioned algorithms can be improved by using the Wong sequences. To calculate the subspaces (2.2.1) and (2.2.2) of the Wong sequences, one may use methods to obtain orthogonal bases for deflating subspaces; see for example [24] and [135].
Furthermore, the QWF – in contrast to the WCF – allows to consider matrix pencils over rational or even symbolic rings and the algorithm is still applicable. In fact, we will show in Proposition 2.2.9 that the number of subspace iterations equals the index of the matrix pencil (cf. Definition 2.2.8); hence in many practical situations only one or two iterations must be carried out.

(vii) A time-varying pendant to the QWF is the standard canonical form developed in [64, 68] and studied in [32, 33]. This form has the same block structure as the QWF, but with time-varying J and N, where N is pointwise strictly lower triangular.

Proof of Theorem 2.2.5: Invertibility of $[V, W]$ follows from Proposition 2.2.3 (ii). The implication

$$\forall\, \alpha \in \mathbb{K}^{n_1} : \left(EV\alpha = 0 \overset{\text{Prop. 2.2.3 (iii)}}{\Longrightarrow} V\alpha = 0 \overset{\operatorname{rk} V = n_1}{\Longrightarrow} \alpha = 0 \right)$$

shows $\operatorname{rk} EV = n_1$, and a similar argument yields $\operatorname{rk} AW = n_2$. Now, invertibility of $[EV, AW]$ is equivalent to $\operatorname{im} EV \cap \operatorname{im} AW = \{0\}$, and the latter is a consequence of

$$\begin{aligned} \forall\, \alpha \in \mathbb{K}^{n_1}\ \forall\, \beta \in \mathbb{K}^{n_2} :\ EV\alpha = AW\beta \implies V\alpha \in E^{-1}(A\mathcal{W}^*) \overset{(2.2.3)}{=} \mathcal{W}^* \\ \overset{\text{Prop. 2.2.3 (ii)}}{\Longrightarrow} V\alpha = 0 \implies \alpha = 0\ \wedge \beta = 0. \end{aligned}$$

Now the subset inequalities (2.2.4) imply that (2.2.7) is solvable and $[EV, AW]^{-1}(sE - A)[V, W]$ is in the form (2.1.1). It remains to prove that N is nilpotent. To this end, we show

$$\forall\, i \in \{0,\dots,k^*\}: \ \operatorname{im} W N^i \subseteq \mathcal{W}_{k^*-i}\,. \tag{2.2.10}$$

The statement is clear for $i=0$. Suppose, for some $i \in \{0,\dots,k^*-1\}$, we have

$$\operatorname{im} W N^i \subseteq \mathcal{W}_{k^*-i}\,. \tag{2.2.11}$$

Then

$$\operatorname{im} AWN^{i+1} \overset{(2.2.7)}{=} \operatorname{im} EWN^i \overset{(2.2.11)}{\subseteq} E\mathcal{W}_{k^*-i} \overset{(2.2.2)}{\subseteq} A\mathcal{W}_{k^*-i-1}$$

and, by invoking Proposition 2.2.3 (iii),

$$\operatorname{im} WN^{i+1} \subseteq \mathcal{W}_{k^*-i-1}.$$

This proves (2.2.10). Finally, (2.2.10) for $i=k^*$ together with the fact that W has full column rank and $\mathcal{W}_0=\{0\}$, implies that $N^{k^*}=0$. □

Example 2.2.7 (Example 2.2.4 revisited)**.**
For V and W as defined in Example 2.2.4 we have

$$[V,W] = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & -1 & 1 & -1 \\ 0 & 1 & 0 & 2 \end{bmatrix} \quad \text{and} \quad [EV,AW] = \begin{bmatrix} 4 & 1 & 4 & -1 \\ 0 & 3 & 2 & -2 \\ -4 & -1 & 4 & -1 \\ -2 & -2 & 0 & 3 \end{bmatrix}$$

and the corresponding transformation $[EV,AW]^{-1}(sE-A)[V,W]$ shows that a QWF of this example is given by

$$s\begin{bmatrix} I_2 & 0 \\ 0 & N \end{bmatrix} - \begin{bmatrix} J & 0 \\ 0 & I_2 \end{bmatrix} \quad \text{where} \quad J := \frac{1}{3}\begin{bmatrix} 2 & 1 \\ -2 & 1 \end{bmatrix}, \quad N := \frac{2}{3}\begin{bmatrix} -1 & 1 \\ -1 & 1 \end{bmatrix}.$$

It follows from Remark 2.2.6 (iii) that the following definition of the index of a regular pencil is well defined since it does not depend on the special choice of N in the QWF.

Definition 2.2.8 (Index of $sE-A$)**.**
Let $sE-A \in \mathbb{K}[s]^{n\times n}$ be a regular matrix pencil and consider the QWF (2.1.1). Then

$$\nu^* \ := \begin{cases} \min\{\, \nu \in \mathbb{N} \mid N^\nu = 0 \,\}, & \text{if } N \text{ exists} \\ 0, & \text{otherwise} \end{cases}$$

is called the *index* of $sE-A$.

The classical definition of the index of a regular matrix pencil (see e.g. [152, Def. 2.9]) is via the WCF. However, invoking Remark 2.2.6 (v), we see that ν^* in Definition 2.2.8 is the same number.

Proposition 2.2.9 (Index of $sE - A$)**.**
If $sE - A \in \mathbb{K}[s]^{n\times n}$ is regular, then the Wong sequence in (2.2.2) *and W and N as in Theorem 2.2.5 satisfy*

$$\forall\, i \in \mathbb{N}_0 : \ \mathcal{W}_i = W \ker N^i \,. \tag{2.2.12}$$

This implies that $\nu^* = k^*$; i.e., the index ν^* coincides with k^* determined by the Wong sequences in (2.2.3)*.*

Proof: We use the notation as in Theorem 2.2.5 and the form (2.1.1), and also the following simple formula

$$\forall\, i \in \mathbb{N}_0 \ : \ \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix} \begin{pmatrix} \{0_{n_1}\} \\ \ker N^i \end{pmatrix} = \begin{pmatrix} \{0_{n_1}\} \\ \ker N^{i+1} \end{pmatrix} . \tag{2.2.13}$$

Next, we conclude, for $\widehat{\mathcal{W}}_0 := \{0\}$ and all $i \in \mathbb{N}$,

$$\begin{aligned}
\widehat{\mathcal{W}}_i \quad &:= \quad [V, W]^{-1}\mathcal{W}_i \overset{(2.2.3)}{=} [V, W]^{-1}E^{-1}A\mathcal{W}_{i-1} \\
&\overset{(2.2.8)}{=} \begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} [EV, AW]^{-1}A[V, W]\widehat{\mathcal{W}}_{i-1} \\
&= \underbrace{\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right) \cdots \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)}_{i\text{-times}} \widehat{\mathcal{W}}_0 \\
&\overset{(2.2.13)}{=} \underbrace{\left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right) \cdots \left(\begin{bmatrix} I & 0 \\ 0 & N \end{bmatrix}^{-1} \begin{bmatrix} J & 0 \\ 0 & I \end{bmatrix}\right)}_{(i-1)\text{-times}} \begin{pmatrix} \{0_{n_1}\} \\ \ker N \end{pmatrix} \\
&\overset{(2.2.13)}{=} \begin{pmatrix} \{0_{n_1}\} \\ \ker N^i \end{pmatrix}
\end{aligned}$$

and hence (2.2.12). □

Example 2.2.10 (Example 2.2.4, 2.2.7 revisited)**.**
For W and N as defined in Example 2.2.4 and 2.2.7, resp., we see that

$N^2 = 0$ and confirm the statement of Proposition 2.2.9:

$$W \ker N = W \operatorname{im} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \operatorname{im} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \mathcal{W}_1.$$

An immediate consequence of Theorem 2.2.5 is

$$\det(sE - A) = \\ \det([EV, AW]) \det(sI_{n_1} - J) \det(sN - I_{n_2}) \det([V, W]^{-1}),$$

and since any nilpotent matrix N satisfies $\det(sN - I_{n_2}) = (-1)^{n_2}$, we arrive at the following corollary.

Corollary 2.2.11 (Properties of the determinant)**.**
Suppose $sE - A \in \mathbb{K}[s]^{n\times n}$ is a regular matrix pencil. Then, using the notation of Theorem 2.2.5 and the form (2.1.1)*, we have:*

(i) $\det(sE - A) = c \det(sI_{n_1} - J)$,
for $c := (-1)^{n_2} \det([EV, AW]) \det([V, W]^{-1}) \neq 0$,

(ii) $\operatorname{spec}(sE - A) = \operatorname{spec}(sI_{n_1} - J)$,

(iii) $\dim \mathcal{V}^* = \deg\big(\det(sE - A)\big)$.

In the remainder of this subsection we characterize $\mathcal{V}^*$ in geometric terms as a largest subspace. [42] already stated that $\mathcal{V}^*$ is the largest subspace such that $A\mathcal{V}^* \subseteq E\mathcal{V}^*$.

Proposition 2.2.12 ($\mathcal{V}^*$ largest subspaces)**.**
Let $sE - A \in \mathbb{K}[s]^{n\times n}$ be a regular matrix pencil. Then $\mathcal{V}^*$ determined by the Wong sequences (2.2.3) *is the largest subspace of $\mathbb{K}^n$ such that $A\mathcal{V}^* \subseteq E\mathcal{V}^*$.*

Proof: We have to show that any subspace $\mathcal{U} \subseteq \mathbb{K}^n$ so that $A\mathcal{U} \subseteq E\mathcal{U}$ satisfies $\mathcal{U} \subseteq \mathcal{V}^*$. Let $u_0 \in \mathcal{U}$. Then

$$\exists\, u_1, \ldots, u_{k^*} \in \mathcal{U}\ \forall\, i = 1, \ldots, k^* :\ Au_{i-1} = Eu_i\,.$$

By Theorem 2.2.5,

$$\exists\,\alpha_0,\dots,\alpha_{k^*}\in\mathbb{K}^{n_1}\ \exists\,\beta_0,\dots,\beta_{k^*}\in\mathbb{K}^{n_2}\ \forall\, i=0,\dots,k^*:\quad u_i=[V,W]\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}$$

and hence

$$\forall\, i=1,\dots,k^*:\ A[V,W]\begin{pmatrix}\alpha_{i-1}\\ \beta_{i-1}\end{pmatrix} = E[V,W]\begin{pmatrix}\alpha_i\\ \beta_i\end{pmatrix}$$

or, equivalently,

$$\begin{aligned}\forall\, i=1,\dots,k^*:\ [EV,AW]\begin{pmatrix}-\alpha_i\\ \beta_{i-1}\end{pmatrix} &= [AV,EW]\begin{pmatrix}-\alpha_{i-1}\\ \beta_i\end{pmatrix}\\ &\overset{(2.2.7)}{=} [EV,AW]\begin{bmatrix}J & 0\\ 0 & N\end{bmatrix}\begin{pmatrix}-\alpha_{i-1}\\ \beta_i\end{pmatrix};\end{aligned}$$

since $[EV,AW]$ is invertible, we arrive at

$$\forall\, i=1,\dots,k^*:\ \beta_{i-1}=N\beta_i$$

and therefore

$$\beta_0=N\beta_1=\ldots=N^{k^*}\beta_{k^*}=0.$$

This yields $u_0=V\alpha_0\in\operatorname{im}V=\mathcal{V}^*$ and proves $\mathcal{U}\subseteq\mathcal{V}^*$. □

2.2.2 Eigenvector chains and WCF

In this subsection we show that the generalized eigenvectors of a regular pencil $sE-A\in\mathbb{C}[s]^{n\times n}$ constitute a basis which transforms $sE-A$ into WCF. From this point of view, the WCF is a generalized Jordan canonical form. The chains of eigenvectors and eigenspaces are derived in terms of the matrices E and A; the QWF is only used in the proofs. This again shows the unifying power of the Wong-sequences and allows for a ‘natural’ proof of the WCF.

Note that eigenvalues and eigenvectors of real or rational matrix pencils are in general complex valued, thus in the following we restrict the analysis to the case $\mathbb{K}=\mathbb{C}$. We recall the well known concept of chains of generalized eigenvectors; for infinite eigenvectors see [23, Def. 2] and also [158, 159].

Definition 2.2.13 (Chains of generalized eigenvectors)**.**
Let $sE-A \in \mathbb{C}[s]^{n\times n}$ be a matrix pencil. Then $(v_1,\ldots,v_k) \in (\mathbb{C}^n\setminus\{0\})^k$ is called a *chain* (*of* $sE-A$ *at eigenvalue* λ) if, and only if,

$$\left.\begin{array}{rrr} \lambda \in \operatorname{spec}(sE-A): & (A-\lambda E)v_1 = 0, & (A-\lambda E)v_2 = Ev_1, \\ & \ldots\;, & (A-\lambda E)v_k = Ev_{k-1}; \\ \lambda = \infty: & Ev_1 = 0, & Ev_2 = Av_1, \\ & \ldots\;, & Ev_k = Av_{k-1}; \end{array}\right\} \tag{2.2.14}$$

the i^{th} vector v_i of the chain is called *generalized eigenvector of order* i *at* λ.

Note that $(v_1,\ldots,v_k)$ is a chain at $\lambda \in \operatorname{spec}(sE-A)$ if, and only if, for all $i \in \{1,\ldots,k\}$,

$$AV_i = EV_i \begin{bmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & \lambda \end{bmatrix} \quad \text{where} \quad V_i := [v_1,\ldots,v_i]\,. \tag{2.2.15}$$

Remark 2.2.14 (Linear relations)**.**
It may be helpful to consider the concept of generalized eigenvectors, in particular for eigenvalues at ∞, from the viewpoint of linear relations, see e.g. [5]:

$\mathcal{R} \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is called a *linear relation* if, and only if, $\mathcal{R}$ is a linear space; its *inverse relation* is $\mathcal{R}^{-1} := \{(y,x) \in \mathbb{C}^n \times \mathbb{C}^n | \,(x,y) \in \mathcal{R}\}$, and the *multiplication* with a relation $\mathcal{S} \subseteq \mathbb{C}^n \times \mathbb{C}^n$ is $\mathcal{RS} := \{(x,y) \in \mathbb{C}^n \times \mathbb{C}^n | \,\exists\, z \in \mathbb{C}^n : \; (x,z) \in \mathcal{S} \wedge (z,y) \in \mathcal{R}\}$. $\lambda \in \mathbb{C}$ is called an *eigenvalue* of a relation $\mathcal{R}$ with eigenvector $x \in \mathbb{C}^n \setminus \{0\}$ if, and only if, $(x,\lambda x) \in \mathcal{R}$; see [214]. Clearly, $\lambda \neq 0$ is an eigenvalue of $\mathcal{R}$ if, and only if, $1/\lambda$ is an eigenvalue of $\mathcal{R}^{-1}$; this justifies to call ∞ an eigenvalue of $\mathcal{R}$ if, and only if, 0 is an eigenvalue of $\mathcal{R}^{-1}$.

In the context of a matrix pencil $sE-A \in \mathbb{C}[s]^{n\times n}$, the matrices A and E induce the linear relations

$$\mathcal{A} := \{(x,Ax) | \,x \in \mathbb{C}^n\} \quad \text{and} \quad \mathcal{E} := \{(x,Ex) | \,x \in \mathbb{C}^n\}\,, \quad \text{resp.}$$

and therefore,

$$\begin{aligned}
\mathcal{E}^{-1} &= \{ (Ex, x) \mid x \in \mathbb{C}^n \}, \\
\mathcal{E}^{-1}\mathcal{A} &= \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid Ax = Ey \}, \\
\mathcal{A}^{-1} &= \{ (Ax, x) \mid x \in \mathbb{C}^n \}, \\
\mathcal{A}^{-1}\mathcal{E} &= \{ (x, y) \in \mathbb{C}^n \times \mathbb{C}^n \mid Ex = Ay \}.
\end{aligned}$$

It now follows that

$$\begin{aligned}
\lambda \in \mathbb{C} \text{ is an eigenvalue of } \mathcal{E}^{-1}\mathcal{A} &\iff \det(\lambda E - A) = 0 \\
\infty \text{ is an eigenvalue of } \mathcal{E}^{-1}\mathcal{A} &\iff 0 \text{ is an eigenvalue of } \mathcal{A}^{-1}\mathcal{E} \\
0 \text{ is an eigenvalue of } \mathcal{A}^{-1}\mathcal{E} &\iff E \text{ is not invertible.}
\end{aligned}$$

In [214] also chains for relations are considered. In the context of the above example this reads: $v_1, \ldots, v_k \in \mathbb{C}^n \setminus \{0\}$ form a (*Jordan*) *chain* at eigenvalue $\lambda \in \mathbb{C} \cup \{\infty\}$ if, and only if,

$$\left.\begin{array}{rlll}
\lambda \in \operatorname{spec}(sE - A): & (v_1, \lambda v_1), & (v_2, v_1 + \lambda v_2), & \\
& \ldots, & (v_k, v_{k-1} + \lambda v_k) & \in \mathcal{E}^{-1}\mathcal{A}; \\
\lambda = \infty: & (0, v_1), & (v_1, v_2), & \\
& \ldots, & (v_{k-1}, v_k) & \in \mathcal{E}^{-1}\mathcal{A}.
\end{array}\right\} \tag{2.2.16}$$

Obviously, (2.2.16) is equivalent to (2.2.14), but the former may be a more 'natural' definition. Linear relations have also been analyzed and exploited for matrix pencils in [25].

In order to decompose $\mathcal{V}^*$, we have to be more specific with the spaces spanned by generalized eigenvectors at eigenvalues.

Definition 2.2.15 (Generalized eigenspaces)**.**
Let $sE - A \in \mathbb{C}[s]^{n \times n}$ be a matrix pencil. Then the sequences of *eigenspaces* (*of* $sE - A$ *at eigenvalue* λ) are defined by $\mathcal{G}_\lambda^0 := \{0\}$ and

$$\forall\, i \in \mathbb{N}_0 \;:\; \mathcal{G}_\lambda^{i+1} := \begin{cases} (A - \lambda E)^{-1}(E\mathcal{G}_\lambda^i), & \text{if } \lambda \in \operatorname{spec}(sE - A) \\ E^{-1}(A\mathcal{G}_\lambda^i), & \text{if } \lambda = \infty. \end{cases}$$

The *generalized eigenspace* (*of* $sE - A$ *at eigenvalue* $\lambda \in \operatorname{spec}(sE - A) \cup \{\infty\}$) is defined by

$$\mathcal{G}_\lambda := \bigcup_{i \in \mathbb{N}_0} \mathcal{G}_\lambda^i.$$

For the multiplicities we use the following notion

$$\begin{aligned}
\operatorname{gm}(\lambda) &:= \dim \mathcal{G}_\lambda^1 \text{ is called the } \textit{geometric multiplicity} \text{ of } \\
&\quad\ \lambda \in \operatorname{spec}(sE-A) \cup \{\infty\}, \\
\operatorname{am}(\lambda) &:= \text{multiplicity of } \lambda \in \operatorname{spec}(sE-A) \cup \{\infty\} \text{ as a zero of} \\
&\quad\ \det(sE-A) \text{ is called the } \textit{algebraic multiplicity} \text{ of } \lambda, \\
\operatorname{am}(\infty) &:= n - \sum_{\lambda \in \operatorname{spec}(sE-A)} \operatorname{am}(\lambda) = n - \deg\big(\det(sE-A)\big) \text{ is} \\
&\quad\ \text{called the } \textit{algebraic multiplicity at } \infty.
\end{aligned}$$

Readily verified properties of the eigenspaces are the following.

Remark 2.2.16 (Eigenspaces).
For any regular $sE - A \in \mathbb{C}[s]^{n\times n}$ and $\lambda \in \operatorname{spec}(sE-A) \cup \{\infty\}$ we have:

(i) For each $i \in \mathbb{N}_0$, $\mathcal{G}_\lambda^i$ is the vector space spanned by the eigenvectors up to order i at λ.

(ii) $\exists\, p^* \in \mathbb{N}_0\ \forall\, j \in \mathbb{N}_0 :\ \mathcal{G}_\lambda^0 \subsetneq \cdots \subsetneq \mathcal{G}_\lambda^{p-1} \subsetneq \mathcal{G}_\lambda^{p^*} = \mathcal{G}_\lambda^{p^*+j}$.

The following result is formulated in terms of the pencil $sE - A$, its proof invokes the QWF.

Proposition 2.2.17 (Eigenvectors and eigenspaces).
Let $sE - A \in \mathbb{C}[s]^{n\times n}$ be regular.

(i) *Every chain $(v_1, \ldots, v_k)$ at any $\lambda \in \operatorname{spec}(sE-A) \cup \{\infty\}$ satisfies, for all $i \in \{1, \ldots, k\}$, $v_i \in \mathcal{G}_\lambda^i \backslash \mathcal{G}_\lambda^{i-1}$.*

(ii) *Let $\lambda \in \operatorname{spec}(sE-A) \cup \{\infty\}$ and $k \in \mathbb{N}$. Then for any $v \in \mathcal{G}_\lambda^k \backslash \mathcal{G}_\lambda^{k-1}$, there exists a unique chain $(v_1, \ldots, v_k)$ such that $v_k = v$.*

(iii) *The vectors of any chain $(v_1, \ldots, v_k)$ at $\lambda \in \operatorname{spec}(sE-A) \cup \{\infty\}$ are linearly independent.*

(iv)

$$\mathcal{G}_\lambda \subseteq \begin{cases} \mathcal{V}^*, & \textit{if} \quad \lambda \in \operatorname{spec}(sE-A) \\ \mathcal{W}^*, & \textit{if} \quad \lambda = \infty. \end{cases}$$

(v)
$$\forall\, \lambda \in \operatorname{spec}(sE - A) \cup \{\infty\} : \; \dim \mathcal{G}_\lambda = \operatorname{am}(\lambda).$$

Proof: *Step 1*: Invoking the notation of Theorem 2.2.5 and of the form (2.1.1), we first show that

$$\forall\, i \in \mathbb{N}_0 : \; \mathcal{G}_\lambda^i = \begin{cases} V \ker(J - \lambda I)^i, & \text{if } \lambda \in \operatorname{spec}(sE - A) \\ \mathcal{W}_i = W \ker N^i, & \text{if } \lambda = \infty. \end{cases} \tag{2.2.17}$$

Suppose $\lambda \in \operatorname{spec}(sE - A)$.

Step 1a: We prove by induction that

$$\forall\, i \in \mathbb{N}_0 : \; \mathcal{G}_\lambda^i \;\subseteq\; V \ker(J - \lambda I)^i. \tag{2.2.18}$$

The claim is clear for $i = 0$. Suppose (2.2.18) holds for $i = k - 1$. Let $v_k \in \mathcal{G}_\lambda^k \setminus \{0\}$ and $v_{k-1} \in \mathcal{G}_\lambda^{k-1}$ such that $(A - \lambda E)v_k = Ev_{k-1}$. By Proposition 2.2.3 (ii) we may set

$$v_k = V\alpha + W\beta \quad \text{for unique} \quad \alpha \in \mathbb{C}^{n_1}, \; \beta \in \mathbb{C}^{n_2}\,.$$

By (2.2.7), $(A - \lambda E)v_k = Ev_{k-1}$ is equivalent to

$$AW(I - \lambda N)\beta = Ev_{k-1} + EV(\lambda I - J)\alpha\,,$$

and so, since by induction hypothesis

$$v_{k-1} \in \mathcal{G}_\lambda^{k-1} \subseteq V \ker(J - \lambda I)^{k-1} \subseteq \mathcal{V}^*,$$

we conclude

$$W(I - \lambda N)\beta \in A^{-1}(E\mathcal{V}^*) \overset{(2.2.1)}{=} \mathcal{V}^*.$$

Now Proposition 2.2.3 (ii) yields, since W has full column rank, $(I - \lambda N)\beta = 0$ and hence, since N is nilpotent, $\beta = 0$. It follows from $v_{k-1} \in V \ker(J - \lambda I)^{k-1}$ that there exists $u \in \mathbb{C}^{n_1}$ such that $v_{k-1} = Vu$ and $(J - \lambda I)^{k-1}u = 0$. Then $EV(J - \lambda I)\alpha = EVu$ and Proposition 2.2.3 (iii) gives, since V has full column rank, $(J - \lambda I)\alpha = u$. Therefore, $v_k = V\alpha$ and $(J - \lambda I)^k\alpha = 0$, hence $v_k \in V \ker(J - \lambda I)^k$ and this completes the proof of (2.2.18).

Step 1b: We prove by induction that

$$\forall\, i \in \mathbb{N}_0 : \; \mathcal{G}_\lambda^i \;\supseteq\; V \ker(J - \lambda I)^i. \tag{2.2.19}$$

The claim is clear for $i = 0$. Suppose (2.2.19) holds for $i = k-1$. Let $v_k \in \ker(J - \lambda I)^k$ and $v_{k-1} \in \ker(J-\lambda I)^{k-1}$ such that $(J - \lambda I)v_k = v_{k-1}$. Since EV has full column rank, this is equivalent to $EV(J - \lambda I)v_k = EVv_{k-1}$ which is, by invoking (2.2.7), equivalent to $(A - \lambda E)Vv_k = EVv_{k-1}$ and then the induction hypothesis yields $Vv_{k-1} \in \mathcal{G}_\lambda^{k-1}$, thus having $Vv_k \in \mathcal{G}_\lambda^k$. This proves (2.2.19) and completes the proof of (2.2.17) for finite eigenvalues.

Step 1c: The statement '$\mathcal{W}_i = W \ker N^i$ for all $i \in \mathbb{N}_0$' follows by Proposition 2.2.9, and '$\mathcal{G}_\infty^i = \mathcal{W}_i$ for all $i \in \mathbb{N}_0$' is clear from the definition.

Step 2: The Assertions (i)–(iv) follow immediately from (2.2.17) and the respective results of the classical eigenvalue theory, see for example [155, Sec. 12.5, 12.7] and [95, Sec. 4.6]. Assertion (v) is a consequence of Corollary 2.2.11 (i) and $\mathrm{am}(\infty) = n - \deg\big(\det(sE - A)\big) = n_2$. This completes the proof of the proposition. □

An immediate consequence of Proposition 2.2.17 and (2.2.17) is the following Theorem 2.2.18. We stress that our proof relies essentially on the relationship between the eigenspaces of $sE - A$ and the eigenspaces of $sI - J$ and $sI - N$ where J and N are as in (2.1.1). Alternatively, we could prove Theorem 2.2.18 by using chains and cyclic subspaces only, however the present proof via the Quasi-Weierstraß form is shorter.

Theorem 2.2.18 (Decomposition and basis of $\mathcal{V}^*$)**.**
Let $sE - A \in \mathbb{C}[s]^{n\times n}$ be regular, $\lambda_1, \ldots, \lambda_k$ be the pairwise distinct zeros of $\det(sE - A)$ and use the notation of Theorem 2.2.5. Then

$$\forall\, \lambda \in \{\lambda_1, \ldots, \lambda_k\}\ \forall\, j \in \{1, \ldots, \mathrm{gm}(\lambda)\}\ \exists\, n_{\lambda,j} \in \mathbb{N}\ \exists \text{ chain } \left(v_{\lambda,j}^1, v_{\lambda,j}^2, \ldots, v_{\lambda,j}^{n_{\lambda,j}}\right) \text{ at } \lambda :$$

$$\mathcal{G}_\lambda = \bigoplus_{j=1}^{\mathrm{gm}(\lambda)} \mathrm{im} \underbrace{\left[v_{\lambda,j}^1, \ldots, v_{\lambda,j}^{n_{\lambda,j}}\right]}_{=:V_{\lambda,j} \in \mathbb{C}^{n \times n_{\lambda,j}}}. \tag{2.2.20}$$

and

$$\boxed{\mathcal{V}^* = \mathcal{G}_{\lambda_1} \oplus \mathcal{G}_{\lambda_2} \oplus \ldots \oplus \mathcal{G}_{\lambda_k}} \quad \text{and} \quad \boxed{\mathcal{W}^* = \mathcal{G}_\infty}.$$

In Corollary 2.2.19 we show that the generalized eigenvectors of a regular matrix pencil $sE - A$ at the finite eigenvalues and at the infinite eigenvalue constitute a basis which transforms $sE - A$ into the well known Weierstraß canonical form. So the WCF can be viewed as a generalized Jordan canonical form.

Corollary 2.2.19 (Weierstraß canonical form)**.**
Let $sE - A \in \mathbb{C}[s]^{n\times n}$ *be regular,* $n_1 := \dim \mathcal{V}^*$*,* $n_2 := n - n_1$ *and* $\lambda_1, \dots, \lambda_k$ *be the pairwise distinct zeros of* $\det(sE - A)$*. Then we may choose*

$$\begin{aligned} V_f &:= \Big[V_{\lambda_1,1},\ \dots,\ V_{\lambda_1,\mathrm{gm}(\lambda_1)},\ V_{\lambda_2,1},\ \dots,\ V_{\lambda_2,\mathrm{gm}(\lambda_2)},\\ &\qquad\qquad\qquad\qquad \dots,\ V_{\lambda_k,1},\ \dots,\ V_{\lambda_k,\mathrm{gm}(\lambda_k)}\Big],\\ V_\infty &:= \Big[V_{\infty,1},\ \dots,\ V_{\infty,\mathrm{gm}(\infty)}\Big], \end{aligned}$$

where $V_{\lambda_i,j}$ *consists of a chain at* λ_i *as in* (2.2.20)*,* $j = 1, \dots, \mathrm{gm}(\lambda_i)$*,* $i = 1, \dots, k$*, resp. For any such* V_f*,* V_∞*, the matrices* $[V_f, V_\infty]$*,* $[EV_f, AV_\infty] \in \mathbb{C}^{n\times n}$ *are invertible and* $[EV_f, AV_\infty]^{-1}(sE-A)[V_f, V_\infty]$ *is in WCF.*

Proof: The existence of V_f and V_∞ satisfying the eigenvector conditions formulated in the corollary follows from Theorem 2.2.18. In view of (2.2.15), it follows from the definition of chains that $[EV_f, AV_\infty]^{-1}(sE - A)[V_f, V_\infty]$ is in the form (2.1.1) for some matrices $J \in \mathbb{C}^{n_1\times n_1}$ and $N \in \mathbb{C}^{n_2\times n_2}$ in Jordan canonical form and nilpotent N. □

Definition 2.2.20 (Canonical form)**.**
Given a group G, a set $\mathcal{S}$, and a group action $\alpha : G \times \mathcal{S} \to \mathcal{S}$ which defines an equivalence relation $s \overset{\alpha}{\sim} s'$, that is $\exists U \in G : \alpha(U, s) = s'$. Then a map $\gamma : \mathcal{S} \to \mathcal{S}$ is called a *canonical form for* α [43] if, and only if,

$$\forall\, s, s' \in \mathcal{S} \ :\ \gamma(s) \overset{\alpha}{\sim} s \quad \wedge \quad \Big[s \overset{\alpha}{\sim} s' \Leftrightarrow \gamma(s) = \gamma(s')\Big].$$

Therefore, the set $\mathcal{S}$ is divided into disjoint orbits (i.e., equivalence classes) and the mapping γ picks a unique representative in each equivalence class.

Remark 2.2.21 (QWF is not canonical)**.**
In the setup of equivalence of regular matrix pencils, using the notation from Definition 2.2.20, the group is $G = \mathbf{Gl}_n(\mathbb{K}) \times \mathbf{Gl}_n(\mathbb{K})$, the considered set is

$$\mathcal{S} = \left\{ \ [E, A] \in \left(\mathbb{K}^{n\times n}\right)^2 \ \middle|\ \det(sE - A) \in \mathbb{K}[s] \setminus \{0\} \ \right\}$$

and the group action $\alpha\big((S,T),[E,A]\big) = [SET, SAT]$ corresponds to $\cong$. The QWF from Theorem 2.2.5 does not provide a mapping γ. That means that the form (2.1.1) is not a unique representative within the equivalence class and hence the QWF is not a canonical form. The WCF however is a canonical form, if we prescribe the order of the eigenvalues and assume that the Jordan blocks corresponding to each eigenvalue (in $\mathbb{C}\sup\{\infty\}$) are arranged according to increasing size. This justifies the name Weierstraß canonical form.

2.3 Quasi-Kronecker form

In this section we derive the quasi-Kronecker form for general matrix pencils $sE - A \in \mathbb{K}[s]^{m\times n}$. In Subsection 2.3.1 we redefine the Wong sequences for this general setting and state the main results of this section: the quasi-Kronecker triangular form (QKTF) and the quasi-Kronecker form. Before we prove these results, some preliminary lemmas and an interim QK(T)F are derived in Subsection 2.3.2. The proofs of the main results are carried out in Subsection 2.3.3. In Subsection 2.3.4 we show that it is easy to obtain the KCF from the QKF and that moreover the complete KCF (except for the transformation matrices) can be obtained from the Wong sequences.

We have to admit that our proof of the KCF does not reach the elegance of the proof of Gantmacher [100], however Gantmacher does not provide any geometrical insight. On the other end of the spectrum, Armentano [7] uses the Wong sequences to obtain a result similar to the QKTF, however his approach is purely geometrical so that it is not directly possible to deduce the transformation matrices which are necessary to obtain the QKTF or QKF. Our result overcomes this drawback because it presents geometrical insights and, at the same time, is constructive.

We like to stress that the representation in this section is self-contained; results from Section 2.2 are not needed, only some standard results from linear algebra are required. The results of this section stem from two joint works with Stephan Trenn [40, 41].

2.3.1 Main results

Recall that, as a consequence of Proposition 2.2.3 and Theorem 2.2.5, for the Wong sequences (2.2.1) and (2.2.2) of a regular pencil $sE - A$ we have

$$\mathcal{V}^* \cap \mathcal{W}^* = \{0\}, \qquad E\mathcal{V}^* \cap A\mathcal{W}^* = \{0\},$$
$$\mathcal{V}^* + \mathcal{W}^* = \mathbb{K}^n, \qquad E\mathcal{V}^* + A\mathcal{W}^* = \mathbb{K}^n.$$

These properties do not hold anymore for a general matrix pencil $sE - A$, see Figure 2.1 for an illustration of the situation.

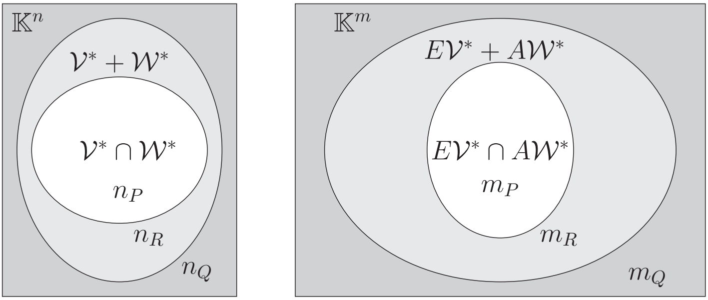


Figure 2.1: *The relationship of the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences of the matrix pencil $sE - A \in \mathbb{K}[s]^{m \times n}$ in the general case; the numbers $n_P, n_R, n_Q, m_P, m_R, m_Q \in \mathbb{N}_0$ denote the (difference of the) dimensions of the corresponding spaces.*

We are now ready to present our first main result which states that the knowledge of the spaces $\mathcal{V}^*$ and $\mathcal{W}^*$ is sufficient to obtain the *quasi-Kronecker triangular form (QKTF)*, which already captures most structural properties of the matrix pencil $sE - A$. With the help of the Wong sequences, Armentano [7] already obtained a similar result, however his aim was to obtain a triangular form where the diagonal blocks are in canonical form. Therefore, his result is more general than ours, however, the price is a more complicated proof and it is also not clear how to obtain the transformation matrices explicitly.

Theorem 2.3.1 (Quasi-Kronecker triangular form).
Let $sE - A \in \mathbb{K}[s]^{m \times n}$ and consider the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences as in (2.2.3). *Choose any full rank matrices $P_1 \in \mathbb{K}^{n \times n_P}$, $R_1^J \in \mathbb{K}^{n \times n_J}$, $R_1^N \in \mathbb{K}^{n \times n_N}$, $Q_1 \in \mathbb{K}^{n \times n_Q}$, $P_2 \in \mathbb{K}^{m \times m_P}$, $R_2^J \in \mathbb{K}^{m \times m_J}$,*

$R_2^N \in \mathbb{K}^{m\times m_N}$, $Q_2 \in \mathbb{K}^{m\times m_Q}$ *such that*

$$\begin{aligned}
\operatorname{im} P_1 &= \mathcal{V}^* \cap \mathcal{W}^*, & (\mathcal{V}^* \cap \mathcal{W}^*) \oplus \operatorname{im} R_1^J &= \mathcal{V}^*,\\
\mathcal{V}^* \oplus \operatorname{im} R_1^N &= \mathcal{V}^* + \mathcal{W}^*, & (\mathcal{V}^* + \mathcal{W}^*) \oplus \operatorname{im} Q_1 &= \mathbb{K}^n,\\
\operatorname{im} P_2 &= E\mathcal{V}^* \cap A\mathcal{W}^*, & (E\mathcal{V}^* \cap A\mathcal{W}^*) \oplus \operatorname{im} R_2^J &= E\mathcal{V}^*,\\
E\mathcal{V}^* \oplus \operatorname{im} R_2^N &= E\mathcal{V}^* + A\mathcal{W}^*, & (E\mathcal{V}^* + A\mathcal{W}^*) \oplus \operatorname{im} Q_2 &= \mathbb{K}^m.
\end{aligned}$$

Then it holds that $T_{\text{trian}} = [P_1, R_1^J, R_1^N, Q_1]$ *and* $S_{\text{trian}} = [P_2, R_2^J, R_2^N, Q_2]^{-1}$ *are invertible and* $S_{\text{trian}}(sE - A)T_{\text{trian}}$ *is in QKTF* (2.1.4)*.*

Remark 2.3.2.

(i) The sizes of the blocks in (2.1.4) are uniquely determined by the matrix pencil $sE-A$ because they only depend on the subspaces constructed by the Wong sequences and not on the choice of bases thereof. It is also possible that $m_P = 0$ (or $n_Q = 0$) which means that there are matrices with no rows (or no columns). On the other hand, if $n_P = 0$, $n_J = 0$, $n_N = 0$ or $m_Q = 0$ then the P-blocks, J-blocks, N-block or Q-blocks are not present at all. Furthermore, it is easily seen that if $sE-A$ fulfills (i), (ii), (iii) or (iv) itself, then $sE - A$ is already in QKTF with $T_{\text{trian}} = P_1 = I$, $T_{\text{trian}} = R_1^J = I$, $T_{\text{trian}} = R_1^N = I$ or $T_{\text{trian}} = Q_1 = I$, and $S_{\text{trian}} = P_2^{-1} = I$, $S_{\text{trian}} = (R_2^J)^{-1} = I$, $S_{\text{trian}} = (R_2^N)^{-1} = I$ or $S_{\text{trian}} = Q_2^{-1} = I$.

(ii) In Theorem 2.3.1 the special choice of $R_2^J = ER_1^J$ and $R_2^N = AR_1^N$, which is feasible due to Steps 2 and 3 of the proof of Theorem 2.3.1 (see Subsection 2.3.3), yields that (2.1.4) simplifies to

$$s\begin{bmatrix} E_P & 0 & E_{PN} & E_{PQ}\\ 0 & I_{n_J} & E_{JN} & E_{JQ}\\ 0 & 0 & N & E_{NQ}\\ 0 & 0 & 0 & E_Q\end{bmatrix} - \begin{bmatrix} A_P & A_{PJ} & 0 & A_{PQ}\\ 0 & A_J & 0 & A_{JQ}\\ 0 & 0 & I_{n_N} & A_{NQ}\\ 0 & 0 & 0 & A_Q\end{bmatrix},$$

where N is nilpotent.

(iii) From Lemma 2.3.7 (see Subsection 2.3.2) we know that $E(\mathcal{V}^* \cap \mathcal{W}^*) = E\mathcal{V}^* \cap A\mathcal{W}^* = A(\mathcal{V}^* \cap \mathcal{W}^*)$, hence

$$E\mathcal{V}^* \cap A\mathcal{W}^* = E(\mathcal{V}^* \cap \mathcal{W}^*) + A(\mathcal{V}^* \cap \mathcal{W}^*).$$

Furthermore, due to (2.2.4),

$$E\mathcal{V}^* + A\mathcal{W}^* = E(\mathcal{V}^* + \mathcal{W}^*) + A(\mathcal{V}^* + \mathcal{W}^*).$$

Hence the subspace pairs $(\mathcal{V}^* \cap \mathcal{W}^*, E\mathcal{V}^* \cap A\mathcal{W}^*)$ and $(\mathcal{V}^* + \mathcal{W}^*, E\mathcal{V}^* + A\mathcal{W}^*)$ are reducing subspaces of the matrix pencil $sE - A$ in the sense of [235] and are in fact the minimal and maximal reducing subspaces.

The QKTF is already useful for the analysis of the matrix pencil $sE - A$ and the associated DAE $\frac{\mathrm{d}}{\mathrm{d}t}Ex = Ax + f$. However, a complete decoupling of the different parts, i.e., a block diagonal form, is more satisfying from a theoretical viewpoint and is also a necessary step to obtain the KCF as a corollary. In the next result we show that we can transform any matrix pencil $sE - A$ into a block diagonal form, which we call *quasi-Kronecker form (QKF)* because all the important features of the KCF are captured. In fact, it turns out that the diagonal blocks of the QKTF (2.1.4) already are the diagonal blocks of the QKF.

Theorem 2.3.3 (Quasi-Kronecker form).
Using the notation from Theorem 2.3.1 the following equations are solvable for matrices $F_1, F_2, G_1, G_2, H_1, H_2, K_1, K_2, L_1, L_2, M_1, M_2$ of appropriate size:

$$\begin{aligned} 0 &= \begin{bmatrix} E_{JQ} \\ E_{NQ} \end{bmatrix} + \begin{bmatrix} E_J & E_{JN} \\ 0 & E_N \end{bmatrix} \begin{bmatrix} G_1 \\ F_1 \end{bmatrix} + \begin{bmatrix} G_2 \\ F_2 \end{bmatrix} E_Q \\ 0 &= \begin{bmatrix} A_{JQ} \\ A_{NQ} \end{bmatrix} + \begin{bmatrix} A_J & A_{JN} \\ 0 & A_N \end{bmatrix} \begin{bmatrix} G_1 \\ F_1 \end{bmatrix} + \begin{bmatrix} G_2 \\ F_2 \end{bmatrix} A_Q \end{aligned} \tag{2.3.1a}$$

$$\begin{aligned} 0 &= (E_{PQ} + E_{PN}F_1 + E_{PJ}G_1) + E_P K_1 + K_2 E_Q \\ 0 &= (A_{PQ} + A_{PN}F_1 + A_{PJ}G_1) + A_P K_1 + K_2 A_Q \end{aligned} \tag{2.3.1b}$$

$$\begin{aligned} 0 &= E_{JN} + E_J H_1 + H_2 E_N \\ 0 &= A_{JN} + A_J H_1 + H_2 A_N \end{aligned} \tag{2.3.1c}$$

$$\begin{aligned} 0 &= [E_{PJ}, E_{PN}] \begin{bmatrix} I & H_1 \\ 0 & I \end{bmatrix} + E_P[M_1, L_1] + [M_2, L_2] \begin{bmatrix} E_J & 0 \\ 0 & E_N \end{bmatrix} \\ 0 &= [A_{PJ}, A_{PN}] \begin{bmatrix} I & H_1 \\ 0 & I \end{bmatrix} + A_P[M_1, L_1] + [M_2, L_2] \begin{bmatrix} A_J & 0 \\ 0 & A_N \end{bmatrix} \end{aligned} \tag{2.3.1d}$$

and for any such matrices let

$$S := \begin{bmatrix} I & -M_2 & -L_2 & -K_2 \\ 0 & I & -H_2 & -G_2 \\ 0 & 0 & I & -F_2 \\ 0 & 0 & 0 & I \end{bmatrix}^{-1} S_{\text{trian}}, \qquad T := T_{\text{trian}} \begin{bmatrix} I & M_1 & L_1 & K_1 \\ 0 & I & H_1 & G_1 \\ 0 & 0 & I & F_1 \\ 0 & 0 & 0 & I \end{bmatrix}.$$

Then S and T are invertible and $S(sE-A)T$ is in QKF (2.1.5)*, where the block diagonal entries are the same as for the QKTF* (2.1.4)*. In particular, the QKF (without the transformation matrices S and T) can be obtained with only the Wong sequences (i.e., without solving* (2.3.1)*). Furthermore, the QKF* (2.1.5) *is unique in the following sense*

$$(E,A) \cong (E',A') \;\Leftrightarrow\; (E_P,A_P) \cong (E'_P,A'_P),\ (E_J,A_J) \cong (E'_J,A'_J),\ (E_N,A_N) \cong (E'_N,A'_N),\ (E_Q,A_Q) \cong (E'_Q,A'_Q), \quad (2.3.2)$$

where $E'_P, A'_P, E'_J, A'_J, E'_N, A'_N, E'_P, A'_P$ are the corresponding blocks of the QKF of the matrix pencil $sE'-A'$.

2.3.2 Preliminaries and interim QKF

In this subsection we provide the preliminary results for the proofs of Theorems 2.3.1 and 2.3.3. To this end, we prove important results concerning the Wong sequences, singular chains, the KCF for full rank pencils and the solvability of Sylvester equations. As a main step towards the QKF we derive the interim quasi-Kronecker (triangular) form (IQK(T)F). In the latter form we do not decouple the regular part of the matrix pencil. This is already a useful and interesting result in its own right, because the decoupling into three parts which have the solution properties (cf. also Section 2.4) 'existence, but non-uniquess' (underdetermined part), 'existence and uniqueness' (regular part) and 'uniqueness, but possible non-existence' (overdetermined part) seems very intuitive.

Standard results from linear algebra

Lemma 2.3.4 (Orthogonal complements and (pre-)images)**.**
For any matrix $M \in \mathbb{K}^{p\times q}$ we have:

(i) *for all subspaces $\mathcal{S} \subseteq \mathbb{K}^p$ it holds that $(M^{-1}\mathcal{S})^\perp = M^*(\mathcal{S}^\perp)$.*

(ii) *for all subspaces $\mathcal{S} \subseteq \mathbb{K}^q$ it holds that $(M\mathcal{S})^\perp = M^{-*}(\mathcal{S}^\perp)$.*

Proof: Property (i) is shown e.g. in [16, Property 3.1.3]. Property (ii) follows from (i) for $M^*, \mathcal{S}^\perp$ instead of $M, \mathcal{S}$ and then taking orthogonal complements. □

Lemma 2.3.5 (Rank of matrices)**.**
Let $A, B \in \mathbb{K}^{m\times n}$ *with* $\operatorname{im} B \subseteq \operatorname{im} A$*. Then for almost all* $c \in \mathbb{K}$*:*

$$\operatorname{rk} A = \operatorname{rk}(A + cB),$$

or, equivalently,

$$\operatorname{im} A = \operatorname{im}(A + cB).$$

In fact, $\operatorname{rk} A > \operatorname{rk}(A + cB)$ *can only hold for at most* $r = \operatorname{rk} A$ *many values of* c*.*

Proof: Consider the Smith form [222] of A:

$$UAV = \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix}$$

with invertible matrices $U \in \mathbb{K}^{m\times m}$ and $V \in \mathbb{K}^{n\times n}$ and $\Sigma_r = \operatorname{diag}(\sigma_1, \sigma_2, \ldots, \sigma_r)$, $\sigma_i \in \mathbb{K} \setminus \{0\}$, $r = \operatorname{rk} A$. Write

$$UBV = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix},$$

where $B_{11} \in \mathbb{K}^{r\times r}$. Since $\operatorname{im} B \subseteq \operatorname{im} A$ it follows that $B_{21} = 0$ and $B_{22} = 0$. Hence we obtain the following implications:

$$\begin{aligned} \operatorname{rk}(A + cB) < \operatorname{rk} A \;&\Rightarrow\; \operatorname{rk}[\Sigma_r + cB_{11}, cB_{12}] < \operatorname{rk}[\Sigma_r, 0] = r \\ &\Rightarrow\; \operatorname{rk}(\Sigma_r + cB_{11}) < r \Rightarrow\; \det(\Sigma_r + cB_{11}) = 0. \end{aligned}$$

Since $\det(\Sigma_r + cB_{11})$ is a polynomial in c of degree at most r but not the zero polynomial (since $\det(\Sigma_r) \neq 0$) it can have at most r zeros. This proves the claim. □

Without proof we record the following well-known result.

Lemma 2.3.6 (Dimension formulae)**.**
Let $\mathcal{S} \subseteq \mathbb{K}^n$ *be any linear subspace and* $M \in \mathbb{K}^{m\times n}$*. Then*

$$\dim M\mathcal{S} = \dim \mathcal{S} - \dim(\ker M \cap \mathcal{S}).$$

Furthermore, for any two linear subspaces $\mathcal{S}, \mathcal{T}$ *of* $\mathbb{K}^n$ *we have*

$$\dim(\mathcal{S} + \mathcal{T}) = \dim \mathcal{S} + \dim \mathcal{T} - \dim(\mathcal{S} \cap \mathcal{T}).$$

The Wong sequences

The next lemma highlights an important property of the intersection of the limits of the Wong sequences.

Lemma 2.3.7 (Property of $\mathcal{V}^* \cap \mathcal{W}^*$)**.**
Let $sE - A \in \mathbb{K}[s]^{m\times n}$ and $\mathcal{V}^*$, $\mathcal{W}^*$ be the limits of the Wong sequences as in (2.2.3)*. Then*

$$E(\mathcal{V}^* \cap \mathcal{W}^*) = E\mathcal{V}^* \cap A\mathcal{W}^* = A(\mathcal{V}^* \cap \mathcal{W}^*).$$

Proof: Clearly, invoking (2.2.4),

$$\begin{aligned} E(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap E\mathcal{W}^* \subseteq E\mathcal{V}^* \cap A\mathcal{W}^* \\ \text{and} \quad A(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq A\mathcal{V}^* \cap A\mathcal{W}^* \subseteq E\mathcal{V}^* \cap A\mathcal{W}^*, \end{aligned}$$

hence it remains to show the converse subspace relationship. To this end choose $x \in E\mathcal{V}^* \cap A\mathcal{W}^*$, which implies existence of $v \in \mathcal{V}^*$ and $w \in \mathcal{W}^*$ such that

$$Ev = x = Aw,$$

hence

$$v \in E^{-1}\{Aw\} \subseteq E^{-1}(A\mathcal{W}^*) = \mathcal{W}^*, \quad w \in A^{-1}\{Ev\} \subseteq A^{-1}(E\mathcal{V}^*) = \mathcal{V}^*.$$

Therefore $v, w \in \mathcal{V}^* \cap \mathcal{W}^*$ and $x = Ev \in E(\mathcal{V}^* \cap \mathcal{W}^*)$ as well as $x = Aw \in A(\mathcal{V}^* \cap \mathcal{W}^*)$ which concludes the proof. □

For the proof of the main result we briefly consider the Wong sequences of the (conjugate) transposed matrix pencil $sE^* - A^*$; these are connected to the original Wong sequences as follows.

Lemma 2.3.8 (Wong-sequences of the transposed matrix pencil)**.**
Consider a matrix pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ with limits of the Wong sequences $\mathcal{V}^*$ and $\mathcal{W}^*$ as in (2.2.3)*. Denote by $\widehat{\mathcal{V}}^*$ and $\widehat{\mathcal{W}}^*$ the limits of the Wong sequences of the (conjugate) transposed matrix pencil $sE^* - A^*$. Then the following holds*

$$\widehat{\mathcal{W}}^* = (E\mathcal{V}^*)^\perp \quad \text{and} \quad \widehat{\mathcal{V}}^* = (A\mathcal{W}^*)^\perp.$$

Proof: We show that for all $i \in \mathbb{N}_0$

$$(E\mathcal{V}_i)^\perp = \widehat{\mathcal{W}}_{i+1} \quad \text{and} \quad (A\mathcal{W}_i)^\perp = \widehat{\mathcal{V}}_i, \tag{2.3.3}$$

from which the claim follows. For $i = 0$ this follows from

$$(E\mathcal{V}_0)^\perp = (\operatorname{im} E)^\perp = \ker E^* = E^{-*}(A^*\{0\}) = \widehat{\mathcal{W}}_1$$

and

$$(A\mathcal{W}_0)^\perp = \{0\}^\perp = \mathbb{R}^m = \widehat{\mathcal{V}}_0.$$

Now suppose that (2.3.3) holds for some $i \in \mathbb{N}_0$. Then

$$\begin{aligned}(E\mathcal{V}_{i+1})^\perp &= \big(EA^{-1}(E\mathcal{V}_i)\big)^\perp \\ &\overset{\text{Lem. 2.3.4 (ii)}}{=} E^{-*}(A^{-1}(E\mathcal{V}_i))^\perp \\ &\overset{\text{Lem. 2.3.4 (i)}}{=} E^{-*}\left(A^\top (E\mathcal{V}_i)^\perp\right) \\ &= E^{-*}\left(A^\top \widehat{\mathcal{W}}_{i+1}\right) = \widehat{\mathcal{W}}_{i+2}.\end{aligned}$$

Analogously it follows $(A\mathcal{W}_{i+1})^\perp = \widehat{\mathcal{V}}_{i+1}$, hence we have inductively shown (2.3.3). □

Singular chains

In this paragraph we introduce the notion of *singular chains* for matrix pencils. This notion is inspired by the theory of linear relations, see [214], where they are a vital tool for analyzing the structure of linear relations. They also play an important role in former works on the KCF, see e.g. [100, Ch. XII] and [7]; however in these works only singular chains of minimal length are considered. We use them here to determine the structure of the intersection $\mathcal{V}^* \cap \mathcal{W}^*$ of the limits of the Wong sequences.

Definition 2.3.9 (Singular chain)**.**
Let $sE - A \in \mathbb{K}[s]^{m\times n}$. For $k \in \mathbb{N}_0$ the tuple $(x_0, \ldots, x_k) \in (\mathbb{K}^n)^{k+1}$ is called a *singular chain of the matrix pencil* $sE - A$ if, and only if,

$$0 = Ax_0,\ Ex_0 = Ax_1,\ \ldots,\ Ex_{k-1} = Ax_k,\ Ex_k = 0$$

or, equivalently, if the polynomial vector $x(s) = x_0 + x_1 s + \ldots + x_k s^k \in \mathbb{K}[s]^n$ satisfies $(sE - A)x(s) = 0$.

Note that with every singular chain $(x_0, x_1, \dots, x_k)$ also the tuple $(0, \dots, 0, x_0, \dots, x_k, 0, \dots, 0)$ is a singular chain of $sE - A$. Furthermore, with every singular chain, each scalar multiple is a singular chain and for two singular chains of the same length the sum of both is a singular chain. A singular chain $(x_0, \dots, x_k)$ is called *linearly independent* if the vectors $x_0, \dots, x_k$ are linearly independent.

Lemma 2.3.10 (Linear independency of singular chains)**.**
Let $sE - A \in \mathbb{K}[s]^{m \times n}$. For every non-trivial singular chain $(x_0, x_1, \dots, x_k)$, $k \in \mathbb{N}_0$, of $sE - A$ there exists $\ell \in \mathbb{N}_0$, $\ell \leq k$, and a linearly independent singular chain $(y_0, y_1, \dots, y_\ell)$ with $\operatorname{span}\{x_0, x_1, \dots, x_k\} = \operatorname{span}\{y_0, y_1, \dots, y_\ell\}$.

Proof: This result is an extension of [214, Lem. 3.1] and our proof resembles some of their ideas.

If $(x_0, x_1, \dots, x_k)$ is already a linearly independent singular chain nothing is to show, therefore, assume existence of a minimal $\ell \in \{0, 1, \dots, k-1\}$ such that $x_{\ell+1} = \sum_{i=0}^{\ell} \alpha_i x_i$ for some $\alpha_i \in \mathbb{K}$, $i = 0, \dots, \ell$. Consider the chains

$$\begin{array}{l}
\alpha_0\,(0,\ 0, \dots, 0,\ 0,\ x_0,\ x_1,\ \dots,\ x_\ell,\ x_{\ell+1},\ \dots,\ x_{k-1}, x_k)\\
\alpha_1\,(0,\ 0, \dots, 0,\ x_0,\ x_1,\ \dots,\ x_\ell,\ x_{\ell+1},\ \dots,\ x_{k-1},\ x_k,\ 0)\\
\alpha_2\,(0,\ 0, \dots, x_0,\ x_1,\ \dots,\ x_\ell,\ x_{\ell+1},\ \dots,\ x_{k-1},\ x_k,\ 0,\ 0)\\
\qquad\vdots\\
\alpha_{\ell-1}\,(0,\ x_0,\ x_1,\ \dots,\ x_{\ell-2}, x_{\ell-1},\ x_\ell,\ x_{\ell+1},\ \dots\ x_k,\ 0,\ \dots,\ 0)\\
\alpha_\ell\,(x_0, x_1, \dots, x_{\ell-2}, x_{\ell-1},\ x_\ell,\ x_{\ell+1},\ \dots\ x_k,\ 0,\ \dots,\ 0,\ 0)
\end{array}$$

and denote its sum by $(z_0, z_1, \dots, z_{k+\ell})$. Note that by construction $z_\ell = \sum_{i=0}^{\ell} \alpha_i x_i = x_{\ell+1}$. Now consider the singular chain

$$(v_0, v_1, \dots, v_{k+\ell+1}) := (x_0, x_1, \dots, x_k, 0, \dots, 0) - (0, z_0, z_1, \dots, z_{\ell+k})$$

which has the property that $v_{\ell+1} = x_{\ell+1} - z_\ell = 0$. In particular $(v_0, v_1, \dots, v_\ell)$ and $(v_{\ell+2}, v_{\ell+3} \dots, v_{k+\ell+1})$ are both singular chains. Furthermore, (we abbreviate $\alpha_i I$ with α_i)

$$\begin{pmatrix} v_0 \\ v_1 \\ v_2 \\ \vdots \\ v_k \\ v_{k+1} \\ \vdots \\ v_{k+\ell+1} \end{pmatrix} = \left[\begin{array}{cccccccccc} I & 0 & & & \cdots & & & & & 0 \\ -\alpha_\ell & I & 0 & & & \cdots & & & & 0 \\ -\alpha_{\ell-1} & -\alpha_\ell & I & 0 & & & \cdots & & & 0 \\ & & & \ddots & & & & & & \\ -\alpha_1 & -\alpha_2 & \cdots & -\alpha_\ell & I & & & & & \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \cdots & -\alpha_\ell & I & & & & \\ 0 & -\alpha_0 & -\alpha_1 & -\alpha_3 & \cdots & -\alpha_\ell & I & & & \\ & & \ddots & & & & & \ddots & & \\ 0 & \cdots & 0 & -\alpha_0 & -\alpha_1 & -\alpha_3 & \cdots & -\alpha_\ell & & I \end{array}\right] \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_k \\ 0 \\ \vdots \\ 0 \end{pmatrix},$$

hence $\operatorname{span}\{v_0, v_1, \ldots, v_{k+\ell+1}\} = \operatorname{span}\{x_0, x_1, \ldots, x_k\} = \operatorname{span}\{v_0, v_1, \ldots, v_k\}$. In particular

$$\operatorname{span}\{v_{k+1}, v_{k+2}, \ldots, v_{k+\ell+1}\} \subseteq \operatorname{span}\{v_0, v_1, \ldots, v_k\},$$

hence, by applying Lemma 2.3.5, there exists $c \neq 0$ such that (note that $\ell < k$)

$$\begin{aligned} &\operatorname{im}[v_0, v_1, \ldots, v_k] \\ &\qquad = \operatorname{im}([v_0, v_1, \ldots, v_k] + c\,[v_{k+1}, v_{k+2}, \ldots, v_{k+\ell+1}, 0, \ldots, 0]). \end{aligned} \tag{2.3.4}$$

Therefore, the singular chain

$$\begin{aligned} &(w_0, w_1, \ldots, w_{k-1}) \\ &\quad := c\,(v_{\ell+2}, \ldots, v_k, v_{k+1}, v_{k+2}, \ldots, v_{k+\ell+1}) + (0, \ldots, 0, v_0, v_1, \ldots, v_\ell) \end{aligned}$$

has the property

$$\begin{aligned} & \operatorname{span}\{w_0, w_1, \ldots, w_{k-1}\} \\ = \; & \operatorname{span}\{v_{\ell+2}, v_{\ell+3}, \ldots, v_k\} \\ & + \operatorname{span}\{cv_{k+1} + v_0, cv_{k+2} + v_1, \ldots, cv_{k+\ell+1} + v_\ell\} \\ \overset{v_{\ell+1}=0}{=} \; & \operatorname{im}([v_0, v_1, \ldots, v_k] + c\,[v_{k+1}, v_{k+2}, \ldots, v_{k+\ell+1}, 0, \ldots, 0]) \\ \overset{(2.3.4)}{=} \; & \operatorname{im}[v_0, v_1, \ldots, v_k] \\ = \; & \operatorname{span}\{v_0, v_1, \ldots, v_k\} = \operatorname{span}\{x_0, x_1, \ldots, x_k\}. \end{aligned}$$

Altogether, we have obtained a shorter singular chain which spans the same subspace as the original singular chain. Repeating this procedure until one obtains a linearly independent singular chain finishes the proof. □

Corollary 2.3.11 (Basis of the singular chain subspace).
Consider a matrix pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ and let the singular chain subspace *be given by*

$$\mathcal{K} := \left\{ x \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists\, k, i \in \mathbb{N}_0\ \exists\, \textit{sing. chain}\ (x_0, \dots, x_{i-1}, x = x_i, x_{i+1}, \\ \dots, x_k) \in (\mathbb{K}^n)^{k+1} \end{array} \right\},$$

i.e., $\mathcal{K}$ is the set of all vectors x appearing somewhere in some singular chain of $sE - A$. Then there exists a linearly independent singular chain $(x_0, x_1, \dots, x_k)$ of $sE - A$ such that

$$\mathcal{K} = \operatorname{span}\{x_0, \dots, x_k\}.$$

Proof: First note that $\mathcal{K}$ is indeed a linear subspace of $\mathbb{K}^n$, since the scalar multiple of every chain is also a chain and the sum of two chains (extending the chains appropriately with zero vectors) is again a chain. Let $y^0, y^1, \dots, y^\ell$ be any basis of $\mathcal{K}$. By the definition of $\mathcal{K}$, for each $i = 0, 1, \dots, \ell$ there exist chains $(y_0^i, y_1^i, \dots, y_{k_i}^i)$ which contain y^i. Let $(v_0, v_1, \dots, v_{\hat{k}})$ with $\hat{k} = k_0 + k_1 + \dots + k_\ell$ be the chain which results by concatenating the chains $(y_0^i, y_1^i, \dots, y_{k_i}^i)$. Clearly, $\operatorname{span}\{v_0, \dots, v_{\hat{k}}\} = \mathcal{K}$, hence Lemma 2.3.10 yields the claim. □

The following result can, in substance, be found in [7]. However, the proof therein is difficult to follow, involving quotient spaces and additional sequences of subspaces. We aim for a presentation which is more straightforward and simpler.

Lemma 2.3.12 (Singular chain subspace and the Wong sequences).
Consider a matrix pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ with the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences as in (2.2.3). *Let the singular chain subspace $\mathcal{K}$ be given as in Corollary 2.3.11, then*

$$\mathcal{V}^* \cap \mathcal{W}^* = \mathcal{K}.$$

Proof: *Step 1:* We show $\mathcal{K} \subseteq \mathcal{V}^* \cap \mathcal{W}^*$. Let $(x_0, \dots, x_k)$ be a singular chain. Clearly we have $x_0 \in A^{-1}(E\{0\}) = \ker A \subseteq \mathcal{V}^*$ and $x_k \in E^{-1}(A\{0\}) = \ker E \subseteq \mathcal{W}^*$, hence inductively we have, for $i = 0, 1, \dots, k-1$ and $j = k, k-1, \dots, 1$

$$x_{i+1} \in A^{-1}(E\{x_i\}) \subseteq A^{-1}(E\mathcal{V}^*) = \mathcal{V}^*$$
$$\text{and}\quad x_{j-1} \in E^{-1}(A\{x_j\}) \subseteq E^{-1}(A\mathcal{W}^*) = \mathcal{W}^*.$$

Therefore,

$$x_0, \dots, x_k \in \mathcal{V}^* \cap \mathcal{W}^*.$$

Step 2: We show $\mathcal{V}^* \cap \mathcal{W}^* \subseteq \mathcal{K}$. Let $x \in \mathcal{V}^* \cap \mathcal{W}^*$, in particular, $x \in \mathcal{W}^* = \mathcal{W}_{l^*}$ for some $l^* \in \mathbb{N}_0$. Hence there exists $x_1 \in \mathcal{W}_{l^*-1}, x_2 \in \mathcal{W}_{l^*-2}, \dots, x_{l^*} \in \mathcal{W}_0 = \{0\}$, such that, for $x_0 := x$,

$$Ex_0 = Ax_1,\ Ex_1 = Ax_2,\ \dots,\ Ex_{l^*-1} = Ax_{l^*}, Ex_{l^*} = 0.$$

Furthermore, since, by Lemma 2.3.7, $E(\mathcal{V}^* \cap \mathcal{W}^*) = A(\mathcal{V}^* \cap \mathcal{W}^*)$ there exist $x_{-1}, x_{-2}, \dots,\ x_{-(l^*+1)} \in \mathcal{V}^* \cap \mathcal{W}^*$ such that

$$Ax_0 = Ex_{-1},\ Ax_{-1} = Ex_{-2},\ \dots,\ Ax_{-(l^*-1)} = Ex_{-l^*},\ Ax_{-l^*} = Ex_{-(l^*+1)}.$$

Let $\tilde{x}_{-(l^*+1)} := -x_{-(l^*+1)} \in \mathcal{V}^* \cap \mathcal{W}^* \subseteq \mathcal{W}^*$ then (with the same argument as above) there exist $\tilde{x}_{-l^*},\ \tilde{x}_{-(l^*-1)},\ \dots,\ \tilde{x}_{-1} \in \mathcal{W}^*$ such that

$$E\tilde{x}_{-(l^*+1)} = A\tilde{x}_{-l^*},\ E\tilde{x}_{-l^*} = A\tilde{x}_{-(l^*-1)} \dots,\ E\tilde{x}_{-2} = A\tilde{x}_{-1},\ E\tilde{x}_{-1} = 0,$$

and thus, defining $\hat{x}_{-i} = x_{-i} + \tilde{x}_{-i}$, for $i = 1, \dots, l^* + 1$, we have $\hat{x}_{-(l^*+1)} = 0$ and we get

$$0 = E\hat{x}_{-(l^*+1)} = A\hat{x}_{-l^*},\ E\hat{x}_{-l^*} = A\hat{x}_{-(l^*-1)},\\ \dots,\ E\hat{x}_{-2} = A\hat{x}_{-1},\ E\hat{x}_{-1} = Ex_{-1} = Ax_0.$$

This shows that $(\hat{x}_{-l^*}, \hat{x}_{-(l^*-1)}, \dots, \hat{x}_{-1}, x_0, x_1, \dots, x_{l^*})$ is a singular chain and $x = x_0 \in \mathcal{K}$. □

The last result in this paragraph relates singular chains with the column rank of the matrix pencil $sE - A$.

Lemma 2.3.13 (Column rank deficit implies singular chains)**.**
Let $sE - A \in \mathbb{K}[s]^{m \times n}$. If $\operatorname{rk}_{\mathbb{C}}(\lambda E - A) < n$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$, then there exists a non-trivial singular chain of $sE - A$.

Proof: It suffices to observe that Definition 2.3.9 coincides (modulo a reversed indexing) with the notion of singular chains in [214] applied to the linear relation $E^{-1}A := \{\ (x, y) \in \mathbb{K}^n \times \mathbb{K}^n \mid Ax = Ey\ \}$. Then the claim follows for $\mathbb{K} = \mathbb{C}$ from [214, Thm. 4.4]. The main idea of the proof there is to choose any $m + 1$ different eigenvalues and corresponding eigenvectors. This is also possible for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{Q}$, hence the proof in [214] is also valid for $\mathbb{K} = \mathbb{R}$ and $\mathbb{K} = \mathbb{Q}$. □

KCF for full rank pencils

In order to derive the KCF as a corollary of the QKF and also for the proof of the solvability of (2.3.1) we need the following lemma, which shows how to obtain the KCF for the special case of full rank pencils.

Lemma 2.3.14 (KCF of full rank rectangular pencil, $m < n$).
Let $sE - A \in \mathbb{K}[s]^{m\times n}$ be such that $m < n$ and let $l := n - m$. Then $\operatorname{rk}_\mathbb{C}(\lambda E - A) = m$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$ if, and only if, there exist numbers $\varepsilon_1, \ldots, \varepsilon_l \in \mathbb{N}_0$ and matrices $S \in \mathbf{Gl}_m(\mathbb{K})$, $T \in \mathbf{Gl}_n(\mathbb{K})$ such that

$$S(sE - A)T = \operatorname{diag}(\mathcal{P}_{\varepsilon_1}(s), \ldots, \mathcal{P}_{\varepsilon_l}(s)),$$

where $\mathcal{P}_\varepsilon(s)$, $\varepsilon \in \mathbb{N}_0$, is as in Definition 2.1.7.

Proof: Sufficiency is clear, hence it remains to show necessity.

If $m = 0$ and $n > 0$, then nothing is to show since $sE - A$ is already in the 'diagonal form' with $\varepsilon_1 = \varepsilon_2 = \ldots = \varepsilon_l = 0$. Hence assume $m > 0$ in the following. The main idea is to reduce the problem to a smaller pencil $sE' - A' \in \mathbb{K}[s]^{m'\times n'}$ with $\operatorname{rk}_\mathbb{C}(\lambda E' - A') = m' < n' < n$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$. Then we can inductively use the transformation to the desired block diagonal structure for the smaller pencil to obtain the block diagonal structure for the original pencil.

By assumption E does not have full column rank, hence there exists a column operation $T_1 \in \mathbf{Gl}_n(\mathbb{K})$ such that

$$ET_1 = \begin{bmatrix} 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{bmatrix}.$$

There are two cases now: Either the first column of AT_1 is zero or not. We consider the two cases separately.

Case 1: The first column of AT_1 is zero. Let $ET_1 =: [0, E']$ and $AT_1 =: [0, A']$. Then, clearly, $\operatorname{rk}_\mathbb{C}(\lambda E - A) = \operatorname{rk}_\mathbb{C}(\lambda E' - A') = m' := m$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$. Furthermore, with $n' := n - 1$, it follows that $n' \geq m'$. Seeking a contradiction, assume $n' = m'$. Then the full rank matrix E' is square and hence invertible. Let $\lambda \in \mathbb{C}$ be any eigenvalue of the matrix $E'^{-1}A'$, thus $0 = \det(\lambda I - E'^{-1}A') = \det(E')^{-1}\det(\lambda E' - A')$, hence $\operatorname{rk}_\mathbb{C}(\lambda E' - A') < m'$, a contradiction. Altogether, this shows that $sE' - A' \in \mathbb{K}[s]^{m'\times n'}$ is a smaller pencil which satisfies the assumption of the lemma, hence we can inductively use the result of the lemma for $sE' - A'$ with transformation matrices S' and T'. Let $S := S'$ and $T := T_1 \left[\begin{smallmatrix} 1 & 0 \\ 0 & T' \end{smallmatrix}\right]$, then $S(sE - A)T$ has the desired block diagonal structure which coincides with the block structure of $sE' - A'$ apart from one additional $\mathcal{P}_0$ block.

Case 2: The first column of AT_1 is not zero. Then there exists a row operation $S_1 \in \mathbf{Gl}_m(\mathbb{K})$ such that

$$S_1(AT_1) = \begin{bmatrix} 1 & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Since E has full row rank, the first row of S_1ET_1 cannot be the zero row, hence there exists a second column operation $T_2 \in \mathbf{Gl}_n(n)$ which does not change the first column such that

$$(S_1ET_1)T_2 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Now let $T_3 \in \mathbf{Gl}_n(\mathbb{K})$ be a column operation which adds multiples of the first column to the remaining columns such that

$$(S_1AT_1T_2)T_3 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & * & * & \cdots & * \\ \vdots & \vdots & & & \vdots \\ 0 & * & * & \cdots & * \end{bmatrix}.$$

Since the first column of $S_1ET_1T_2$ is zero, the column operation T_3 has no effect on the matrix $S_1ET_1T_2$. Let

$$S_1ET_1T_2T_3 =: \left[\begin{array}{c|cccc} 0 & 1 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & E' & & \\ 0 & & & & \end{array}\right], \quad S_1AT_1T_2T_3 =: \left[\begin{array}{c|cccc} 1 & 0 & 0 & \cdots & 0 \\ \hline 0 & & & & \\ \vdots & & A' & & \\ 0 & & & & \end{array}\right],$$

with $sE' - A' \in \mathbb{K}[s]^{m' \times n'}$ and $m' := m - 1$, $n' := n - 1$, in particular $m' < n'$. Seeking a contradiction, assume $\operatorname{rk}_{\mathbb{C}} \lambda E' - A' < m'$ for some

$\lambda \in \mathbb{C} \cup \{\infty\}$. If $\lambda = \infty$ then this implies that E' does not have full row rank which would also imply that E does not have full row rank, which is not the case. Hence we may choose a vector $v' \in \mathbb{C}^{m'}$ such that $v'(\lambda E' - A') = 0$. Let $v := [0, v']S_1$. Then a simple calculation reveals $v(\lambda E - A) = [0, v'(\lambda E' - A')](T_1T_2T_3)^{-1} = 0$, which contradicts full complex rank of $\lambda E - A$. As in the first case we can now inductively use the result of the lemma for the smaller matrix pencil $sE' - A'$ to obtain transformations S' and T' which put $sE' - A'$ in the desired block diagonal form. With $S := \left[\begin{smallmatrix} 1 & 0 \\ 0 & S' \end{smallmatrix}\right] S_1$ and $T := T_1T_2T_3 \left[\begin{smallmatrix} 1 & 0 \\ 0 & T' \end{smallmatrix}\right]$ we obtain the same block diagonal structure for $sE - A$ as for $sE' - A'$ apart from the first block which is $\mathcal{P}_{\varepsilon_1+1}$ instead of $\mathcal{P}_{\varepsilon_1}$. □

The following corollary follows directly from Lemma 2.3.14 by transposing the respective matrices.

Corollary 2.3.15 (KCF of full rank rectangular pencils, $m > n$).
Let $sE - A \in \mathbb{K}[s]^{m\times n}$ be such that $m > n$ and let $l := m - n$. Then $\mathrm{rk}_{\mathbb{C}}(\lambda E - A) = n$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$ if, and only if, there exist numbers $\eta_1, \ldots, \eta_l \in \mathbb{N}_0$ and matrices $S \in \mathbf{Gl}_m(\mathbb{K})$, $T \in \mathbf{Gl}_n(\mathbb{K})$ such that

$$S(sE - A)T = \mathrm{diag}\,(\mathcal{Q}_{\eta_1}(s), \ldots, \mathcal{Q}_{\eta_l}(s)),$$

where $\mathcal{Q}_\eta(s)$, $\eta \in \mathbb{N}_0$, is as in Definition 2.1.7.

Solvability of linear matrix equations

In generalization of the method presented in [71, Sec. 6] we reduce the problem of solvability of (2.3.1) to the problem of solving a generalized Sylvester equation

$$AXB - CXD = E. \tag{2.3.5}$$

To this end the following lemma is crucial.

Lemma 2.3.16 (Solutions of Sylvester equation).
Let $A, C \in \mathbb{K}^{m\times n}$, $B, D \in \mathbb{K}^{p\times q}$, $E, F \in \mathbb{K}^{m\times q}$ and consider the system of matrix equations with 'unknowns' $Y \in \mathbb{K}^{n\times q}$ and $Z \in \mathbb{K}^{m\times p}$

$$\begin{aligned} 0 &= E + AY + ZD, \\ 0 &= F + CY + ZB. \end{aligned} \tag{2.3.6}$$

Suppose there exists $\lambda \in \mathbb{K}$ *and* $M_\lambda \in \mathbb{K}^{q\times p}$ *such that* $M_\lambda(B-\lambda D) = I$, *in particular* $p \geq q$. *Then, for any solution* $X \in \mathbb{K}^{n\times p}$ *of the matrix equation*

$$AXB - CXD = -E - (\lambda E - F)M_\lambda D,$$

the matrices

$$\begin{aligned} Y &= X(B - \lambda D) \\ Z &= -(C - \lambda A)X - (F - \lambda E)M_\lambda \end{aligned}$$

solve (2.3.6).

Proof: We calculate

$$\begin{aligned} &E + AY + ZD \\ &= E + AX(B - \lambda D) - (C - \lambda A)XD - (F - \lambda E)M_\lambda D \\ &= E - AX\lambda D + \lambda AXD - (F - \lambda E)M_\lambda D - E - (\lambda E - F)M_\lambda D \\ &= 0, \end{aligned}$$

$$\begin{aligned} &F + CY + ZB \\ &= F + CX(B - \lambda D) - (C - \lambda A)XB - (F - \lambda E)M_\lambda B \\ &= F + CXB - CXB - (F - \lambda E)M_\lambda B - \lambda\,(E + (\lambda E - F)M_\lambda D) \\ &= (F - \lambda E) - (F - \lambda E)M_\lambda B - \lambda(\lambda E - F)M_\lambda D \\ &= (F - \lambda E)(I_q - M_\lambda(B - \lambda D)) \\ &= 0. \end{aligned}$$

□

The following result is well known and since it only considers regular matrix pencils we do not repeat its proof here. We use the augmented spectrum $\mathrm{spec}_\infty(sE - A)$ of a regular pencil $sE - A \in \mathbb{K}[s]^{n\times n}$.

Lemma 2.3.17 (Solvability of generalized Sylvester equation: regular case, [71,114])**.** *Let* $A, C \in \mathbb{K}^{n\times n}$, $B, D \in \mathbb{K}^{p\times p}$, $E \in \mathbb{K}^{n\times p}$ *and consider the generalized Sylvester equation* (2.3.5). *Assume that* $(sB - D)$ *and* $(sC - A)$ *are both regular and*

$$\mathrm{spec}_\infty(sB - D) \cap \mathrm{spec}_\infty(sC - A) = \emptyset.$$

Then (2.3.5) *is solvable.*

Finally, we can state and prove the result about the solvability of the generalized Sylvester equation which is needed for the proof of Theorem 2.3.3.

Lemma 2.3.18 (Solvability of generalized Sylvester equation with special properties). *Let* $A, C \in \mathbb{K}^{m\times n}$*,* $m \leq n$*,* $B, D \in \mathbb{K}^{p\times q}$*,* $p > q$*,* $E \in \mathbb{K}^{m\times q}$ *and consider the generalized Sylvester equation* (2.3.5)*. Assume that* $(\lambda B - D)$ *has full rank for all* $\lambda \in \mathbb{C} \cup \{\infty\}$ *and that either* $(\lambda C - A)$ *has full rank for all* $\lambda \in \mathbb{C} \cup \{\infty\}$ *or that* $(sC - A)$ *is regular. Then* (2.3.5) *is solvable.*

Proof: The proof follows the one of [114, Thm. 2]. By Corollary 2.2.19, Lemma 2.3.14 and Lemma 2.3.15, we already know that we can put the pencils $sC - A$ and $sB - D$ into WCF or KCF, resp. Therefore, choose invertible S_1, T_1, S_2, T_2 such that $sC_0 - A_0 = S_1(sC - A)T_1$ and $sB_0 - D_0 = S_2(sB - D)T_2$ are in WCF (KCF). Hence, with $X_0 = T_1^{-1} X S_1^{-1}$ and $E_0 = S_1 E T_2$, equation (2.3.5) is equivalent to

$$A_0X_0B_0 - C_0X_0D_0 = E_0.$$

Let $sC_0-A_0 = \operatorname{diag}(sC_0^1-A_0^1, \ldots, sC_0^{n_1}-A_0^{n_1})$, $n_1 \in \mathbb{N}_0$, and $sB_0-D_0 = \operatorname{diag}(sB_0^1 - D_0^1, \ldots, sB_0^{n_2} - D_0^{n_2})$, $n_2 \in \mathbb{N}_0$, corresponding to the block structure of the KCF as in Definition 2.1.7, then (2.3.5) is furthermore equivalent to the set of equations

$$A_0^i X_0^{ij} B_0^j - C_0^i X_0^{ij} D_0^j = E_0^{ij}, \quad i = 1, \ldots, n_1,\ j = 1, \ldots, n_2,$$

where X_0^{ij} and E_0^{ij} are the corresponding sub-blocks of X_0 and E_0 respectively. Note that by assumption each pencil $sC_0^i - A_0^i$ is a $\mathcal{P}_\varepsilon(s)$, $\mathcal{J}_\rho^\lambda(s)$ or $\mathcal{N}_\sigma(s)$ block and all pencils $sB_0^j - D_0^j$ are $\mathcal{Q}_\eta(s)$ blocks. If $sC_0^i - A_0^i$ is a $\mathcal{P}_\varepsilon(s)$ block we consider a reduced equation by deleting the first column of $sC_0^i - A_0^i$ which results in a regular $\mathcal{J}_\varepsilon^0(s)$ block with augmented spectrum $\{0\}$ and by deleting the last row of $sB_0^j - D_0^j$ which results in a regular $\mathcal{N}_\eta(s)$ block with augmented spectrum $\{\infty\}$. Hence, we use Lemma 2.3.17 to obtain solvability of the reduced problem. Filling the reduced solution for X_0^{ij} by a zero row and a zero column results in a solution for the original problem. If $sC_0^i - A_0^i$ is a $\mathcal{J}_\rho^\lambda(s)$ block we can apply the same trick (this time we only delete the last row of $sB_0^j - D_0^j$) to arrive at the same conclusion, as ∞ is not in the augmented spectrum of $sC_0^i - A_0^i$. Finally, if $sC_0^i - A_0^i$ is a $\mathcal{N}_\sigma(s)$ block with augmented spectrum $\{\infty\}$ we have to reduce $sB_0^j - D_0^j$ by deleting the first row which results in a $\mathcal{J}_\eta^0(s)$ block with augmented spectrum $\{0\}$. This concludes the proof. ☐

Interim quasi-Kronecker form

As a first step towards the QKF we derive the interim quasi-Kronecker triangular form (IQKTF). In this form we decouple the underdetermined part, the regular part and the overdetermined part of the matrix pencil.

Theorem 2.3.19 (Interim quasi-Kronecker triangular form)**.**
Let $sE - A \in \mathbb{K}[s]^{m\times n}$ and consider the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the Wong sequences as in Definition 2.2.1. Choose any full rank matrices $P_1 \in \mathbb{K}^{n\times n_P}$, $P_2 \in \mathbb{K}^{m\times m_P}$, $R_1 \in \mathbb{K}^{n\times n_R}$, $R_2 \in \mathbb{K}^{m\times m_R}$, $Q_1 \in \mathbb{K}^{n\times n_Q}$, $Q_2 \in \mathbb{K}^{m\times m_Q}$ such that

$$
\begin{aligned}
\operatorname{im} P_1 &= \mathcal{V}^* \cap \mathcal{W}^*, & \mathcal{V}^* \cap \mathcal{W}^* \oplus \operatorname{im} R_1 &= \mathcal{V}^* + \mathcal{W}^*,\\
& & (\mathcal{V}^* + \mathcal{W}^*) \oplus \operatorname{im} Q_1 &= \mathbb{K}^n,\\
\operatorname{im} P_2 &= E\mathcal{V}^* \cap A\mathcal{W}^*, & E\mathcal{V}^* \cap A\mathcal{W}^* \oplus \operatorname{im} R_2 &= E\mathcal{V}^* + A\mathcal{W}^*,\\
& & (E\mathcal{V}^* + A\mathcal{W}^*) \oplus \operatorname{im} Q_2 &= \mathbb{K}^m.
\end{aligned}
$$

Then $T_{\mathrm{trian}} = [P_1, R_1, Q_1] \in \mathbf{Gl}_n(\mathbb{K})$, $S_{\mathrm{trian}} = [P_2, R_2, Q_2]^{-1} \in \mathbf{Gl}_m(\mathbb{K})$ and $S_{\mathrm{trian}}(sE - A)T_{\mathrm{trian}}$ is in IQKTF (2.1.2)*.*

Proof: We proceed in several steps.

Step 1: We show the block-triangular form (2.1.2). By the choice of P_1, R_1, Q_1 and P_2, R_2, Q_2 it follows immediately that T_{trian} and S_{trian} are invertible. Note that $S_{\mathrm{trian}}(sE - A)T_{\mathrm{trian}}$ is in IQKTF (2.1.2) if, and only if, the following equations are solvable for given E, A and $P_1, R_1, Q_1, P_2, R_2, Q_2$:

$$
\begin{aligned}
EP_1 &= P_2E_P, & AP_1 &= P_2A_P,\\
ER_1 &= P_2E_{PR} + R_2E_R, & AR_1 &= P_2A_{PR} + R_2A_R,\\
EQ_1 &= P_2E_{PQ} + R_2E_{RQ} + Q_2E_Q, & AQ_1 &= P_2A_{PQ} + R_2A_{RQ} + Q_2A_Q.
\end{aligned}
$$

The solvability of the latter is a consequence of the following subspace inclusions

$$
\begin{aligned}
E(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap A\mathcal{W}^*, & A(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap A\mathcal{W}^*,\\
E(\mathcal{V}^* + \mathcal{W}^*) &\subseteq E\mathcal{V}^* + A\mathcal{W}^*, & A(\mathcal{V}^* + \mathcal{W}^*) &\subseteq E\mathcal{V}^* + A\mathcal{W}^*,\\
E\mathbb{K}^n &\subseteq \mathbb{K}^m, & A\mathbb{K}^n &\subseteq \mathbb{K}^m,
\end{aligned}
$$

which clearly hold due to (2.2.4).

Step 2: We show (i).

Step 2a: Full row rank of E_P and A_P. From Lemma 2.3.7 it follows that

$$\operatorname{im} P_2 E_P = \operatorname{im} E P_1 = \operatorname{im} P_2 \quad \text{and} \quad \operatorname{im} P_2 A_P = \operatorname{im} A P_1 = \operatorname{im} P_2$$

hence, invoking the full column rank of P_2, $\operatorname{im} E_P = \mathbb{K}^{m_P} = \operatorname{im} A_P$, which implies full row rank of E_P and A_P. In particular this shows full row rank of $\lambda E_P - A_P$ for $\lambda = 0$ and $\lambda = \infty$.

Step 2b: Full row rank of $\lambda E_P - A_P$ for all $\lambda \in \mathbb{C} \setminus \{0\}$. Seeking a contradiction, assume existence of $\lambda \in \mathbb{C} \setminus \{0\}$ with $\operatorname{rk}_{\mathbb{C}}(\lambda E_P - A_P) < m_P$. Then there exists $v \in \mathbb{C}^{m_P}$ such that $v^*(\lambda E_P - A_P) = 0$. Full column rank of $P_2 \in \mathbb{K}^{m \times m_P}$ implies existence of $w \in \mathbb{C}^m$ such that $w^* P_2 = v^*$, hence

$$0 = v^*(\lambda E_P - A_P) = w^*(\lambda P_2 E_P - P_2 A_P) = w^*(\lambda E - A) P_1.$$

Invoking Lemma 2.3.12, there exists a linearly independent singular chain $(x_0, x_1, \ldots, x_k)$ such that

$$\operatorname{span}\{x_0, x_1, \ldots, x_k\} = \operatorname{im} P_1 = \mathcal{V}^* \cap \mathcal{W}^*.$$

In particular, $x_i \in \operatorname{im} P_1$ for $i = 0, 1, \ldots$, implies

$$\forall i \in \{0, 1, \ldots, k\} : \quad w^*(\lambda E - A) x_i = 0.$$

Since $E x_k = 0$ it follows that $w^* A x_k = 0$ and inductively it follows

$$0 = w^*(\lambda E x_{i-1} - A x_{i-1}) = w^*(\lambda A x_i - A x_{i-1}) = -w^* A x_{i-1}$$

and, therefore,

$$0 = w^* A P_1 = w^* P_2 A_P = v^* A_P.$$

This shows that $A_P \in \mathbb{K}^{m_P \times n_P}$ does not have full row rank over $\mathbb{C}$ which implies also a row rank defect over $\mathbb{K}$. This is the sought contradiction because the full row rank of A_P was already shown in Step 2a.

Step 3: We show (ii). For notational convenience let $\mathcal{K} := \mathcal{V}^* \cap \mathcal{W}^*$.

Step 3a: We show that $m_R = n_R$. Invoking

$$\ker E \cap \mathcal{K} = \ker E \cap \mathcal{V}^*, \quad \ker A \cap \mathcal{K} = \ker A \cap \mathcal{W}^*, \tag{2.3.7}$$

and Lemmas 2.3.6, 2.3.7, the claim follows from

$$\begin{aligned}
m_R &= \operatorname{rk} R_2 \\
&= \dim(E\mathcal{V}^* + A\mathcal{W}^*) - \dim(E\mathcal{V}^* \cap A\mathcal{W}^*) \\
&= \dim E\mathcal{V}^* + \dim A\mathcal{W}^* - 2\dim(E\mathcal{V}^* \cap A\mathcal{W}^*) \\
&= \dim \mathcal{V}^* - \dim(\ker E \cap \mathcal{V}^*) + \dim \mathcal{W}^* - \dim(\ker A \cap \mathcal{W}^*) \\
&\quad - \dim E\mathcal{K} - \dim A\mathcal{K} \\
&= \dim \mathcal{V}^* - \dim(\ker E \cap \mathcal{V}^*) + \dim \mathcal{W}^* - \dim(\ker A \cap \mathcal{W}^*) \\
&\quad - \dim \mathcal{K} + \dim(\ker E \cap \mathcal{K}) - \dim \mathcal{K} + \dim(\ker A \cap \mathcal{K}) \\
&\overset{(2.3.7)}{=} \dim \mathcal{V}^* + \dim \mathcal{W}^* - 2\dim \mathcal{K} \\
&= \dim(\mathcal{V}^* + \mathcal{W}^*) - \dim(\mathcal{V}^* \cap \mathcal{W}^*) \\
&= \operatorname{rk} R_1 = n_R.
\end{aligned}$$

Step 3b: We show that $\det(sE_R - A_R) \not\equiv 0$. Seeking a contradiction, assume $\det(sE_R - A_R)$ is the zero polynomial. Then $\lambda E_R - A_R$ has a column rank defect for all $\lambda \in \mathbb{C} \cup \{\infty\}$, hence

$$\forall\, \lambda \in \mathbb{C} \cup \{\infty\} : \ \operatorname{rk}_{\mathbb{C}}(\lambda E_R - A_R) < n_R.$$

Now, Lemma 2.3.13 ensures existence of a nontrivial singular chain $(y_0, y_1, \ldots, y_k)$ of the matrix pencil $sE_R - A_R$.
We show that there exists a singular chain $(x_0, x_1, \ldots, x_k, x_{k+1}, \ldots, x_{\hat{k}})$ of $sE - A$ such that $x_i = [P_1, R_1]\left(\begin{smallmatrix} z_i \\ y_i \end{smallmatrix}\right)$ for $i = 0, \ldots, k$. To this end denote some right inverse of A_P (invoking full row rank of A_P as shown in Step 2a) with A_P^+ and let

$$z_0 = -A_P^+ A_{PR} y_0, \quad z_{i+1} = A_P^+(E_P z_i + E_{PR} y_i - A_{PR} y_{i+1}),\ i = 0, \ldots, k,$$

where $y_{k+1} = 0$. Then it follows that

$$\begin{aligned}
Ax_i = A[P_1, R_1]\begin{pmatrix} z_i \\ y_i \end{pmatrix} &= AT_{\text{trian}}\begin{pmatrix} z_i \\ y_i \\ 0 \end{pmatrix} = S_{\text{trian}}^{-1}\begin{bmatrix} A_P & A_{PR} & A_{PQ} \\ 0 & A_R & A_{RQ} \\ 0 & 0 & A_Q \end{bmatrix}\begin{pmatrix} z_i \\ y_i \\ 0 \end{pmatrix} \\
&= S_{\text{trian}}^{-1}\begin{pmatrix} A_P z_i + A_{PR} y_i \\ A_R y_i \\ 0 \end{pmatrix}
\end{aligned}$$

and, analogously,

$$Ex_i = S_{\text{trian}}^{-1} \begin{pmatrix} E_P z_i + E_{PR} y_i \\ E_R y_i \\ 0 \end{pmatrix},$$

hence $Ax_0 = 0$ and $Ex_i = Ax_{i+1}$ for $i = 0, \dots, k$. Note that, if we set $x_{k+1} = P_1 z_{k+1}$, then $x_{k+1} \in \mathcal{V}^* \cap \mathcal{W}^* \subseteq \mathcal{W}^*$ and identically as shown in the first part of Step 2 of the proof of Lemma 2.3.12 there exist $x_{k+2}, \dots, x_{\hat{k}}$, $\hat{k} > k$, such that $Ex_{k+1} = Ax_{k+2}, \dots, Ex_{\hat{k}-1} = Ax_{\hat{k}}, Ex_{\hat{k}} = 0$ and, therefore, $(x_0, x_1, \dots, x_{\hat{k}})$ is a singular chain of $sE - A$. Lemma 2.3.12 implies that $\{x_0, x_1, \dots, x_{\hat{k}}\} \subseteq \operatorname{im} P_1$, hence $x_i = [P_1, R_1] \left(\begin{smallmatrix} z_i \\ y_i \end{smallmatrix}\right)$ implies $y_i = 0$ for all $i \in \{0, \dots, k\}$, which contradicts non-triviality of $(y_0, \dots, y_k)$.

Step 4: We show (iii). We will consider the (conjugate) transposed matrix pencil $sE^* - A^*$ with corresponding Wong sequences and will show that the block (E_Q^*, A_Q^*) will play the role of the block (E_P, A_P). Therefore, denote the limits of the Wong-sequences of $sE^* - A^*$ by $\widehat{\mathcal{V}}^*$ and $\widehat{\mathcal{W}}^*$. Let

$$\widehat{Q}_1 := ([0,\ 0,\ I_{n_Q}][P_1, R_1, Q_1]^{-1})^*, \quad \widehat{Q}_2 := ([0,\ 0,\ I_{m_Q}][P_2, R_2, Q_2]^{-1})^*,$$

then

$$\widehat{Q}_i^* Q_i = I \quad \text{and} \quad \operatorname{im} \widehat{Q}_i = (\operatorname{im}[P_i, R_i])^\perp, \quad \text{for } i = 1, 2.$$

In fact, the latter follows from $n - n_Q = n_P + n_R$ and

$$\widehat{Q}_1^*[P_1, R_1] = [0,\ 0,\ I_{n_Q}] \begin{bmatrix} I_{n_P} & 0 \\ 0 & I_{n_R} \\ 0 & 0 \end{bmatrix} = 0$$

for $i = 1$ and analogously for $i = 2$. We will show in the following that

$$\begin{aligned} E^* \widehat{Q}_2 &= \widehat{Q}_1 E_Q^*, & A^* \widehat{Q}_2 &= \widehat{Q}_1 A_Q^*, \\ \operatorname{im} \widehat{Q}_2 &= \widehat{\mathcal{V}}^* \cap \widehat{\mathcal{W}}^*, & \operatorname{im} \widehat{Q}_1 &= E^* \widehat{\mathcal{V}}^* \cap A^* \widehat{\mathcal{V}}^*, \end{aligned}$$

then the arguments from Step 2 can be applied to $sE_Q^* - A_Q^*$ and the claim is shown.

Step 4a: We show $E^*\widehat{Q}_2 = \widehat{Q}_1 E_Q^*$ and $A^*\widehat{Q}_2 = \widehat{Q}_1 A_Q^*$. Using that $S_{\text{trian}}(sE - A)T_{\text{trian}}$ is in the form (2.1.2) we obtain

$$\widehat{Q}_2^* E = \underbrace{\widehat{Q}_2^*[P_2, R_2, Q_2]}_{=[0\ 0\ I]} \begin{bmatrix} E_P & E_{PR} & E_{PQ} \\ 0 & E_R & E_{RQ} \\ 0 & 0 & E_Q \end{bmatrix} [P_1, R_1, Q_1]^{-1}$$
$$= [0\ 0\ E_Q][P_1, R_1, Q_1]^{-1} = E_Q \widehat{Q}_1^*,$$

hence $E^*\widehat{Q}_2 = \widehat{Q}_1 E_Q^*$. Analog arguments show that $A^*\widehat{Q}_2 = \widehat{Q}_1 A_Q^*$.

Step 4b: We show $\operatorname{im} \widehat{Q}_2 = \widehat{\mathcal{V}}^* \cap \widehat{\mathcal{W}}^*$. By construction and Lemma 2.3.8

$$\operatorname{im} \widehat{Q}_2 = (\operatorname{im}[P_2, R_2])^\perp = (E\mathcal{V}^* + A\mathcal{W}^*)^\perp = (E\mathcal{V}^*)^\perp \cap (A\mathcal{W}^*)^\perp = \widehat{\mathcal{V}}^* \cap \widehat{\mathcal{W}}^*.$$

Step 4c: We show $\operatorname{im} \widehat{Q}_1 = E^*\widehat{\mathcal{V}}^* \cap A^*\widehat{\mathcal{V}}^*$. Lemma 2.3.8 applied to (E^*, A^*) gives

$$(E^*\widehat{\mathcal{V}}^*)^\perp = \mathcal{W}^* \quad \text{and} \quad (A^*\widehat{\mathcal{W}}^*)^\perp = \mathcal{V}^*$$

or, equivalently,

$$E^*\widehat{\mathcal{V}}^* = \mathcal{W}^{*\perp} \quad \text{and} \quad A^*\widehat{\mathcal{W}}^* = \mathcal{V}^{*\perp}.$$

Hence

$$\operatorname{im} \widehat{Q}_1 = (\operatorname{im}[P_1, R_1])^\perp = (\mathcal{V}^* + \mathcal{W}^*)^\perp = \mathcal{V}^{*\perp} \cap \mathcal{W}^{*\perp} = A^*\widehat{\mathcal{W}}^* \cap E^*\widehat{\mathcal{V}}^*.$$

This concludes the proof of the theorem. □

As a corollary of Theorem 2.3.19 we show that it is possible to get rid of the off-diagonal blocks in the IQKTF by solving a set of generalized Sylvester equations using Lemma 2.3.18

Corollary 2.3.20 (Interim quasi-Kronecker form)**.**
Using the notation from Theorem 2.3.19 the following equations are solvable for matrices $F_1, F_2, G_1, G_2, H_1, H_2$ of appropriate size:

$$\begin{aligned} 0 &= E_{RQ} + E_R F_1 + F_2 E_Q \\ 0 &= A_{RQ} + A_R F_1 + F_2 A_Q \end{aligned} \tag{2.3.8a}$$

$$\begin{aligned} 0 &= E_{PR} + E_P G_1 + G_2 E_R \\ 0 &= A_{PR} + A_P G_1 + G_2 A_R \end{aligned} \tag{2.3.8b}$$

$$\begin{aligned} 0 &= (E_{PQ} + E_{PR} F_1) + E_P H_1 + H_2 E_Q \\ 0 &= (A_{PQ} + A_{PR} F_1) + A_P H_1 + H_2 A_Q \end{aligned} \tag{2.3.8c}$$

and for any such matrices let

$$\begin{aligned} S &:= \begin{bmatrix} I & -G_2 & -H_2 \\ 0 & I & -F_2 \\ 0 & 0 & I \end{bmatrix}^{-1} S_{\text{trian}} \\ &= [P_2, R_2 - P_2G_2, Q_2 - P_2H_2 - R_2F_2]^{-1} \quad \textit{and} \\ T &:= T_{\text{trian}} \begin{bmatrix} I & G_1 & H_1 \\ 0 & I & F_1 \\ 0 & 0 & I \end{bmatrix} = [P_1, R_1 + P_1G_1, Q_1 + P_1H_1 + R_1F_1]. \end{aligned}$$

Then $S \in \mathbf{Gl}_m(\mathbb{K})$, $T \in \mathbf{Gl}_n(\mathbb{K})$ and $S(sE - A)T$ is in IQKF (2.1.3)*, where the block diagonal entries are the same as for the IQKTF* (2.1.2)*. In particular, the IQKF (without the transformation matrices S and T) can be obtained with only the Wong sequences (i.e., without solving* (2.3.8)*). Furthermore, the IQKF* (2.1.3) *is unique in the following sense*

$$\begin{aligned} &(E, A) \cong (E', A') \;\Leftrightarrow \\ &(E_P, A_P) \cong (E'_P, A'_P),\ (E_R, A_R) \cong (E'_R, A'_R),\ (E_Q, A_Q) \cong (E'_Q, A'_Q), \end{aligned} \tag{2.3.9}$$

where $E'_P, A'_P, E'_R, A'_R, E'_P, A'_P$ are the corresponding blocks of the IQKF of the matrix pencil $sE' - A'$.

Proof: By the properties of the pencils $sE_P - A_P$, $sE_R - A_R$ and $sE_Q - A_Q$ there exist $\lambda \in \mathbb{K}$ and full rank matrices N_λ^P, N_λ^R, M_λ^R and M_λ^Q such that $(\lambda E_P - A_P)N_\lambda^P = I$, $(\lambda E_R - A_R)N_\lambda^R = I$, $M_\lambda^R(\lambda E_R - A_R) = I$ and $M_\lambda^Q(\lambda E_Q - A_Q) = I$. Hence Lemma 2.3.16 shows that it suffices to consider solvability of the following generalized Sylvester equations

$$E_RX_1A_Q - A_RX_1E_Q = -E_{RQ} - (\lambda E_{RQ} - A_{RQ})M_\lambda^Q E_Q \tag{2.3.10a}$$
$$E_PX_2A_R - A_PX_2E_R = -E_{PR} - (\lambda E_{PR} - A_{PR})M_\lambda^R E_R \tag{2.3.10b}$$
$$\begin{aligned} E_PX_3A_Q - A_PX_3E_Q = &-(E_{PQ} + E_{PR}F_1) - (\lambda(E_{PQ} + E_{PR}F_1) \\ &- (A_{PQ} + A_{PR}F_1))M_\lambda^Q E_Q, \end{aligned} \tag{2.3.10c}$$

where F_1 is any solution of (2.3.8a), whose existence will follow from solvability of (2.3.10a). Furthermore, the properties of $sE_P - A_P$, $sE_Q - A_Q$ and $sE_R - A_R$ imply that Lemma 2.3.18 is applicable to the equations (2.3.10) (where (2.3.10b) must be considered in the (conjugate) transposed form) and ensures existence of solutions. Now, a simple calculation shows that for any solution of (2.3.8) the second part of the statement of Corollary 2.3.20 holds.

Finally, to show uniqueness in the sense of (2.3.9) assume first that $(E, A) \cong (E', A')$. Then there exist invertible matrices S' and T' such that $(E', A') = (S'ET', S'AT')$. It is easily seen, that the Wong sequences $\mathcal{V}_i'$, $\mathcal{W}_i'$, $i \in \mathbb{N}_0$, of the pencil $sE' - A'$ fulfill

$$\mathcal{V}_i' = {T'}^{-1}\mathcal{V}_i, \quad \mathcal{W}_i' = {T'}^{-1}\mathcal{W}_i, \quad E'\mathcal{V}_i' = S'E\mathcal{V}_i, \quad A'\mathcal{W}_i' = S'A\mathcal{W}_i. \tag{2.3.11}$$

Hence, using the notation of Theorem 2.3.19, there exist invertible matrices M_P, M_R, M_Q, N_P, N_R, N_Q such that

$$\begin{aligned} [P_1', R_1', Q_1'] &= {T'}^{-1}[P_1M_P, R_1M_R, Q_1M_Q], \\ [P_2', R_2', Q_2'] &= S'[P_2N_P, R_2N_R, Q_2N_Q], \end{aligned}$$

in particular the block sizes of the corresponding IQKFs coincide. Therefore,

$$\begin{aligned} &s\begin{bmatrix} E_P & * & * \\ 0 & E_R & * \\ 0 & 0 & E_Q \end{bmatrix} - \begin{bmatrix} A_P & * & * \\ 0 & A_R & * \\ 0 & 0 & A_Q \end{bmatrix} = [P_2, R_2, Q_2]^{-1}(sE - A)[P_1, R_1, Q_1] \\ &= \begin{bmatrix} N_P & 0 & 0 \\ 0 & N_R & 0 \\ 0 & 0 & N_Q \end{bmatrix} [P_2', R_2', Q_2']S'(sE - A)T'[P_1', R_2', Q_2'] \begin{bmatrix} M_P & 0 & 0 \\ 0 & M_R & 0 \\ 0 & 0 & M_Q \end{bmatrix}^{-1} \\ &= \begin{bmatrix} N_P & 0 & 0 \\ 0 & N_R & 0 \\ 0 & 0 & N_Q \end{bmatrix} [P_2', R_2', Q_2'](sE' - A')[P_1', R_2', Q_2'] \begin{bmatrix} M_P^{-1} & 0 & 0 \\ 0 & M_R^{-1} & 0 \\ 0 & 0 & M_Q^{-1} \end{bmatrix} \\ &= s\begin{bmatrix} N_PE_P'M_P^{-1} & * & * \\ 0 & N_RE_R'M_R^{-1} & * \\ 0 & 0 & N_QE_Q'M_Q^{-1} \end{bmatrix} - \begin{bmatrix} N_PA_P'M_P^{-1} & * & * \\ 0 & N_RA_RM_R^{-1} & * \\ 0 & 0 & N_QA_QM_Q^{-1} \end{bmatrix}. \end{aligned}$$

Hence the necessity part of (2.3.9) is shown. Sufficiency follows from the simple observation that equivalence of the IQKFs implies equivalence of the original matrix pencils. □

2.3.3 Proofs of the main results

In this subsection we prove the main results about the QKTF and the QKF, where, compared to the IQKTF and IQKF, the regular part is decoupled into an ODE part and a nilpotent part. In the IQKTF and IQKF we directly completed a basis of $\mathcal{V}^* \cap \mathcal{W}^*$ to a basis of $\mathcal{V}^* + \mathcal{W}^*$, and a basis of $E\mathcal{V}^* \cap A\mathcal{W}^*$ to a basis of $E\mathcal{V}^* + A\mathcal{W}^*$, resp. For the QKTF and the QKF we consider $\mathcal{V}^*$ and $E\mathcal{V}^*$ as intermediate steps.

Proof of Theorem 2.3.1

Step 1: We show that $S_{\text{trian}}(sE - A)T_{\text{trian}}$ is in the form (2.1.4) and prove (i) and (iv). It follows from (2.2.4) that

$$\begin{aligned} E(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap A\mathcal{W}^*, & A(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq E\mathcal{V}^* \cap A\mathcal{W}^*, \\ E\mathcal{V}^* &= E\mathcal{V}^*, & A\mathcal{V}^* &\subseteq E\mathcal{V}^*, \\ E(\mathcal{V}^* + \mathcal{W}^*) &\subseteq E\mathcal{V}^* + A\mathcal{W}^*, & A(\mathcal{V}^* + \mathcal{W}^*) &\subseteq E\mathcal{V}^* + A\mathcal{W}^*, \\ E\mathbb{K}^n &\subseteq \mathbb{K}^m, & A\mathbb{K}^n &\subseteq \mathbb{K}^m. \end{aligned}$$

These inclusions imply solvability of

$$\left.\begin{aligned} &EP_1 = P_2E_P, && AP_1 = P_2A_P, \\ &ER_1^J = P_2E_{PJ} + R_2^JE_J, && AR_1^J = P_2A_{PJ} + R_2^JA_J, \\ &ER_1^N = P_2E_{PN} + R_2^JE_{JN} + R_2^NE_N, && AR_1^N = P_2A_{PN} + R_2^JA_{JN} + R_2^NA_N, \\ &EQ_1 = P_2E_{PQ} + R_2^JE_{JQ} + R_2^NE_{NQ} + Q_2E_Q, && AQ_1 = P_2A_{PQ} + R_2^JA_{JQ} + R_2^NA_{NQ} + Q_2A_Q. \end{aligned}\right\} \tag{2.3.12}$$

which is equivalent to $S_{\text{trian}}(sE - A)T_{\text{trian}}$ being in the form (2.1.4). The properties (i) and (iv) immediately follow from Theorem 2.3.19 as the choice of bases here is more special.

Step 2: We show $(E\mathcal{V}^* \cap A\mathcal{W}^*) \oplus \operatorname{im} ER_1^J = E\mathcal{V}^*$. As $\operatorname{im} R_1^J \subseteq \mathcal{V}^*$ it follows that $(E\mathcal{V}^* \cap A\mathcal{W}^*) + \operatorname{im} ER_1^J \subseteq E\mathcal{V}^*$. Invoking $E\mathcal{W}^* \subseteq A\mathcal{W}^*$, the opposite inclusion is immediate from

$$\begin{aligned} E\mathcal{V}^* = E((\mathcal{V}^* \cap \mathcal{W}^*) \oplus \operatorname{im} R_1^J) &\subseteq E(\mathcal{V}^* \cap \mathcal{W}^*) + \operatorname{im} ER_1^J \\ &\subseteq (E\mathcal{V}^* + A\mathcal{W}^*) + \operatorname{im} ER_1^J. \end{aligned}$$

It remains to show that the intersection is trivial. To this end let $x \in (E\mathcal{V}^* \cap A\mathcal{W}^*) \cap \operatorname{im} ER_1^J$, i.e., $x = Ey$ with $y \in \operatorname{im} R_1^J$. Further, $x \in E\mathcal{V}^* \cap A\mathcal{W}^* = E(\mathcal{V}^* \cap \mathcal{W}^*)$ (where the subspace equality follows from Lemma 2.3.7) and this yields that $x = Ez$ with $z \in \mathcal{V}^* \cap \mathcal{W}^*$, thus $z - y \in \ker E \subseteq \mathcal{W}^*$. Hence, since $z \in \mathcal{W}^*$, it follows $y \in \mathcal{W}^* \cap \operatorname{im} R_1^J = \{0\}$.

Step 3: We show $E\mathcal{V}^* \oplus \operatorname{im} AR_1^N = E\mathcal{V}^* + A\mathcal{W}^*$. We immediately see that, since $A\mathcal{V}^* \subseteq E\mathcal{V}^*$,

$$E\mathcal{V}^* + A\mathcal{W}^* = E\mathcal{V}^* + A\mathcal{V}^* + A\mathcal{W}^* = E\mathcal{V}^* + A(\mathcal{V}^* + \mathcal{W}^*) =$$
$$E\mathcal{V}^* + A(\mathcal{V}^* + \operatorname{im} R_1^N) = E\mathcal{V}^* + A\mathcal{V}^* + A \operatorname{im} R_1^N = E\mathcal{V}^* + \operatorname{im} AR_1^N.$$

In order to show that the intersection is trivial, let $x \in E\mathcal{V}^* \cap \operatorname{im} AR_1^N$, i.e., $x = Ay = Ez$ with $y \in \operatorname{im} R_1^N$ and $z \in \mathcal{V}^*$. Therefore, $y \in A^{-1}(E\mathcal{V}^*) = \mathcal{V}^*$ and $y \in \operatorname{im} R_1^N$, thus $y = 0$.

Step 4: We show $m_J = n_J$ and $m_N = n_N$. By Step 2 and Step 3 we have that $m_J = \operatorname{rk} ER_1^J \le n_J$ and $m_N = \operatorname{rk} AR_1^N \le n_N$. In order to see that we have equality in both cases observe that: $ER_1^J v = 0$ for some $v \in \mathbb{K}^{n_J}$ implies $R_1^J v \in \operatorname{im} R_1^J \cap \ker E = \{0\}$, since $\ker E \subseteq \mathcal{W}^*$, and hence $v = 0$ as R_1^J has full column rank; $AR_1^N v = 0$ for some $v \in \mathbb{K}^{n_N}$ implies $R_1^N v \in \operatorname{im} R_1^N \cap \ker A = \{0\}$, since $\ker A \subseteq \mathcal{V}^*$, and hence $v = 0$ as R_1^N has full column rank.

Step 5: We show that E_J and A_N are invertible. For the first, assume that there exists $v \in \mathbb{K}^{n_J} \setminus \{0\}$ such that $E_J v = 0$. Then $ER_1^J v \overset{(2.3.12)}{=} P_2 E_{PJ} v$ and hence $ER_1^J v \in \operatorname{im} ER_1^J \cap \operatorname{im} P_2 \overset{\text{Step 2}}{=} \{0\}$, a contradiction with the fact that ER_1^J has full column rank (as shown in Step 4). In order to show that A_N is invertible, let $v \in \mathbb{K}^{n_N} \setminus \{0\}$ be such that $A_N v = 0$. Then $AR_1^N v \overset{(2.3.12)}{=} P_2 A_{PN} v + R_2^J A_{JN} v$ and hence $AR_1^N v \in \operatorname{im} AR_1^N \cap \operatorname{im}[P_2, R_2^J] \overset{\text{Step 3}}{=} \{0\}$, a contradiction with the fact that AR_1^N has full column rank (as shown in Step 4).

Step 6: It only remains to show that $A_N^{-1} E_N$ is nilpotent. In order to prove this we will show that, for ℓ^* as in (2.2.3),

$$\forall\, i \in \{0, \ldots, \ell^*\}: \ \mathcal{V}^* \oplus \operatorname{im} R_1^N (A_N^{-1} E_N)^i \subseteq \mathcal{V}^* + \mathcal{W}_{\ell^* - i}. \qquad (2.3.13)$$

We show this by induction. For $i = 0$ the assertion is clear from the choice of R_1^N. Suppose (2.3.13) holds for some $i \in \{0, \ldots, \ell^* - 1\}$. Then

$$\begin{aligned} A(\mathcal{V}^* + \operatorname{im} R_1^N (A_N^{-1} E_N)^{i+1}) &\subseteq A\mathcal{V}^* + \operatorname{im} AR_1^N (A_N^{-1} E_N)^{i+1} \\ &\overset{(2.3.12)}{\subseteq} E\mathcal{V}^* + \operatorname{im}(P_2 A_{PN} + R_2^J A_{JN} + R_2^N A_N)(A_N^{-1} E_N)^{i+1} \\ &\subseteq E\mathcal{V}^* + \underbrace{\operatorname{im} P_2 A_{PN} (A_N^{-1} E_N)^{i+1}}_{\subseteq E\mathcal{V}^*} + \underbrace{\operatorname{im} R_2^J A_{JN} (A_N^{-1} E_N)^{i+1}}_{\subseteq E\mathcal{V}^*} \\ &\quad + \operatorname{im} R_2^N E_N (A_N^{-1} E_N)^i \end{aligned}$$

$$
\begin{aligned}
&\overset{(2.3.12)}{\subseteq} \quad E\mathcal{V}^* + \operatorname{im}(ER_1^N - P_2E_{PN} - R_2^J E_{JN})(A_N^{-1}E_N)^i \\
&\subseteq \quad E\mathcal{V}^* + \operatorname{im} ER_1^N(A_N^{-1}E_N)^i + \underbrace{\operatorname{im} P_2E_{PN}(A_N^{-1}E_N)^i}_{\subseteq E\mathcal{V}^*} \\
&\qquad + \underbrace{\operatorname{im} R_2^J E_{JN}(A_N^{-1}E_N)^i}_{\subseteq E\mathcal{V}^*} \\
&\subseteq \quad E(\mathcal{V}^* + \operatorname{im} R_1^N(A_N^{-1}E_N)^i) \\
&\overset{(2.3.13)}{\subseteq} \quad E\mathcal{V}^* + E\mathcal{W}_{\ell^*-i} \subseteq E\mathcal{V}^* + A\mathcal{W}_{\ell^*-i-1}
\end{aligned}
$$

and hence

$$
\begin{aligned}
\mathcal{V}^* + \operatorname{im} R_1^N(A_N^{-1}E_N)^{i+1} &\subseteq A^{-1}(E\mathcal{V}^* + A\mathcal{W}_{\ell^*-i-1}) \\
&\subseteq A^{-1}(E\mathcal{V}^*) + \mathcal{W}_{\ell^*-i-1} \subseteq \mathcal{V}^* + \mathcal{W}_{\ell^*-i-1}.
\end{aligned}
$$

Furthermore, we have

$$
\mathcal{V}^* \cap \operatorname{im} R_1^N(A_N^{-1}E_N)^{i+1} \subseteq \mathcal{V}^* \cap \operatorname{im} R_1^N = \{0\}
$$

and hence we have proved (2.3.13). Now (2.3.13) for $i = \ell^*$ yields $R_1^N(A_N^{-1}E_N)^{\ell^*} = 0$, and since R_1^N has full column rank we may conclude that $(A_N^{-1}E_N)^{\ell^*} = 0$. □

Proof of Theorem 2.3.3

We may choose $\lambda \in \mathbb{C}$ and M_λ of appropriate size such that $M_\lambda(A_N - \lambda E_N) = I$ and, due to Lemma 2.3.16, for the solvability of (2.3.1c) it then suffices to consider solvability of

$$
E_JXA_N - A_JXE_N = -E_{JN} - (\lambda E_{JN} - A_{JN})M_\lambda E_N,
$$

which however is immediate from Lemma 2.3.17. Solvability of the other equations (2.3.1a), (2.3.1b), (2.3.1d) then follows as in the proof of Corollary 2.3.20.

Uniqueness in the sense of (2.3.2) can be established along lines similar to the proof of Corollary 2.3.20. □

2.3.4 KCF, elementary divisors and minimal indices

An analysis of the matrix pencils $sE_P - A_P$ and $sE_Q - A_Q$ in (2.1.5), invoking Lemma 2.3.14 and Corollary 2.3.15, together with Corollary 2.2.19 allows now to obtain the KCF as a corollary.

Corollary 2.3.21 (Kronecker canonical form)**.**
For every matrix pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ there exist transformation matrices $S \in \mathbf{Gl}_m(\mathbb{C})$ and $T \in \mathbf{Gl}_n(\mathbb{C})$ such that $S(sE-A)T \in \mathbb{C}[s]^{m\times n}$ is in KCF (2.1.6)*.*

The numbers ρ_i, σ_i, ε_i and η_i in Definition 2.1.7 are usually called (degrees of) elementary divisors and minimal indices and play an important role in the analysis of matrix pencils, see e.g. [100,166,167,175]. More precisely, $\rho_1, \ldots, \rho_b$ are the degrees of the finite elementary divisors, $\sigma_1, \ldots, \sigma_c$ are the degrees of the infinite elementary divisors, $\varepsilon_1, \ldots, \varepsilon_a$ are the columns minimal indices and $\eta_1, \ldots, \eta_d$ are the row minimal indices. The minimal indices completely determine (under the standing assumption that $0 \le \varepsilon_1 \le \ldots \le \varepsilon_a$ and $0 \le \eta_1 \le \ldots \le \eta_d$) the singular part of the KCF. The following result shows that the minimal indices can be determined via the Wong sequences. Another method for determining the minimal indices via the control theoretic version of the Wong sequences can be found in [167, Prop. 2.2], however the minimal indices there correspond to a feedback canonical form.

Theorem 2.3.22 (Minimal indices)**.**
Let $sE - A \in \mathbb{K}[s]^{m\times n}$, consider the Wong sequences (2.2.3) *and the notation from Corollary 2.3.21 so that $S(sE-A)T$ is in KCF* (2.1.6)*. Let $\mathcal{K} = \mathcal{V}^* \cap \mathcal{W}^*$, $\widehat{\mathcal{W}}_i := (E\mathcal{V}_{i-1})^\perp$, $i = 1, 2, \ldots$, and $\widehat{\mathcal{K}} = (A\mathcal{W}^*)^\perp \cap (E\mathcal{V}^*)^\perp$. Then*

$$a = \dim(\mathcal{W}_1 \cap \mathcal{K}), \quad d = \dim(\widehat{\mathcal{W}}_1 \cap \widehat{\mathcal{K}})$$

and with, for $i = 0, 1, 2, \ldots$,

$$\begin{aligned}\Delta_i &:= \dim(\mathcal{W}_{i+2} \cap \mathcal{K}) - \dim(\mathcal{W}_{i+1} \cap \mathcal{K}),\\ \widehat{\Delta}_i &:= \dim(\widehat{\mathcal{W}}_{i+2} \cap \widehat{\mathcal{K}}) - \dim(\widehat{\mathcal{W}}_{i+1} \cap \widehat{\mathcal{K}}),\end{aligned}$$

it holds that

$$\varepsilon_1 = \ldots = \varepsilon_{a-\Delta_0} = 0, \quad \varepsilon_{a-\Delta_{i-1}+1} = \ldots = \varepsilon_{a-\Delta_i} = i, \quad i = 1, 2, \ldots,$$

and

$$\eta_1 = \ldots = \eta_{d-\widehat{\Delta}_0} = 0, \quad \eta_{d-\widehat{\Delta}_{i-1}+1} = \ldots = \eta_{d-\widehat{\Delta}_i} = i, \quad i = 1, 2, \ldots.$$

Furthermore, the minimal indices are invariant under matrix pencil equivalence. In particular, the KCF is unique.

Proof: First consider two equivalent matrix pencils $sE-A$ and $sE'-A'$ and the relationship of the corresponding Wong sequences (2.3.11). From this it follows that the subspace dimension used for the calculations of a, d, Δ_i and $\widehat{\Delta}_i$, $i = 0, 1, 2, \ldots$, are invariant under equivalence transformations. Hence it suffices to show the claim when the matrix pair is already in KCF (2.1.6). In particular, uniqueness of the KCF then also follows from this invariance.

Note that the Wong sequences of a matrix pencil in IQKF (2.1.3) fulfill

$$\mathcal{V}_i = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathcal{V}_i^P \oplus \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathcal{V}_i^R \oplus \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \mathcal{V}_i^Q, \quad \mathcal{W}_i = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \mathcal{W}_i^P \oplus \begin{bmatrix} 0 \\ I \\ 0 \end{bmatrix} \mathcal{W}_i^R \oplus \begin{bmatrix} 0 \\ 0 \\ I \end{bmatrix} \mathcal{W}_i^Q,$$

where $\mathcal{V}_i^P$, $\mathcal{W}_i^P$, $\mathcal{V}_i^R$, $\mathcal{W}_i^R$, $\mathcal{V}_i^Q$, $\mathcal{W}_i^Q$, $i \in \mathbb{N}_0$, are the Wong sequences corresponding to the matrix pencils $sE_P - A_P$, $sE_R - A_R$ and $sE_Q - A_Q$. Furthermore, due to Corollary 2.3.20 and the uniqueness result therein, it follows that

$$\mathcal{V}^* \cap \mathcal{W}^* = \operatorname{im} \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Altogether it is therefore sufficient to show the claim for the singular block $sE_P - A_P$ with its Wong sequences $\mathcal{V}_i^P$ and $\mathcal{W}_i^P$; the result for the singular block $sE_Q - A_Q$ follows by considering the (conjugate) transpose and invoking Lemma 2.3.8. Since $\mathcal{K}^P := \mathcal{V}^{P^*} \cap \mathcal{W}^{P^*} = \mathbb{R}^{n_p}$ we have $\mathcal{W}_i^P \cap \mathcal{K}^P = \mathcal{W}_i^P$ and the claim simplifies again.

The remaining proof is quite simple but the notation is rather cumbersome, therefore we accompany the proof with an illustrative example:

$$sE_P - A_P = s \begin{bmatrix} \bar{0} & \bar{0} & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} \bar{0} & \bar{0} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}. \tag{2.3.14}$$

In this example $a = 5$, $\varepsilon_1 = \varepsilon_2 = 0$ (indicated by the bars in (2.3.14)), $\varepsilon_3 = \varepsilon_4 = 1$ and $\varepsilon_5 = 3$. Denote by $\kappa_1, \kappa_2, \ldots, \kappa_a$ the position where a new $\mathcal{P}_\varepsilon(s)$ block begins in $sE_P - A_P$, for the example this is $\kappa_1 = 1$, $\kappa_2 = 2$, $\kappa_3 = 3$, $\kappa_4 = 5$, $\kappa_5 = 7$. By definition

$$\mathcal{W}_1^P = \ker E_P = \operatorname{im}[e_{\kappa_1}, e_{\kappa_2}, \ldots, e_{\kappa_a}] =: \operatorname{im} W_1^P,$$

where e_ℓ denotes the ℓ-th unit vector. This shows $a = \dim \mathcal{W}_1^P = \dim(\mathcal{W}_1 \cap \mathcal{K})$.

Let ν_i, $i = 0, 1, 2, \ldots, m_P$ be the number of $\mathcal{P}_\varepsilon(s)$-blocks of size i in $sE_P - A_P$, for our example (2.3.14) this means $\nu_0 = 2$, $\nu_1 = 2$, $\nu_2 = 0$, $\nu_3 = 1$, $\nu_4 = 0$, $\nu_5 = 0$. Then

$$A_P W_1^P = [\underbrace{0, \ldots, 0}_{\nu_0}, \underbrace{e_1, \ldots, e_{\nu_1}}_{\nu_1}, \underbrace{e_{\nu_1+1}, e_{\nu_1+3}, \ldots, e_{\nu_1+2(\nu_2-1)}}_{\nu_2}, \ldots,$$
$$\underbrace{e_{\nu_1+2\nu_2+\ldots+(m_P-1)\nu_{m_P-1}+1}, \ldots, e_{\nu_1+2\nu_2+\ldots+(m_P-1)\nu_{m_P-1}+m_P(\nu_{m_P}-1)}}_{\nu_{m_P}}].$$

For (2.3.14) this reads as

$$A_P W_1^P = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&0&0\\ 0&0&0&0&0 \end{bmatrix}.$$

Denote by β_1 the smallest index such that $\varepsilon_{\beta_1} \geq 1$, if it exists, $\beta_1 = a+1$ otherwise (for our example, $\beta_1 = 3$). It then follows that

$$\begin{aligned} \mathcal{W}_2^P &= E_P{}^{-1}(A_P \mathcal{W}_1^P) \\ &= \mathcal{W}_1^P \oplus \operatorname{im}[e_{\kappa_{\beta_1}+1}, e_{\kappa_{\beta_1+1}+1}, \ldots, e_{\kappa_a+1}] \\ &= \operatorname{im}[e_{\kappa_1}, \ldots, e_{\kappa_{\beta_1-1}}, e_{\kappa_{\beta_1}}, e_{\kappa_{\beta_1}+1}, e_{\kappa_{\beta_1+1}}, e_{\kappa_{\beta_1+1}+1}, \ldots, e_{\kappa_a}, e_{\kappa_a+1}] \\ &=: \operatorname{im} W_2^P. \end{aligned}$$

For the example (2.3.14) this is

$$\mathcal{W}_2^P = \mathcal{W}_1^P \oplus \operatorname{im}[e_4, e_6, e_8] = \operatorname{im}[e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8].$$

Since $\Delta_0 = \dim \mathcal{W}_2^P - \dim \mathcal{W}_1^P = a - (\beta_1 - 1)$ and $\varepsilon_1 = \ldots = \varepsilon_{\beta_1-1} = 0$ we have shown that $\varepsilon_1 = \ldots = \varepsilon_{a-\Delta_0} = 0$.

With β_i being the smallest index such that $\varepsilon_{\beta_i} \geq i$ (or $\beta_i = a+1$ if it does not exist) and with an analogue argument as above we inductively conclude that

$$\mathcal{W}_{i+1}^P = \mathcal{W}_i^P \oplus \operatorname{im}[e_{\kappa_{\beta_i}+i}, e_{\kappa_{\beta_i+1}+i}, \ldots, e_{\kappa_a+i}], \quad i = 2, 3, \ldots$$

In the example we obtain $\beta_2 = \beta_3 = 5$ and $\beta_4 = \beta_5 = \ldots = 6$ and

$$\begin{aligned} &\mathcal{W}_3^P = \mathcal{W}_2^P \oplus \operatorname{im}[e_9], \\ &\mathcal{W}_4^P = \mathcal{W}_3^P \oplus \operatorname{im}[e_{10}], \\ &\mathcal{W}_5^P = \mathcal{W}_4^P, \quad \mathcal{W}_6^P = \mathcal{W}_5^P, \ \ldots. \end{aligned}$$

Hence $\Delta_{i-1} = \dim \mathcal{W}_{i+1} - \dim \mathcal{W}_i = a - (\beta_i - 1)$ and by induction $\Delta_i = a - (\beta_{i+1} - 1)$. By definition, $\varepsilon_{\beta_{i-1}} = \ldots = \varepsilon_{\beta_i - 1} = i$ and therefore $\varepsilon_{a-\Delta_{i-1}} = \ldots = \varepsilon_{a-(\Delta_i - 1)} = i$. □

Note that $\Delta_{i-1} = \Delta_i$ or $\widehat{\Delta}_{i-1} = \widehat{\Delta}_i$ is possible for some i. In that case there are no minimal indices of value i because the corresponding index range is empty. Furthermore, once $\Delta_i = 0$ or $\widehat{\Delta}_{\widehat{i}} = 0$ for some index i or $\widehat{i}$ then $\Delta_{i+j} = 0$ and $\widehat{\Delta}_{\widehat{i}+\widehat{j}} = 0$ for all $j \geq 0$ and $\widehat{j} \geq 0$. In particular, there are no minimal indices with values larger than i and $\widehat{i}$, respectively.

The proof of Theorem 2.3.22 uses the KCF and therefore, in particular it uses Lemma 2.3.14 and Corollary 2.3.15 which provide an explicit method to obtain the KCF for the full rank matrix pencils $sE_P - A_P$ and $sE_Q - A_Q$ in the QKF (2.1.5). However, if one is only interested in the singular part of the KCF (without the necessary transformation), Theorem 2.3.22 shows that knowledge of the Wong sequences is already sufficient; there is no need to carry out the tedious calculations of Lemma 2.3.14 and Corollary 2.3.15.

In the following we show that the index (of the nilpotent part) of the matrix pencil and the degrees of the infinite elementary divisors may be obtained from the Wong sequences as well.

Proposition 2.3.23 (Index and infinite elementary divisors).
Consider the Wong sequences $\mathcal{V}_i$ and $\mathcal{W}_i$ and the notation from Theorem 2.3.1 and the form (2.1.4). *Let*

$$\nu := \min \{ \, i \in \mathbb{N}_0 \; | \; \mathcal{V}^* + \mathcal{W}_i = \mathcal{V}^* + \mathcal{W}_{i+1} \, \} \, .$$

If $\nu \geq 1$, then ν is the index of nilpotency of $A_N^{-1}E_N$, i.e., $(A_N^{-1}E_N)^\nu = 0$ and $(A_N^{-1}E_N)^{\nu-1} \neq 0$. If $\nu = 0$, then $n_N = 0$, i.e., the pencil $sE_N - A_N$ is absent in the form (2.1.5).

Furthermore, if $\nu \geq 1$, let

$$\Delta_i := \dim(\mathcal{V}^* + \mathcal{W}_{i+1}) - \dim(\mathcal{V}^* + \mathcal{W}_i) \geq 0, \quad i = 0, 1, 2, \ldots \nu.$$

Then $\Delta_{i-1} \geq \Delta_i$, $i = 1, 2, \ldots, \nu$ and, for $c = \Delta_0$, let the numbers $\sigma_1, \sigma_2, \ldots, \sigma_c \in \mathbb{N}_0$ be given by (where in case of $\Delta_{i-1} = \Delta_i$ the respective index range is empty)

$$\sigma_{c-\Delta_{i-1}+1} = \ldots = \sigma_{c-\Delta_i} = i, \quad i = 1, 2, \ldots, \nu.$$

Then $(E_N, A_N) \cong (N, I)$ where $N = \operatorname{diag}(N_{\sigma_1}, N_{\sigma_2}, \ldots, N_{\sigma_c})$.

Proof: As in the proof of Theorem 2.3.22 we may assume, without loss of generality, that $sE - A$ is in KCF (2.1.6). Decomposing the Wong sequences into the four parts corresponding to each type of blocks in the QKF (2.1.5), that is

$$\mathcal{V}_i = \mathcal{V}_i^P \times \mathcal{V}_i^J \times \mathcal{V}_i^N \times \mathcal{V}_i^Q, \quad \mathcal{W}_i = \mathcal{W}_i^P \times \mathcal{W}_i^J \times \mathcal{W}_i^N \times \mathcal{W}_i^Q,$$

and supposing that $sE_P - A_P$, $sE_J - A_J = sI - J$, $sE_N - A_N = sN - I$ and $sE_Q - A_Q$ are in KCF, we find that:

(i) $\mathcal{V}_1^P = A_P^{-1}(\operatorname{im} E_P) = A_P^{-1}\mathbb{K}^{n_P} = \mathbb{K}^{n_P} \implies \forall\, i \geq 0: \ \mathcal{V}_i^P = \mathbb{K}^{n_P}$.

(ii) $\mathcal{V}_1^J = J^{-1}\mathbb{K}^{n_J} = \mathbb{K}^{n_J} \implies \forall\, i \geq 0: \ \mathcal{V}_i^J = \mathbb{K}^{n_J}$.

(iii) $\mathcal{V}_1^N = \operatorname{im} N$ and $\mathcal{V}_{i+1}^N = N\mathcal{V}_i^N \implies \forall\, i \geq 0: \ \mathcal{V}_i^N = \operatorname{im} N^i$.

(iv) For the derivation of $\mathcal{V}_i^Q$, we assume for a moment that $sE_Q - A_Q$ consists only of one block, that is $sE_Q - A_Q = \mathcal{Q}_\eta(s)$ for some $\eta \in \mathbb{N}_0$. Then

$$\begin{aligned}\mathcal{V}_1^Q = A_Q^{-1}(\operatorname{im} E_Q) &= \left\{ x \in \mathbb{K}^{n_Q} \ \middle|\ \exists\, y \in \mathbb{K}^{n_Q}: \ \begin{pmatrix} x \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ y \end{pmatrix} \right\} \\ &= \left\{ x \in \mathbb{K}^{n_Q} \ \middle|\ x_1 = 0 \right\},\end{aligned}$$

and, iteratively, $\mathcal{V}_i^Q = \{ x \in \mathbb{K}^{n_Q} \mid x_1 = \ldots = x_i = 0 \}$. For a higher number of blocks the calculation is analogous, but what matters is that there exists some $j \in \mathbb{N}_0$ such that $\mathcal{V}_j^Q = \{0\}$.

The above yields that

$$\mathcal{V}^* = \mathbb{K}^{n_P} \times \mathbb{K}^{n_J} \times \{0\}^{n_N} \times \{0\}^{n_Q}.$$

Now observe that:

(i) $\mathcal{W}_1^N = \ker N$ and $\mathcal{W}_{i+1}^N = N^{-1}\mathcal{W}_i^N \implies \forall\, i \geq 0: \ \mathcal{W}_i^N = \ker N^i$.

(ii) $\mathcal{W}_1^Q = \ker E_Q = \{0\} \implies \forall\, i \geq 0: \ \mathcal{W}_i^Q = \{0\}^{n_Q}$.

The assertion of the proposition is then immediate from

$$\mathcal{V}^* + \mathcal{W}_i = \mathbb{K}^{n_P} \times \mathbb{K}^{n_J} \times \ker N^i \times \{0\}^{n_Q}, \quad i \geq 0. \qquad \square$$

Remark 2.3.24 (Wong sequences determine infinite and singular structure)**.**
From Proposition 2.3.23 and Theorem 2.3.22 we see that the degrees of the infinite elementary divisors and the row and column minimal indices corresponding to a matrix pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ are fully determined by the Wong sequences corresponding to $sE - A$. It can also be seen from the representation of the Wong sequences for a matrix pencil in KCF that the degrees of the finite elementary divisors cannot be deduced from the Wong sequences. However, they can be derived from a modification of the second Wong sequence (similar to Definition 2.2.15) as shown in the following.

Proposition 2.3.25 (Finite elementary divisors)**.**
Consider the Wong sequences $\mathcal{V}_i$ and $\mathcal{W}_i$ and the notation from Theorem 2.3.1 and the form (2.1.4)*. Consider, for $\lambda \in \mathbb{C}$, the sequence*

$$\mathcal{W}_0^\lambda := \{0\}, \quad \mathcal{W}_{i+1}^\lambda := (A - \lambda E)^{-1}(E\mathcal{W}_i^\lambda) \subseteq \mathbb{K}^n. \tag{2.3.15}$$

Then we have, for all $\lambda \in \mathbb{C}$, the characterization

$$\lambda \notin \operatorname{spec}(sE_J - A_J) \quad \Longleftrightarrow \quad \mathcal{W}_1^\lambda \subseteq \mathcal{W}^*. \tag{2.3.16}$$

Consider now the notation from Corollary 2.3.21 so that $S(sE - A)T$ is in KCF (2.1.6) *and reorder $\mathcal{J}_{\rho_1}^{\lambda_1}(s)$, …, $\mathcal{J}_{\rho_b}^{\lambda_b}(s)$ as $\mathcal{J}_{\rho_{1,1}}^{\alpha_1}(s)$, …, $\mathcal{J}_{\rho_{b_1,1}}^{\alpha_1}(s)$, $\mathcal{J}_{\rho_{1,2}}^{\alpha_2}(s)$, …, $\mathcal{J}_{\rho_{b_2,2}}^{\alpha_2}(s)$, …, $\mathcal{J}_{\rho_{1,k}}^{\alpha_k}(s)$,…, $\mathcal{J}_{\rho_{b_k,k}}^{\alpha_k}(s)$ with $\rho_{1,j} \le \ldots \le \rho_{b_j,j}$ for all $j = 1,\ldots,k$ and α_1, …, α_k pairwise distinct, where*

$$\mathcal{J}_{\rho_{i,j}}^{\lambda_j}(s) = (s - \lambda_j)I - N_{\rho_{i,j}} \in \mathbb{C}^{\rho_{i,j}\times\rho_{i,j}}[s], \; j = 1\ldots,k, \; i = 1,\ldots,b_j.$$

Let

$$\Delta_i^j := \dim(\mathcal{W}^*+\mathcal{W}_{i+1}^{\alpha_j}) - \dim(\mathcal{W}^*+\mathcal{W}_i^{\alpha_j}), \;\; j = 1,\ldots,k, \;\; i = 0,1,2,\ldots.$$

Then $\Delta_0^j = b_j$, $\Delta_{i-1}^j \ge \Delta_i^j$ and

$$\rho_{b_j-\Delta_{i-1}^j+1,j} = \ldots = \rho_{b_j-\Delta_i^j,j} = i, \quad j = 1,\ldots,k, \quad i = 1,2,3,\ldots.$$

Proof: Similar to the proof of Proposition 2.3.23 we may consider $sE - A$ in KCF. Then

$$\mathcal{W}^* = \mathbb{K}^{n_P} \times \{0\} \times \mathbb{K}^{n_N} \times \{0\}.$$

The proof now follows from the observation that, for all $\lambda \in \mathbb{C}$ and $i \in \mathbb{N}_0$,

$$\mathcal{W}^* + \mathcal{W}_i^\lambda = \mathbb{K}^{n_P} \times \left(\bigtimes_{\substack{j=1,\dots,k \\ l=1,\dots,b_k}} \left(\ker \mathcal{J}_{\rho_{l,j}}^{\alpha_j}(\lambda) \right)^i \right) \times \mathbb{K}^{n_N} \times \{0\}^{n_Q}$$

and $\ker \mathcal{J}_{\rho_{l,j}}^{\alpha_j}(\lambda) = \{0\}$ for $\lambda \neq \alpha_j$. □

Remark 2.3.26 (Jordan canonical form).
In case of a pencil $sI - A$, the following simplifications can be made in Proposition 2.3.25: $\mathcal{W}^* = \{0\}$, and hence $\mathcal{W}_i^\lambda = \ker(A - \lambda I)^i$. Then (2.3.16) becomes the classical eigenvalue definition

$$\lambda \in \sigma(A) \iff \ker(A - \lambda I) \neq \{0\},$$

Furthermore,

$$\Delta_i^j = \dim \ker(A - \lambda_j I)^{i+1} - \dim \ker(A - \lambda_j I)^i,$$

and hence we obviously obtain the Jordan canonical form of A, that is the sizes and numbers of Jordan blocks, from Proposition 2.3.25.

Remark 2.3.27 (Determination of the KCF).
The results presented in this subsection show that the KCF of a pencil $sE - A$ (without the corresponding transformation matrices) is completely determined by the Wong sequences:

(i) The row and column minimal indices η_i and ε_i are given by Theorem 2.3.22, which directly give the KCF of the singular part of the matrix pencil.

(ii) The degrees σ_i of the infinite elementary divisors are given by Proposition 2.3.23 yielding the KCF of the matrix pencil $sE_N - A_N$.

(iii) Finally, the finite eigenvalues can be determined by deriving the roots of $\det(\lambda E_J - A_J)$ or using (2.3.16), and the degrees ρ_i of the finite elementary divisors (corresponding to the above eigenvalues) are given by Proposition 2.3.25. This yields the Jordan canonical form of $E_J^{-1} A_J$ completing the KCF.

2.4 Solution theory

In this section we study the DAE

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + f(t) \tag{2.4.1}$$

corresponding to the matrix pencil $sE - A \in \mathbb{R}[s]^{m\times n}$. Note that we restrict ourselves here to the field $\mathbb{K} = \mathbb{R}$, because 1) the vast majority of DAEs arising from modeling physical phenomena are not complex-valued, 2) all the results for $\mathbb{K} = \mathbb{R}$ carry over to $\mathbb{K} = \mathbb{C}$ without modification (the converse is not true in general), 3) the case $\mathbb{K} = \mathbb{Q}$ is rather artificial when considering solutions of the DAE (2.4.1), because then we had to consider functions $f : \mathbb{R} \to \mathbb{Q}$ or even $f : \mathbb{Q} \to \mathbb{Q}$.

In Subsection 2.4.1 we show that a complete characterization of the solutions of the DAE (2.4.1) can be given in terms of the QKF, hence only using the Wong sequences. This result is based on using a unimodular extension of the underdetermined (overdetermined) part of $sE - A$ and the inverse of it. While this is a simple and general characterization, the drawback of this approach is that, due to non-uniqueness of the extension and hence the unknown degree of the inverse, we have to require that all functions involved are infinitely times (weakly) differentiable. To resolve this problem we use the KCF in Subsection 2.4.2 and derive a characterization with minimal smoothness requirements.

The results of Subsection 2.4.1 are partly contained in a joint work with Stephan Trenn [40].

2.4.1 Solutions in terms of QKF

In this subsection we will show (see Theorem 2.4.8 and Remark 2.4.9) that the Wong sequences are sufficient to completely characterize the solution behavior of the DAE (2.4.1) including the characterization of consistent initial values as well as constraints on the inhomogeneity f. Although (versions of) the Wong sequences appear frequently in the literature (see Section 2.5 (i)), it seems that their relevance for a complete solution theory of DAEs associated to a singular matrix pencil has been overlooked and the present characterization is novel.

We first have to decide in which (function) space we consider the DAE (2.4.1). We consider the space $\mathcal{L}^1_{\mathrm{loc}}$, because we like to allow for step functions as inhomogeneities and, in the later chapters, as inputs of

DAE control systems. However, we require the solution $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ to be smooth in the sense that $Ex \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$. This leads to certain smoothness requirements on the inhomogeneity and excludes the possibility of derivatives of step functions to appear in the solution, which would require distributional solutions. A distributional solution setup (from a control theoretic point of view) has been considered in [101, 188, 200] for instance, see also [77]. A typical argumentation in these works is that inconsistent initial values cause distributional solutions in a way that the state trajectory is composed of a continuous function and a linear combination of Dirac's delta impulse and some of its derivatives. However, some frequency domain considerations in [182] refute this approach (see [229] for an overview on inconsistent initialization). This justifies that we do only consider integrable solutions. For a mathematically rigorous approach to distributional solution theory of linear DAEs we refer to [227, 228] by Trenn, see also Section 2.5 (iii). We have motivated the following definition.

Definition 2.4.1 (Solutions of DAEs)**.**
Let $sE - A \in \mathbb{R}[s]^{m\times n}$, $t_0 \in \mathbb{R}$, $x^0 \in \mathbb{R}^n$ and $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$. Then $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ is called a *solution of the initial value problem* (2.4.1)*,* $x(t_0) = x^0$*,* if, and only if, $Ex \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$, $x(0) = x^0$, and (2.4.1) is satisfied for almost all $t \in \mathbb{R}$.

x is called a *solution of the DAE* (2.4.1) if, and only if, there exist $t_0 \in \mathbb{R}$, $x^0 \in \mathbb{R}^n$ such that x is a solution of the initial value problem (2.4.1), $x(t_0) = x^0$.

If $f \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^m)$, then the initial value x^0 is called *consistent* at t_0 for (2.4.1) if, and only if, the initial value problem (2.4.1), $x(t_0) = x^0$, has at least one solution $x \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n)$.

Note that the existence of a solution of (2.4.1) may automatically require a higher (weak) differentiability of (components of) the inhomogeneity f. This is also reflected in the definition of consistent initial values: since solutions are in general only unique almost everywhere, more smoothness of f and the solution x is required here.

Before stating our main results concerning the solution theory of the DAE (2.4.1), we need a preliminary result about polynomial matrices. A (square) polynomial matrix $U(s) \in \mathbb{K}[s]^{n\times n}$ is called *unimodular* if, and only if, it is invertible over the ring $\mathbb{K}[s]^{n\times n}$, i.e., there exists $V(s) \in \mathbb{K}[s]^{n\times n}$ such that $U(s)V(s) = I$. We will say that $P(s) \in \mathbb{K}[s]^{m\times n}$ can be extended to a unimodular matrix if, and only if, in the case $m < n$, there exists $Q(s) \in \mathbb{K}[s]^{(n-m)\times n}$ such that $\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix}$ is unimodular, in the case $m > n$, there exists $Q(s) \in \mathbb{K}[s]^{m\times(m-n)}$ such that $[P(s), Q(s)]$ is unimodular and, in the case $m = n$, $P(s)$ itself is unimodular.

Lemma 2.4.2 (Unimodular extension).
A matrix $P(s) \in \mathbb{K}[s]^{m\times n}$ can be extended to a unimodular matrix if, and only if, $\operatorname{rk}_{\mathbb{C}} P(\lambda) = \min\{m, n\}$ *for all* $\lambda \in \mathbb{C}$.

Proof: Necessity is clear, hence it remains to show that under the full rank assumption a unimodular extension is possible. Note that $\mathbb{K}[s]$ is a principal ideal domain, hence we can consider the Smith form [222] of $P(s)$ given by

$$P(s) = U(s) \begin{bmatrix} \Sigma_r(s) & 0 \\ 0 & 0 \end{bmatrix} V(s),$$

where $U(s), V(s)$ are unimodular matrices and $\Sigma(s) = \operatorname{diag}(\sigma_1(s), \ldots, \sigma_r(s))$, $r \in \mathbb{N}_0$, with nonzero diagonal entries. Note that $\operatorname{rk}_{\mathbb{C}} P(\lambda) = \operatorname{rk}_{\mathbb{C}} \Sigma(\lambda)$ for all $\lambda \in \mathbb{C}$, hence the full rank condition implies $r = \min\{m, n\}$ and $\sigma_1(s), \ldots, \sigma_r(s)$ are constant (nonzero) polynomials. For $m = n$ this already shows the claim. For $m > n$, i.e., $P(s) = U(s) \begin{bmatrix} \Sigma_n(s) \\ 0 \end{bmatrix} V(s)$, the sought unimodular extension is given by

$$[P(s), Q(s)] = U(s) \left[\begin{array}{c|c} \Sigma_n(s) & 0 \\ 0 & I \end{array}\right] \begin{bmatrix} V(s) & 0 \\ 0 & I \end{bmatrix}$$

and, for $m < n$,

$$\begin{bmatrix} P(s) \\ Q(s) \end{bmatrix} = \begin{bmatrix} U(s) & 0 \\ 0 & I \end{bmatrix} \left[\begin{array}{cc} \Sigma_m(s) & 0 \\ \hline 0 & I \end{array}\right] V(s). \qquad \square$$

We are now in a position to derive unimodular matrices which constitute a right (left) inverse and kernel of a given matrix pencil.

Lemma 2.4.3 (Existence of unimodular inverse).
Consider a matrix pencil $sE - A \in \mathbb{K}[s]^{m\times n}$ with $m \neq n$ and $\operatorname{rk} \lambda E - A = \min\{m, n\}$ *for all $\lambda \in \mathbb{C}$. Then there exist polynomial matrices $M(s) \in \mathbb{K}[s]^{n\times m}$ and $K(s) \in \mathbb{K}[s]^{n'\times m'}$, $n', m' \in \mathbb{N}_0$, such that, if $m < n$, $[M(s), K(s)]$ is unimodular and*

$$(sE - A)[M(s), K(s)] = [I_m, 0],$$

or, if $m > n$, $\begin{bmatrix} M(s) \\ K(s) \end{bmatrix}$ *is unimodular and*

$$\begin{bmatrix} M(s) \\ K(s) \end{bmatrix} (sE - A) = \begin{bmatrix} I_n \\ 0 \end{bmatrix}.$$

Proof: Let $Q(s)$ be any unimodular extension of $sE - A$ according to Lemma 2.4.2. If $m < n$, choose $[M(s), K(s)] = \begin{bmatrix} sE - A \\ Q(s) \end{bmatrix}^{-1}$ and if $m > n$, let $\begin{bmatrix} M(s) \\ K(s) \end{bmatrix} := [sE - A, Q(s)]^{-1}$. □

In order to prove a complete characterization of the solutions of (2.4.1) we need the following lemmas, which characterize the solutions of DAEs in the case of full rank pencils. Note that we have to assume a certain differentiability of the inhomogeneity f and the solution x for the characterization, which is due to the unknown degree of the unimodular right (left) inverse and kernel. Smoothness issues are investigated in more detail in Subsection 2.4.2.

Lemma 2.4.4 (Full row rank pencils)**.**
Let $sE - A \in \mathbb{R}[s]^{m \times n}$ *such that* $m < n$ *and* $\operatorname{rk}_{\mathbb{C}}(\lambda E - A) = m$ *for all* $\lambda \in \mathbb{C} \cup \{\infty\}$. *According to Lemma 2.4.3 choose* $M(s) \in \mathbb{R}[s]^{n \times m}$ *and* $K(s) \in \mathbb{R}[s]^{n \times (n-m)}$ *such that* $(sE - A)[M(s), K(s)] = [I, 0]$ *and* $[M(s), K(s)]$ *is unimodular. Then, for all inhomogeneities* $f \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^m)$, $x \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$ *is a solution of* (2.4.1) *if, and only if, there exists* $u \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n-m})$ *such that*

$$x = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) + K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u).$$

Furthermore, if $f \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^m)$, *then, for any* $t_0 \in \mathbb{R}$, *all initial values* $x^0 \in \mathbb{R}^n$ *are consistent at* t_0 *for* (2.4.1).

Proof: *Step 1:* We show that $x = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) + K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u)$ is a solution of (2.4.1) for any $u \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n-m})$. This is clear since

$$(\tfrac{\mathrm{d}}{\mathrm{d}t}E - A)\left(M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) + K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u)\right) = f + 0 = f$$

and we may define $t_0 := 0$ and $x^0 := x(0)$.

Step 2: We show that any solution x of (2.4.1) can be represented as above. To this end let $u := [0, I][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n-m})$, which is well-defined due to the unimodularity of $[M(s), K(s)]$. Then

$$f \overset{\text{a.e.}}{=} (\tfrac{\mathrm{d}}{\mathrm{d}t}E - A)x \overset{\text{a.e.}}{=} (\tfrac{\mathrm{d}}{\mathrm{d}t}E - A)[M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x$$
$$\overset{\text{a.e.}}{=} [I, 0][M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})]^{-1}x,$$

and therefore, invoking also continuity of all involved functions, it follows that

$$M(\tfrac{\mathrm{d}}{\mathrm{d}t})f + K(\tfrac{\mathrm{d}}{\mathrm{d}t})u = [M(\tfrac{\mathrm{d}}{\mathrm{d}t}), K(\tfrac{\mathrm{d}}{\mathrm{d}t})] \begin{bmatrix} [I, 0][M(\frac{\mathrm{d}}{\mathrm{d}t}), K(\frac{\mathrm{d}}{\mathrm{d}t})]^{-1}x \\ [0, I][M(\frac{\mathrm{d}}{\mathrm{d}t}), K(\frac{\mathrm{d}}{\mathrm{d}t})]^{-1}x \end{bmatrix} = x.$$

Step 3: Let $f \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^m)$. We show that every initial value is consistent. Write $K(s) = K_0 + K_1 s + \ldots + K_p s^p$, $p \in \mathbb{N}_0$, and let $\mathcal{K}$ be the singular chain subspace of $sE - A$ as in Corollary 2.3.11.

Step 3a: We show $\operatorname{im}[K_0, K_1, \ldots, K_p] = \mathcal{K} = \mathbb{R}^n$. Remark 2.3.2 and Lemma 2.3.12 yields $\mathbb{R}^n = \mathcal{V}^* \cap \mathcal{W}^* = \mathcal{K}$. From $(sE - A)K(s) = 0$ it follows that

$$0 = AK_0, \; EK_0 = AK_1, \ldots, \; EK_{p-1} = AK_p, \; EK_p = 0,$$

hence the i-th column vectors of $K_0, K_1, \ldots, K_p$, $i = 1, \ldots, n-m$, form a singular chain. This shows $\operatorname{im}[K_0, K_1, \ldots, K_p] \subseteq \mathcal{K}$.
For showing the converse inclusion, we first prove $\operatorname{im} K_0 = \ker A$. From $AK_0 = (\lambda E - A)K(\lambda)\big|_{\lambda=0} = 0$ it follows that $\operatorname{im} K_0 \subseteq \ker A$. By unimodularity of $[M(s), K(s)]$ it follows that $K(0) = K_0$ must have full rank, i.e. $\dim \operatorname{im} K_0 = n - m$. Full rank of $(sE - A)$ for all $s \in \mathbb{C}$ also implies full rank of A, hence $\dim \ker A = n - m$ and $\operatorname{im} K_0 = \ker A$ is shown.
Let $(x_0, x_1, \ldots, x_l)$, $l \in \mathbb{N}_0$, be a singular chain. Then $Ax_0 = 0$, i.e. $x_0 \in \ker A = \operatorname{im} K_0$. Proceeding inductively, assume $x_0, x_1, \ldots, x_i \in \operatorname{im}[K_0, K_1, \ldots, K_i]$, for some $i \in \mathbb{N}_0$ with $0 \le i < l$. For notational convenience set $K_j = 0$ for all $j > p$. From $Ax_{i+1} = Ex_i \in \operatorname{im}[EK_0, EK_1, \ldots, EK_i] = \operatorname{im}[AK_1, AK_2, \ldots, AK_{i+1}]$ it follows that $x_{i+1} \in \ker A + \operatorname{im}[K_1, K_2, \ldots, K_{i+1}] = \operatorname{im}[K_0, K_1, \ldots, K_{i+1}]$. This shows that each singular chain is contained in $\operatorname{im}[K_0, K_1, \ldots, K_p]$.

Step 3b: We show existence of a solution $x \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^n)$ such that $x(t_0) = x^0$. By Step 3a there exist $u_0, u_1, \ldots, u_p \in \mathbb{R}^{n-m}$ such that

$$K_0 u_0 + K_1 u_1 + \ldots + K_p u_p = x^0 - M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f)(t_0). \tag{2.4.2}$$

Let

$$u(t) := u_0 + (t - t_0)u_1 + \frac{(t - t_0)^2}{2}u_2 + \ldots + \frac{(t - t_0)^p}{p!}u_p, \quad t \in \mathbb{R}.$$

Then we have that $u \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^{n-m})$ and

$$K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u)(t_0) = K_0u_0 + K_1u_1 + \ldots + K_pu_p \overset{(2.4.2)}{=} x^0 - M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f)(t_0), \tag{2.4.3}$$

which implies that the solution $x = M(\frac{\mathrm{d}}{\mathrm{d}t})(f) + K(\frac{\mathrm{d}}{\mathrm{d}t})(u) \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^n)$ satisfies

$$x(t_0) = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f)(t_0) + K(\tfrac{\mathrm{d}}{\mathrm{d}t})(u)(t_0) \overset{(2.4.3)}{=} x^0.$$

□

Remark 2.4.5 (Full row rank and eigenvalue ∞)**.**
A careful analysis of the proof of Lemma 2.4.4 reveals that for the solution formula the full row rank of $\lambda E - A$ for $\lambda = \infty$ is not necessary. The latter is only necessary to show that all initial value problems have a solution.

Lemma 2.4.6 (Full column rank pencils)**.**
Let $sE - A \in \mathbb{R}[s]^{m\times n}$ such that $m > n$ and $\mathrm{rk}_{\mathbb{C}}(\lambda E - A) = n$ for all $\lambda \in \mathbb{C} \cup \{\infty\}$. According to Lemma 2.4.3 choose $M(s) \in \mathbb{R}[s]^{n\times m}$ and $K(s) \in \mathbb{R}[s]^{(m-n)\times m}$ such that $\left[\begin{smallmatrix} M(s) \\ K(s) \end{smallmatrix}\right](sE - A) = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} M(s) \\ K(s) \end{smallmatrix}\right]$ is unimodular. Then, for all $f \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^m)$, $x \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$ is a solution of (2.4.1) *if, and only if,*

$$x = M(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) \quad \textit{and} \quad K(\tfrac{\mathrm{d}}{\mathrm{d}t})(f) = 0.$$

Furthermore, every component or linear combination of f is restricted in some way, more precisely $K(s)F$ has no zero column for any invertible $F \in \mathbb{R}^{m\times m}$.

Proof: The characterization of the solution follows from the equivalence

$$(\tfrac{\mathrm{d}}{\mathrm{d}t}E - A)x \overset{\text{a.e.}}{=} f \quad \iff \quad \underbrace{\begin{bmatrix} M(\frac{\mathrm{d}}{\mathrm{d}t}) \\ K(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix}(\tfrac{\mathrm{d}}{\mathrm{d}t}E - A)}_{=\left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]}x \overset{\text{a.e.}}{=} \begin{bmatrix} M(\frac{\mathrm{d}}{\mathrm{d}t})f \\ K(\frac{\mathrm{d}}{\mathrm{d}t})f \end{bmatrix}$$

and continuity of the involved functions. To show that $K(s)F$ does not have any zero column, write $K(s) = K_0 + K_1s + \ldots + K_ps^p$.

Since $(sE^\top - A^\top)K(s)^\top = 0$ it follows with the same arguments as in Step 3a of Lemma 2.4.4 that $\operatorname{im}[K_0^\top, K_1^\top, \ldots, K_p^\top] = \mathbb{R}^m$. Hence, $\ker[K_0^\top, K_1^\top, \ldots, K_p^\top]^\top = \{0\}$ which shows that the only $v \in \mathbb{R}^m$ with $K_i v = 0$ for all $i = 1, \ldots, p$ is $v = 0$. This shows that $K(s)F$ does not have a zero column for any invertible $F \in \mathbb{R}^{m\times m}$. □

Remark 2.4.7 (Full column rank and eigenvalue ∞)**.**
Analogously, as pointed out in Remark 2.4.5, the condition that $\lambda E - A$ must have full column rank for $\lambda = \infty$ is not needed to characterize the solution. It is only needed to show that the inhomogeneity is 'completely' restricted.

The following theorem gives a complete characterization of the solution behavior of the DAE (2.4.1) just based on the QKF (2.1.5) without a knowledge of a more detailed structure (e.g., some staircase form or the KCF of some of the blocks).

Theorem 2.4.8 (Complete characterization of solutions in terms of QKF)**.**
Let $sE - A \in \mathbb{R}[s]^{m\times n}$ and use the notation from Theorem 2.3.3 so that $S(sE - A)T$ is in QKF (2.1.5)*. Let $f \in \mathcal{W}^{\infty,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^m)$ and $(f_P^\top, f_J^\top, f_N^\top, f_Q^\top)^\top := Sf$, where the splitting corresponds to the block sizes in* (2.1.5)*. According to Lemma 2.4.3 choose unimodular matrices $[M_P(s), K_P(s)] \in \mathbb{R}[s]^{n_P\times(m_P+(n_P-m_P))}$, $\begin{bmatrix} M_Q(s) \\ K_Q(s)\end{bmatrix} \in \mathbb{R}[s]^{(n_Q+(m_Q-n_Q))\times m_Q}$ such that*

$$(sE_P - A_P)[M_P(s), K_P(s)] = [I, 0] \quad \textit{and} \quad \begin{bmatrix} M_Q(s) \\ K_Q(s)\end{bmatrix}(sE_Q - A_Q) = \begin{bmatrix} I \\ 0\end{bmatrix}.$$

Then there exists a solution $x \in \mathcal{W}^{\infty,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ of the DAE (2.4.1) *if, and only if,*

$$\boxed{K_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q) = 0.} \tag{2.4.4}$$

If $f \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^m)$ and (2.4.4) *holds, then an initial value $x^0 = T\left({x_P^0}^\top, {x_J^0}^\top, {x_N^0}^\top, {x_Q^0}^\top\right)^\top$ is consistent at $t_0 \in \mathbb{R}$ for* (2.4.1) *if, and only if,*

$$\boxed{x_Q^0 = \left(M_Q(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_Q)\right)(t_0)} \quad \textit{and}$$
$$\boxed{x_N^0 = -\left(\sum_{k=0}^{n_N-1} \left(A_N^{-1}E_N\right)^k (\tfrac{\mathrm{d}}{\mathrm{d}t})^k(f_N)\right)(t_0).} \tag{2.4.5}$$

If (2.4.4) *and* (2.4.5) *hold, then* $x = T\big(x_P{}^\top, x_J{}^\top, x_N{}^\top, x_Q{}^\top\big)^\top \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$ *is a solution of the initial value problem* (2.4.1), $x(t_0) = x^0$, *if, and only if, we have*

$$\boxed{\begin{array}{rcl} x_P &=& M_P(\frac{\mathrm{d}}{\mathrm{d}t})(f_P) + K_P(\frac{\mathrm{d}}{\mathrm{d}t})(u_{x_P^0}), \\ x_J &=& e^{E_J^{-1}A_J(\cdot - t_0)}x_J^0 + \int_{t_0}^{\cdot} e^{E_J^{-1}A_J(\cdot - s)} f_J(s)\,\mathrm{d}s\,, \\ x_N &=& -\sum_{k=0}^{n_N-1}\big(A_N^{-1}E_N\big)^k(\frac{\mathrm{d}}{\mathrm{d}t})^k(f_N), \\ x_Q &=& M_Q(\frac{\mathrm{d}}{\mathrm{d}t})(f_Q). \end{array}} \tag{2.4.6}$$

where $u_{x_P^0} \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n_P - m_P})$ *is such that*

$$\big(K_P(\tfrac{\mathrm{d}}{\mathrm{d}t})(u_{x_P^0})\big)(t_0) = x_P^0 - \big(M_P(\tfrac{\mathrm{d}}{\mathrm{d}t})(f_P)\big)(t_0). \tag{2.4.7}$$

Proof: The first statement of the theorem about existence of solutions of (2.4.1) follows from Lemma 2.4.6. In order to prove the second statement about consistent initial values let $f \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^m)$ and observe that:

Step 1: Assume that x^0 is consistent at t_0 for (2.4.1). Then there exists a solution $x \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n)$ of (2.4.1), $x(t_0) = x^0$. Clearly, by Lemma 2.4.6, $x_Q^0 = x_Q(t_0) = \big(M_Q(\frac{\mathrm{d}}{\mathrm{d}t})(f_Q)\big)(t_0)$. For the second condition we calculate

$$\begin{aligned} x_N^0 = x_N(t_0) &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(A_N^{-1}E_N\big)x_N(t_0) - f_N(t_0) \\ &= \tfrac{\mathrm{d}}{\mathrm{d}t}\big(A_N^{-1}E_N\big)\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\big(A_N^{-1}E_N\big)x_N(t_0) - f_N(t_0)\right) - f_N(t_0) \\ &= \ldots \\ &= -\left(\sum_{k=0}^{n_N-1}\big(A_N^{-1}E_N\big)^k(\tfrac{\mathrm{d}}{\mathrm{d}t})^k(f_N)\right)(t_0). \end{aligned}$$

Step 2: Assume that (2.4.5) holds true. We show that there exists a solution $x \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n)$ of (2.4.1), $x(t_0) = x^0$. Clearly x as in (2.4.6) is a solution of (2.4.1), where existence of $u_{x_P^0} \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^{n_p - m_p})$ is guaranteed by Lemma 2.4.4, and the initial value conditions in (2.4.5) are satisfied.

The last statement of the theorem about the representation of solutions of initial value problems follows from the above considerations and the fact that due to Lemma 2.4.4 it is always possible to choose $u_{x_P^0} \in \mathcal{W}^{\infty,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n_P - m_P})$ such that (2.4.7) is satisfied. □

Remark 2.4.9 (Characterization in terms of QKTF)**.**
A similar statement as in Theorem 2.4.8 is also possible if we only consider the QKTF (2.1.4). The corresponding conditions for the Q-part remain the same, in the condition for the N-part the inhomogeneity f_N is replaced by $f_N - (\frac{\mathrm{d}}{\mathrm{dt}}E_{NQ} - A_{NQ})(x_Q)$, in the J-part the inhomogeneity f_J is replaced by $f_J - (\frac{\mathrm{d}}{\mathrm{dt}}E_{JN} - A_{JN})(x_N) - (\frac{\mathrm{d}}{\mathrm{dt}}E_{JQ} - A_{JQ})(x_Q)$ and in the P-part the inhomogeneity f_P is replaced by $f_P - (\frac{\mathrm{d}}{\mathrm{dt}}E_{PJ} - A_{PJ})(x_J) - (E_{PN}\frac{\mathrm{d}}{\mathrm{dt}} - A_{PN})(x_N) - (E_{PQ}\frac{\mathrm{d}}{\mathrm{dt}} - A_{PQ})(x_Q)$.

Remark 2.4.10 (Solutions on (finite) time intervals)**.**
The solution of a DAE (2.4.1) on some time interval $I \subsetneq \mathbb{R}$ can be defined in a straightforward manner (compare Definition 2.4.1). From Theorem (2.4.8) we can infer that any solution x on some finite time interval $I \subsetneq \mathbb{R}$ can be extended to a solution on the whole real axis provided the inhomogeneity f is defined on $\mathbb{R}$.

2.4.2 Solutions in terms of KCF

In this subsection we aim to derive a characterization of solutions of (2.4.1) without any further smoothness requirements on the solution using the KCF. However, we cannot use the KCF of the whole pencil; we use the QKF and transform all parts except for the ODE part into KCF. This can be achieved by combining Theorem 2.3.3, Lemma 2.3.14, Corollary 2.3.15 and the Jordan canonical form of a nilpotent matrix and leads to the following result.

Corollary 2.4.11.
For any $sE - A \in \mathbb{R}[s]^{m\times n}$ there exist $S \in \mathbf{Gl}_m(\mathbb{R})$ and $T \in \mathbf{Gl}_n(\mathbb{R})$ such that, using the notation from Definition 2.1.7,

$$S(sE - A)T = \operatorname{diag}\big(sI_k - J, \mathcal{P}_{\varepsilon_1}(s), \ldots, \mathcal{P}_{\varepsilon_a}(s), \mathcal{N}_{\sigma_1}(s), \ldots, \mathcal{N}_{\sigma_c}(s), \mathcal{Q}_{\eta_1}(s), \ldots, \mathcal{Q}_{\eta_d}(s)\big), \quad (2.4.8)$$

where $k \in \mathbb{N}_0$ and $J \in \mathbb{R}^{k\times k}$.

This leads to the separate consideration of the DAEs corresponding to each type of blocks:

(i) The ODE block $sI - J$ corresponds to the ODE $\frac{\mathrm{d}}{\mathrm{d}t}x(t) = Jx(t) + f(t)$ whose solution satisfies

$$x(t) = e^{J(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{J(t-\tau)}f(\tau)d\tau, \quad t \in \mathbb{R}.$$

In particular, solvability is guaranteed by $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m)$. The initial value $x(t_0) \in \mathbb{R}^n$ can be chosen arbitrarily; the prescription of $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m)$ and $x(t_0) \in \mathbb{R}^n$ guarantees uniqueness of the solution.

(ii) Solutions of the DAE corresponding to a block $\mathcal{N}_\sigma(s) = sN_\sigma - I$, namely $\frac{\mathrm{d}}{\mathrm{d}t}N_\sigma x(t) = x(t) + f(t)$, can be calculated as done in the proof of Theorem 2.4.8:

$$\begin{aligned} x &\overset{\text{a.e.}}{=} \tfrac{\mathrm{d}}{\mathrm{d}t}N_\sigma x - f \overset{\text{a.e.}}{=} \tfrac{\mathrm{d}}{\mathrm{d}t}N_\sigma\left(\tfrac{\mathrm{d}}{\mathrm{d}t}N_\sigma x - f\right) - f \\ &\overset{\text{a.e.}}{=} (\tfrac{\mathrm{d}}{\mathrm{d}t}N_\sigma)^2\left(\tfrac{\mathrm{d}}{\mathrm{d}t}N_\sigma x - f\right) - \tfrac{\mathrm{d}}{\mathrm{d}t}N_\sigma f - f \overset{\text{a.e.}}{=} \ldots \overset{\text{a.e.}}{=} -\sum_{k=0}^{\sigma-1}(\tfrac{\mathrm{d}}{\mathrm{d}t}N_\sigma)^k f. \end{aligned} \tag{2.4.9}$$

Here, only the property of x being a solution of (2.4.1) (and hence $N_\sigma x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\sigma)$) and no further smoothness is used. However, the solution requires a certain smoothness of the inhomogeneity, expressed by

$$\sum_{k=0}^{\sigma-2} N_\sigma(\tfrac{\mathrm{d}}{\mathrm{d}t}N_\sigma)^k f \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\sigma). \tag{2.4.10}$$

In fact, we can prove that for $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\sigma)$ there exists a solution $x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\sigma)$ of $\frac{\mathrm{d}}{\mathrm{d}t}N_\sigma x(t) = x(t) + f(t)$ if, and only if, condition (2.4.10) holds: The '$\Rightarrow$'-part follows from pre-multiplying (2.4.9) with N_σ and the fact that $N_\sigma x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\sigma)$. The '$\Leftarrow$'-part follows by defining $x := -\sum_{k=0}^{\sigma-1}(\frac{\mathrm{d}}{\mathrm{d}t}N_\sigma)^k f$ and observing that by (2.4.10) x is indeed well-defined and satisfies $x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\sigma)$ and $N_\sigma x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\sigma)$, thus x is a solution of $\frac{\mathrm{d}}{\mathrm{d}t}N_\sigma x(t) = x(t) + f(t)$.

Note that, similar to Theorem 2.4.8, for $f \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^\sigma)$ an initial value $x^0 \in \mathbb{R}^\sigma$ is consistent at $t_0 \in \mathbb{R}$ for $\frac{\mathrm{d}}{\mathrm{d}t}N_\sigma x(t) = x(t) + f(t)$

if, and only if,

$$x^0 = -\left(\sum_{k=0}^{\sigma-1} (\tfrac{\mathrm{d}}{\mathrm{d}t} N_\sigma)^k (f)\right)(t_0).$$

However, solutions to initial value problems may even exist for inconsistent initial values, see Example 2.4.12.

(iii) An underdetermined block $\mathcal{P}_\varepsilon(s) = sL_{\varepsilon+1} - K_{\varepsilon+1}$ corresponds to a DAE $\frac{\mathrm{d}}{\mathrm{d}t} L_{\varepsilon+1} x = K_{\varepsilon+1} x + f$. Using the structure of $L_{\varepsilon+1}$ and $K_{\varepsilon+1}$ and

$$x_- := \begin{pmatrix} x_2 \\ \vdots \\ x_{\varepsilon+1} \end{pmatrix},$$

we may rewrite this equation as

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x_-(t) = N_\varepsilon x_-(t) + e_1 x_1(t) + f(t)$$

and e_1 is the first unit vector. Hence, a solution exists for all $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^\varepsilon)$ and all $x_1 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ as well as $x_2(t_0), \ldots, x_{\varepsilon+1}(t_0)$. This system is therefore underdetermined in the sense that one component as well as all initial values can be freely chosen. Hence any existing solution for fixed inhomogeneity f and fixed initial value $x(t_0)$ is far from being unique.

(iv) An overdetermined block $\mathcal{Q}_\eta(s) = sL_{\eta+1}^\top - K_{\eta+1}^\top$ corresponds to a DAE $\frac{\mathrm{d}}{\mathrm{d}t} L_{\eta+1}^\top x = K_{\eta+1}^\top x + f$. Using the structure of $L_{\eta+1}^\top$ and $K_{\eta+1}^\top$ and denoting

$$x_+ := \begin{pmatrix} x \\ 0 \end{pmatrix},$$

we obtain the equivalent equation

$$\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\eta+1} x_+(t) = x_+(t) + f(t).$$

Hence we obtain $x_+ \overset{\text{a.e.}}{=} -\sum_{k=0}^{\eta} (\frac{\mathrm{d}}{\mathrm{d}t} N_{\eta+1})^k f$, which gives

$$x \overset{\text{a.e.}}{=} -[I_\eta, 0] \sum_{k=0}^{\eta} (\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\eta+1})^k f$$

together with the consistency condition on the inhomogeneity (which is a special case of (2.4.4)):

$$e_{\eta+1}^\top \sum_{k=0}^{\eta} (\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\eta+1})^k f \overset{\text{a.e.}}{=} 0. \tag{2.4.11}$$

The smoothness condition

$$\sum_{k=0}^{\eta-1} N_{\eta+1} (\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\eta+1})^k f \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{\eta+1})$$

is therefore not enough to guarantee existence of a solution; the additional constraint formed by (2.4.11) has to be satisfied, too. Furthermore, as in (ii), consistency of the initial value x^0 at t_0 is restricted by the inhomogeneity f.

Example 2.4.12 (Inconsistent initial values)**.**
Consider the simple DAE

$$0 = x(t) + f(t) \tag{2.4.12}$$

for $f \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R})$. Clearly, $x^0 \in \mathbb{R}$ is consistent at $t_0 \in \mathbb{R}$ for (2.4.12) if, and only if, $x^0 = -f(t_0)$. However, the solution of (2.4.12) is only unique almost everywhere since any $y \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R})$ which satisfies $y(t) = -f(t)$ for almost all $t \in \mathbb{R}$ is a solution of (2.4.12). Therefore, any initial value problem (2.4.12), $x(t_0) = x^0$, with arbitrary $x^0 \in \mathbb{R}$ has a solution, but only $x^0 = -f(t_0)$ is consistent.

Summarizing the above considerations we obtain the following theorem which characterizes existence of solutions of a DAE (2.4.1) by smoothness and consistency conditions on the inhomogeneity. The smoothness conditions reflect the limits of the solution setup using integrable functions only: if any component of the inhomogeneity does not obey these conditions this directly leads to Dirac impulses in the solution and hence the requirement for a distributional solution setup, see also Section 2.5 (iii). We omit the characterization of consistent initial values; this can be concluded from (i)–(iv) above.

Theorem 2.4.13 (Characterization of solutions of the DAE in terms of KCF)**.**

Let $sE - A \in \mathbb{R}[s]^{m \times n}$ *and use the notation from Corollary 2.4.11 so that* (2.4.8) *holds. Let* $f \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m)$ *and, according to the block structure of* (2.4.8),

$$\left(f_J^\top, f_{P,1}^\top, \ldots, f_{P,a}^\top, f_{N,1}^\top, \ldots, f_{N,c}^\top, f_{Q,1}^\top, \ldots, f_{Q,d}^\top\right)^\top := Sf.$$

Then there exists a solution $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ *of* (2.4.1) *if, and only if, the smoothness conditions*

$$\begin{aligned} &\forall\, i = 1, \ldots, c: \ \sum_{k=0}^{\sigma_i - 2} N_{\sigma_i} (\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\sigma_i})^k f_{N,i} \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^{\sigma_i}) \qquad \textit{and} \\ &\forall\, i = 1, \ldots, d: \ \sum_{k=0}^{\eta_i - 1} N_{\eta_i+1} (\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\eta_i+1})^k f_{Q,i} \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^{\eta_i+1}) \end{aligned} \tag{2.4.13}$$

and the consistency condition

$$\forall\, i = 1, \ldots, d: \ e_{\eta_i+1}^\top \sum_{k=0}^{\eta_i} (\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\eta_i+1})^k f_{Q,i} \overset{a.e.}{=} 0$$

are satisfied.

Note that the smoothness conditions (2.4.13) in Theorem 2.4.13 implicitly require a higher (weak) differentiability of certain components of f so that (2.4.13) is satisfied.

2.5 Notes and References

(i) The Wong sequences can be traced back to Dieudonné [87], however his focus is only on the first of the two Wong sequences. Bernhard [42] and Armentano [7] used the Wong sequences to carry out a geometric analysis of matrix pencils. In [185] the first Wong sequence is introduced as a 'fundamental geometric tool in the characterization of the subspace of consistent initial conditions' of a regular DAE. Both Wong sequences are introduced in [184] where the authors obtain a quasi-Kronecker staircase form; however, they did not consider both Wong sequences in combination so that the important role of the spaces $\mathcal{V}^* \cap \mathcal{W}^*$, $\mathcal{V}^* + \mathcal{W}^*$, $E\mathcal{V}^* \cap A\mathcal{W}^*$, $E\mathcal{V}^* + A\mathcal{W}^*$ (see Definition 2.2.1, Figure 2.1 and Theorem 2.3.1) is not highlighted. They also appear in [3,4,151,233]. In control theory, modified versions of the Wong sequences (where $\operatorname{im} B$ is added to $E\mathcal{V}_i$ and $A\mathcal{W}_i$ resp.) have been studied extensively for not necessarily regular DAEs, see e.g. [3, 11, 13, 14, 92, 159, 167, 188, 200] and they have been found to be an appropriate tool to construct invariant subspaces, the reachable space and provide a Kalman decomposition, to name but a few features.

(ii) The notion of consistent initial values as in Definition 2.4.1 is most important for DAEs and therefore as old as the theory of DAEs itself, see e.g. [100]. The space of consistent initial values is often used and sometimes called viability kernel [45], see also [9, 10].

(iii) Since solutions of a DAE (2.4.1) might involve derivatives of the inhomogeneities, jumps in the inhomogeneity might lead to non-existence of solutions due to a lack of differentiability. This is characterized by the smoothness conditions in Theorem 2.4.13. However, this is not a 'structural non-existence' since every smooth approximation of the jump could lead to well defined solutions. Therefore, one might extend the solution space by considering distributions (or generalized functions) as formally introduced by Schwartz [218]. The advantage of this larger solution space is that each distribution is 'smooth', in particular the unit step function (Heaviside function) has a derivative: the Dirac impulse. Unfortunately, the whole space of distributions is too large, for example it is in general not possible to speak of an initial value, because evaluation of a distribution at a specific time is not defined. To overcome this obstacle one might consider the smaller space of *piecewise-smooth distributions* $\mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ as introduced in [227, 228]. For piecewise-smooth distributions a left- and right-sided evaluation is possible, i.e., for $D \in \mathbb{D}_{\mathrm{pw}\mathcal{C}^\infty}$ the values $D(t-) \in \mathbb{R}$ and $D(t+) \in \mathbb{R}$ are well defined for all $t \in \mathbb{R}$.

(iv) Distributional solutions for time-invariant DAEs have already been considered by Cobb [75] and Geerts [101, 102] and for time-varying DAEs by Rabier and Rheinboldt [202] and Kunkel and Mehrmann [152].

3 Controllability

In this chapter we consider linear differential-algebraic control systems of the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t),$$

with $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$; the set of these systems is denoted by $\Sigma_{l,n,m}$, and we write $[E, A, B] \in \Sigma_{l,n,m}$. We do not assume that the pencil $sE - A \in \mathbb{R}[s]^{l\times n}$ is regular.

The function $u : \mathbb{R} \to \mathbb{R}^m$ is called *input*; $x : \mathbb{R} \to \mathbb{R}^n$ is called *(generalized) state*. Note that, strictly speaking, $x(t)$ is in general not a state in the sense that the free system (i.e., $u \equiv 0$) satisfies a semigroup property [145, Sec. 2.2]. We will, however, speak of the state $x(t)$ for sake of brevity, especially since $x(t)$ contains the full information about the system at time t. Furthermore, one might argue that (especially in the behavioral setting) it is not correct to call u an 'input', because due to the implicit nature of (3.1.1) it may be that actually some components of u are uniquely determined and some components of x are free, and only the free variables should be called inputs in the behavioral setting. However, the controllability concepts given in Definition 3.1.5 explicitly distinguish between x and u and not between free and determined variables. We feel that, in some cases, it might still be the choice of the designer to assign the input variables, that is u, and if some of these are determined, then the input space has to be restricted in an appropriate way.

By virtue of Definition 2.4.1, a trajectory $(x, u) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ is said to be a *solution* of $[E, A, B]$ if, and only if, it belongs to the *behavior* of $[E, A, B]$:

$$\mathfrak{B}_{[E,A,B]} := \big\{(x,u) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m) \mid Ex \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^l) \text{ and } (x,u) \text{ satisfies } \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t) \text{ for almost all } t \in \mathbb{R}\big\}.$$

Note that, by linearity of the system $[E, A, B]$, $\mathfrak{B}_{[E,A,B]}$ is a vector space. This chapter is organized as follows.

In Section 3.1 the concepts of impulse controllability, controllability at infinity, R-controllability, controllability in the behavioral sense, strong and complete controllability, as well as strong and complete reachability and stabilizability in the behavioral sense, strong and complete stabilizability will be defined and described in time-domain. In the DAE literature these notions are not consistently treated; for a clarification see Section 3.7. Some relations between the concepts are also included in Section 3.1.

In Section 3.2 we concentrate on decompositions under state space transformation and, further, under state space, input and feedback transformations. We introduce the concepts of system and feedback equivalence and state decompositions under these equivalences, which for instance generalize the Brunovský form. It is also discussed when these forms are canonical and what properties (regarding controllability and stabilizability) the appearing subsystems have.

The generalized Brunovský form enables us to give short proofs of equivalent criteria, in particular generalizations of the Hautus test, for the controllability concepts in Section 3.3, the most of which are of course well-known - we discuss the relevant literature.

In Section 3.4 we revisit the concept of feedback for DAE systems and prove new results concerning the equivalence of stabilizability of DAE control systems and the existence of a feedback which stabilizes the closed-loop system. Stabilization via control in the behavioral sense is investigated as well.

In Section 3.5 we give a brief summary of some selected results of the geometric theory using invariant subspaces which lead to a representation of the reachable space and criteria for controllability at infinity, impulse controllability, controllability in the behavioral sense, complete and strong controllability.

Finally, in Section 3.6 the results regarding the Kalman decomposition for DAE systems are stated and it is shown how the controllability concepts can be related to certain properties of the Kalman decomposition.

The results of this chapter have been published in a joint work with Timo Reis [38].

3.1 Controllability concepts

In this section we introduce and investigate controllability concepts for control systems

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \tag{3.1.1}$$

where $[E, A, B] \in \Sigma_{l,n,m}$. The following spaces play a fundamental role in this chapter:

(a) The *space of consistent initial states*

$$\mathcal{V}_{[E,A,B]} = \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\,(x,u) \in \mathfrak{B}_{[E,A,B]} : \\ x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \;\wedge\; x(0) = x^0 \end{array} \right\}.$$

(b) The *space of consistent initial differential variables*

$$\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} = \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\,(x,u) \in \mathfrak{B}_{[E,A,B]} : \\ x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \;\wedge\; Ex(0) = Ex^0 \end{array} \right\}.$$

(c) The *reachable space at time* $t \geq 0$

$$\mathcal{R}^t_{[E,A,B]} = \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists\,(x,u) \in \mathfrak{B}_{[E,A,B]} : x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \\ \wedge\; x(0) = 0 \;\wedge\; x(t) = x^0 \end{array} \right\}$$

and the *reachable space*

$$\mathcal{R}_{[E,A,B]} = \bigcup_{t \geq 0} \mathcal{R}^t_{[E,A,B]}.$$

(d) The *controllable space at time* $t \geq 0$

$$\mathcal{C}^t_{[E,A,B]} = \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists\,(x,u) \in \mathfrak{B}_{[E,A,B]} : x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \\ \wedge\; x(0) = x^0 \;\wedge\; x(t) = 0 \end{array} \right\}$$

and the *controllable space*

$$\mathcal{C}_{[E,A,B]} = \bigcup_{t \geq 0} \mathcal{C}^t_{[E,A,B]}.$$

Remark 3.1.1 (Fundamental spaces)**.**

(i) Note that by linearity of the system, $\mathcal{V}_{[E,A,B]}$, $\mathcal{V}^{\text{diff}}_{[E,A,B]}$, $\mathcal{R}^t_{[E,A,B]}$ and $\mathcal{C}^t_{[E,A,B]}$ are linear subspaces of $\mathbb{R}^n$. We will show that $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]} = \mathcal{C}^{t_1}_{[E,A,B]} = \mathcal{C}^{t_2}_{[E,A,B]}$ for all $t_1, t_2 > 0$, see Lemma 3.1.4. This implies $\mathcal{R}_{[E,A,B]} = \mathcal{R}^t_{[E,A,B]} = \mathcal{C}^t_{[E,A,B]} = \mathcal{C}_{[E,A,B]}$ for all $t > 0$.

(ii) In the definition of the above spaces we require that $x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$. If we had assumed that for $(x,u) \in \mathfrak{B}_{[E,A,B]}$ the state x belongs to $\mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$, and not necessarily to $\mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$, then any initial state $x(0) = x^0 \in \mathbb{R}^n$ would be consistent for the equation $0 = x$. However, it is desirable that the only consistent initial value of $0 = x$ is $x^0 = 0$, which is guaranteed by $x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$.
Likewise, the reachable and controllable spaces for the equation $0 = x$ should be $\{0\}$ as well, which again is reflected by the requirement $x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$.

(iii) We have that $\mathcal{V}_{[E,A,B]} \subseteq \mathcal{V}^{\text{diff}}_{[E,A,B]}$ and the inclusion is in general strict (cf. Lemma 3.1.4). Consider for example

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{pmatrix}x_1(t)\\ x_2(t)\end{pmatrix} + \begin{bmatrix}0\\ 0\end{bmatrix}u(t),$$

where

$$\mathcal{V}_{[E,A,B]} = \{0\} \quad \text{and} \quad \mathcal{V}^{\text{diff}}_{[E,A,B]} = \operatorname{im}\begin{bmatrix}0\\ 1\end{bmatrix}.$$

Note that the sets of solutions in the definitions of $\mathcal{V}_{[E,A,B]}$ and $\mathcal{V}^{\text{diff}}_{[E,A,B]}$ coincide, but in $\mathcal{V}^{\text{diff}}_{[E,A,B]}$ we may take x^0 'larger', so that it is not an initial value.

Since the matrices in (3.1.1) are time-invariant, the behavior is *shift-invariant*, that is $(\sigma_\tau x, \sigma_\tau u) \in \mathfrak{B}_{[E,A,B]}$ for all $\tau \in \mathbb{R}$ and $(x,u) \in \mathfrak{B}_{[E,A,B]}$. Therefore, we have for all $t \in \mathbb{R}$ that

$$\mathcal{V}_{[E,A,B]} = \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\,(x,u) \in \mathfrak{B}_{[E,A,B]} : \\ x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \;\wedge\; x(t) = x^0 \end{array} \right\}, \tag{3.1.2}$$

$$\mathcal{V}^{\text{diff}}_{[E,A,B]} = \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{l} \exists\,(x,u) \in \mathfrak{B}_{[E,A,B]} : \\ x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \;\wedge\; Ex(t) = Ex^0 \end{array} \right\}.$$

In the following three lemmas we clarify some of the relations between the fundamental spaces, before we state the controllability concepts.

Lemma 3.1.2 (Inclusions for reachable spaces).
For $[E, A, B] \in \Sigma_{l,n,m}$ *and* $t_1, t_2 > 0$ *with* $t_1 < t_2$*, the following holds true:*

(a) $\mathcal{R}^{t_1}_{[E,A,B]} \subseteq \mathcal{R}^{t_2}_{[E,A,B]}$.

(b) If $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]}$*, then* $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t}_{[E,A,B]}$ *for all* $t \in \mathbb{R}$ *with* $t > t_1$.

Proof:

(a) Let $\bar{x} \in \mathcal{R}^{t_1}_{[E,A,B]}$. By definition, there exists some $(x, u) \in \mathfrak{B}_{[E,A,B]}$ with $x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$, $x(0) = 0$ and $x(t_1) = \bar{x}$. Define $(x_1, u_1) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m$ such that

$$(x_1(t), u_1(t)) = \begin{cases} (x(t - t_2 + t_1), u(t - t_2 + t_1)), & \text{if } t > t_2 - t_1 \\ (0, 0), & \text{if } t \leq t_2 - t_1 \end{cases}$$

Then $x(0) = 0$ implies that x_1 is continuous at $t_2 - t_1$. Since, furthermore,

$$\begin{aligned} x_1|_{(-\infty, t_2 - t_1]} &\in \mathcal{W}^{1,1}_{\mathrm{loc}}((-\infty, t_2 - t_1]; \mathbb{R}^n) \\ \text{and } x_1|_{[t_2 - t_1, \infty)} &\in \mathcal{W}^{1,1}_{\mathrm{loc}}([t_2 - t_1, \infty); \mathbb{R}^n), \end{aligned}$$

we have $(x_1, u_1) \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^m)$. By shift-invariance, $\frac{\mathrm{d}}{\mathrm{d}t} E x_1(t) = A x_1(t) + B u_1(t)$ holds true for almost all $t \in \mathbb{R}$, i.e., $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$. Then, due to $x_1(0) = 0$ and $\bar{x} = x(t_1) = x_1(t_2)$, we obtain $\bar{x} \in \mathcal{R}^{t_2}_{[E,A,B]}$.

(b) *Step 1:* We show that $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]}$ implies $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_1 + 2(t_2 - t_1)}_{[E,A,B]}$: By (a), it suffices to show the inclusion "$\supseteq$". Assume that $\bar{x} \in \mathcal{R}^{t_1 + 2(t_2 - t_1)}_{[E,A,B]}$, i.e., there exists some $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$ with $x_1 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$, $x_1(0) = 0$ and $x_1(t_1 + 2(t_2 - t_1)) = \bar{x}$. Since $x_1(t_2) \in \mathcal{R}^{t_2}_{[E,A,B]} = \mathcal{R}^{t_1}_{[E,A,B]}$, there exists some $(x_2, u_2) \in \mathfrak{B}_{[E,A,B]}$

with $x_2 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$, $x_2(0) = 0$ and $x_2(t_1) = x_1(t_2)$. Now consider the trajectory

$$(x(t), u(t)) = \begin{cases} (x_2(t), u_2(t)), & \text{if } t < t_1, \\ (x_1(t + (t_2 - t_1)), u_1(t + (t_2 - t_1))), & \text{if } t \geq t_1. \end{cases}$$

Since x is continuous at t_1, we can apply the same arguments as in the proof of (a) to infer that $x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ and $(x, u) \in \mathfrak{B}_{[E,A,B]}$. The result to be shown in the present step is now a consequence of $x(0) = x_2(0) = 0$ and

$$\bar{x} = x_1(t_1 + 2(t_2 - t_1)) = x(t_2) \in \mathcal{R}^{t_2}_{[E,A,B]} = \mathcal{R}^{t_1}_{[E,A,B]}.$$

Step 2: We show (b): From the result shown in the first step, we may inductively conclude that $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]}$ implies $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_1+i(t_2-t_1)}_{[E,A,B]}$ for all $i \in \mathbb{N}$. Let $t \in \mathbb{R}$ with $t > t_1$. Then there exists some $i \in \mathbb{N}$ with $t \leq t_1 + i(t_2 - t_1)$. Then statement (a) implies

$$\mathcal{R}^{t_1}_{[E,A,B]} \subseteq \mathcal{R}^{t}_{[E,A,B]} \subseteq \mathcal{R}^{t_1+i(t_2-t_1)}_{[E,A,B]},$$

and, by $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_1+i(t_2-t_1)}_{[E,A,B]}$, we obtain the desired result. □

Now we present some relations between controllable and reachable spaces of $[E, A, B] \in \Sigma_{l,n,m}$ and its *backward system* $[-E, A, B] \in \Sigma_{l,n,m}$. It can easily be verified that

$$\mathfrak{B}_{[-E,A,B]} = \left\{ \ (\varrho x, \varrho u) \ \middle| \ (x, u) \in \mathfrak{B}_{[E,A,B]} \ \right\}. \tag{3.1.3}$$

Lemma 3.1.3 (Reachable and controllable spaces of the backward system)**.**
For $[E, A, B] \in \Sigma_{l,n,m}$ and $t > 0$, we have

$$\mathcal{R}^{t}_{[E,A,B]} = \mathcal{C}^{t}_{[-E,A,B]}, \ \textit{and} \ \ \mathcal{C}^{t}_{[E,A,B]} = \mathcal{R}^{t}_{[-E,A,B]}.$$

Proof: Both assertions follow immediately from the fact that $(x, u) \in \mathfrak{B}_{[E,A,B]}$, if, and only if, $(\sigma_t(\varrho x), \sigma_t(\varrho u)) \in \mathfrak{B}_{[-E,A,B]}$. □

The previous lemma enables us to show that the controllable and reachable spaces of $[E, A, B] \in \Sigma_{l,n,m}$ are even equal. We further prove that both spaces do not depend on time $t > 0$.

Lemma 3.1.4 (Initial conditions and controllable spaces)**.**
For $[E, A, B] \in \Sigma_{l,n,m}$*, the following holds true:*

(a) $\mathcal{R}^{t_1}_{[E,A,B]} = \mathcal{R}^{t_2}_{[E,A,B]}$ *for all* $t_1, t_2 > 0$*.*

(b) $\mathcal{R}^{t}_{[E,A,B]} = \mathcal{C}^{t}_{[E,A,B]}$ *for all* $t > 0$*.*

(c) $\mathcal{V}^{\mathrm{diff}}_{[E,A,B]} = \mathcal{V}_{[E,A,B]} + \ker_{\mathbb{R}} E$*.*

Proof:

(a) By Lemma 3.1.2 (a), we have

$$\mathcal{R}^{\frac{t_1}{n+1}}_{[E,A,B]} \subseteq \mathcal{R}^{\frac{2t_1}{n+1}}_{[E,A,B]} \subseteq \cdots \subseteq \mathcal{R}^{\frac{nt_1}{n+1}}_{[E,A,B]} \subseteq \mathcal{R}^{t_1}_{[E,A,B]} \subseteq \mathbb{R}^n,$$

and thus

$$0 \leq \dim \mathcal{R}^{\frac{t_1}{n+1}}_{[E,A,B]} \leq \dim \mathcal{R}^{\frac{2t_1}{n+1}}_{[E,A,B]} \leq \cdots \leq \dim \mathcal{R}^{\frac{nt_1}{n+1}}_{[E,A,B]} \leq \dim \mathcal{R}^{t_1}_{[E,A,B]} \leq n.$$

As a consequence, there has to exist some $j \in \{1, \ldots, n+1\}$ with

$$\dim \mathcal{R}^{\frac{jt_1}{n+1}}_{[E,A,B]} = \dim \mathcal{R}^{\frac{(j+1)t_1}{n+1}}_{[E,A,B]}.$$

Together with the subset inclusion, this yields

$$\mathcal{R}^{\frac{jt_1}{n+1}}_{[E,A,B]} = \mathcal{R}^{\frac{(j+1)t_1}{n+1}}_{[E,A,B]}.$$

Lemma 3.1.2 (b) then implies the desired statement.

(b) Let $\bar{x} \in \mathcal{R}^{t}_{[E,A,B]}$. Then there exists some $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$ with $x_1 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$, $x_1(0) = 0$ and $x_1(t) = \bar{x}$. Since, by (a), we have $x_1(2t) \in \mathcal{R}^{t}_{[E,A,B]}$, there also exists some $(x_2, u_2) \in \mathfrak{B}_{[E,A,B]}$ with $x_2 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$, $x_2(0) = 0$ and $x_2(t) = x_1(2t)$. By linearity and shift-invariance, we have

$$(x, u) := (\sigma_t x_1 - x_2, \sigma_t u_1 - u_2) \in \mathfrak{B}_{[E,A,B]} \quad \wedge \quad x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n).$$

The inclusion $\mathcal{R}^t_{[E,A,B]} \subseteq \mathcal{C}^t_{[E,A,B]}$ then follows from

$$x(0) = x_1(t) - x_2(0) = \bar{x}, \quad x(t) = x_1(2t) - x_2(t) = 0.$$

To prove the opposite inclusion, we make use of the previously shown subset relation and Lemma 3.1.3 to infer that

$$\mathcal{C}^t_{[E,A,B]} = \mathcal{R}^t_{[-E,A,B]} \subseteq \mathcal{C}^t_{[-E,A,B]} = \mathcal{R}^t_{[E,A,B]}.$$

(c) We first show that $\mathcal{V}^{\text{diff}}_{[E,A,B]} \subseteq \mathcal{V}_{[E,A,B]} + \ker_{\mathbb{R}} E$: Assume that $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$, i.e., $Ex^0 = Ex(0)$ for some $(x,u) \in \mathfrak{B}_{[E,A,B]}$ with $x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$. By $x(0) \in \mathcal{V}_{[E,A,B]}$, $x(0) - x^0 \in \ker_{\mathbb{R}} E$, we obtain

$$x^0 = x(0) + (x^0 - x(0)) \in \mathcal{V}_{[E,A,B]} + \ker_{\mathbb{R}} E.$$

To prove $\mathcal{V}_{[E,A,B]} + \ker_{\mathbb{R}} E \subseteq \mathcal{V}^{\text{diff}}_{[E,A,B]}$, assume that $x^0 = x(0) + \bar{x}$ for some $(x,u) \in \mathfrak{B}_{[E,A,B]}$ with $x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ and $\bar{x} \in \ker_{\mathbb{R}} E$. Then $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$ is a consequence of $Ex^0 = E(x(0) + \bar{x}) = Ex(0)$. □

By Lemma 3.1.4 it is sufficient to consider only the spaces $\mathcal{V}_{[E,A,B]}$ and $\mathcal{R}_{[E,A,B]}$ in the following. We are now in the position to define the central notions of controllability, reachability and stabilizability considered in this thesis.

Definition 3.1.5.
The system $[E,A,B] \in \Sigma_{l,n,m}$ is called

(a) *controllable at infinity*

$$\begin{aligned} :\Longleftrightarrow \quad & \forall\, x^0 \in \mathbb{R}^n \ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \\ & \qquad x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \ \wedge \ x(0) = x^0 \\ & \text{(or, equivalently, } \mathcal{V}_{[E,A,B]} = \mathbb{R}^n). \end{aligned}$$

(b) *impulse controllable*

$$\begin{aligned} :\Longleftrightarrow \quad & \forall\, x^0 \in \mathbb{R}^n \ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \\ & \qquad x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \ \wedge \ Ex^0 = Ex(0) \\ & \text{(or, equivalently, } \mathcal{V}^{\text{diff}}_{[E,A,B]} = \mathbb{R}^n). \end{aligned}$$

(c) *controllable within the set of reachable states (R-controllable)*

$$:\iff \forall\, x_0, x_f \in \mathcal{V}_{[E,A,B]}\ \exists\, t > 0\ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \\ x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)\ \wedge\ x(0) = x_0\ \wedge\ x(t) = x_f.$$

(d) *controllable in the behavioral sense*

$$:\iff \forall\, (x_1,u_1),(x_2,u_2) \in \mathfrak{B}_{[E,A,B]}\ \exists\, T > 0\ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \\ (x(t),u(t)) = \begin{cases} (x_1(t),u_1(t)), & \text{if } t < 0, \\ (x_2(t),u_2(t)), & \text{if } t > T. \end{cases}$$

(e) *stabilizable in the behavioral sense*

$$:\iff \forall\, (x,u) \in \mathfrak{B}_{[E,A,B]}\ \exists\, (\tilde{x},\tilde{u}) \in \mathfrak{B}_{[E,A,B]} : \\ (x,u)|_{(-\infty,0)} \overset{\text{a.e.}}{=} (\tilde{x},\tilde{u})|_{(-\infty,0)} \\ \wedge\ \lim_{t\to\infty} \operatorname{ess-sup}_{[t,\infty)} \|(\tilde{x},\tilde{u})\| = 0.$$

(f) *completely reachable*

$$:\iff \exists\, t > 0\ \forall\, x_f \in \mathbb{R}^n\ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \\ x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)\ \wedge\ x(0) = 0\ \wedge\ x(t) = x_f \\ (\text{or, equivalently, } \mathcal{R}^t_{[E,A,B]} = \mathbb{R}^n \text{ for some } t > 0).$$

(g) *completely controllable*

$$:\iff \exists\, t > 0\ \forall\, x_0, x_f \in \mathbb{R}^n\ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \\ x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)\ \wedge\ x(0) = x_0\ \wedge\ x(t) = x_f.$$

(h) *completely stabilizable*

$$:\iff \forall\, x_0 \in \mathbb{R}^n\ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \\ x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)\ \wedge\ x(0) = x_0\ \wedge\ \lim_{t\to\infty} x(t) = 0.$$

(i) *strongly reachable*

$$:\iff \exists\, t > 0\ \forall\, x_f \in \mathbb{R}^n\ \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \\ Ex(0) = 0\ \wedge\ Ex(t) = Ex_f.$$

(j) *strongly controllable*

$$:\iff \exists\, t>0\ \forall\, x_0, x_f \in \mathbb{R}^n\ \exists\,(x,u)\in \mathfrak{B}_{[E,A,B]}:\\ Ex(0)=Ex_0 \ \wedge\ Ex(t)=Ex_f.$$

(k) *strongly stabilizable* (or merely *stabilizable*)

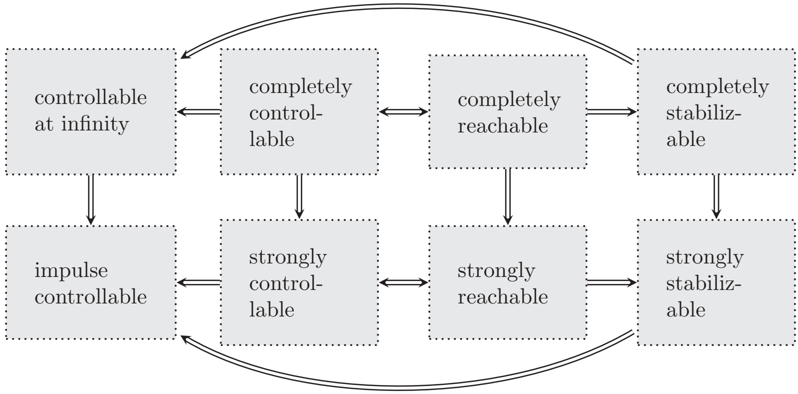
$$:\iff \forall\, x_0 \in \mathbb{R}^n\ \exists\,(x,u)\in \mathfrak{B}_{[E,A,B]}:\\ Ex(0)=Ex_0 \ \wedge\ \lim_{t\to\infty} Ex(t)=0.$$

For the development of the controllability and stabilizability concepts of Definition 3.1.5 see Section 3.7. In particular, for an explanation of the origin of the names 'controllability at infinity' and 'impulse controllability' see Section 3.7 (v).

Remark 3.1.6 (Controllability concepts)**.**
In the definition of behavioral controllability and stabilizability and strong reachability, controllability and stabilizability, different from the other concepts we do not require that $x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$. In the behavioral setting this is indeed not needed and for the strong controllability concepts we have that the considered function Ex satisfies $Ex \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^l)$. The resulting continuity of Ex also justifies the limit in the definition of strong stabilizability.

The following dependencies hold true between the concepts from Definition 3.1.5. Some further relations will be derived in Section 3.3.

Proposition 3.1.7.
For any $[E,A,B]\in\Sigma_{l,n,m}$ the following implications hold true:



If "$\Rightarrow$" holds, then "$\Leftarrow$" does, in general, not hold.

Proof: Since it is easy to construct counterexamples for any direction where in the diagram only "$\Rightarrow$" holds, we skip their presentation. The following implications are immediate consequences of Definition 3.1.5:

completely controllable $\Rightarrow$ controllable at infinity $\Rightarrow$ impulse controllable,

completely controllable $\Rightarrow$ strongly controllable $\Rightarrow$ impulse controllable,

completely controllable $\Rightarrow$ completely reachable $\Rightarrow$ strongly reachable,

strongly controllable $\Rightarrow$ strongly reachable,

completely stabilizable $\Rightarrow$ controllable at infinity,

strongly stabilizable $\Rightarrow$ impulse controllable,

completely stabilizable $\Rightarrow$ strongly stabilizable.

It remains to prove the following assertions:

(a) completely reachable $\Rightarrow$ completely controllable,

(b) strongly reachable ⇒ strongly controllable,

(c) completely reachable ⇒ completely stabilizable,

(d) strongly reachable ⇒ strongly stabilizable.

(a) Let $x_0, x_f \in \mathbb{R}^n$. Then, by complete reachability of $[E, A, B]$, there exist $t > 0$ and some $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$ with $x_1(0) = 0$ and $x_1(t) = x_0$. Further, there exists $(x_2, u_2) \in \mathfrak{B}_{[E,A,B]}$ with $x_2(0) = 0$ and $x_2(t) = x_f - x_1(2t)$. By linearity and shift-invariance, we have

$$(x, u) := (\sigma_t x_1 + x_2, \sigma_t u_1 + u_2) \in \mathfrak{B}_{[E,A,B]}.$$

On the other hand, this trajectory fulfills $x(0) = x_1(t) + x_2(0) = x_0$ and $x(t) = x_1(2t) + x_2(t) = x_f$.

(b) The proof of this statement is analogous to (a).

(c) By (a) it follows that the system is completely controllable. Complete controllability implies that there exists some $t > 0$, such that for all $x_0 \in \mathbb{R}^n$ there exists $(x_1, u_1) \in \mathfrak{B}_{[E,A,B]}$ with $x_1(0) = x_0$ and $x_1(t) = 0$. Then, since (x, u) with

$$(x(\tau), u(\tau)) = \begin{cases} (x_1(\tau), u_1(\tau)), & \text{if } \tau \leq t \\ (0, 0), & \text{if } \tau \geq t \end{cases}$$

satisfies $(x, u) \in \mathfrak{B}_{[E,A,B]}$ (cf. the proof of Lemma 3.1.2(a)) , the system $[E, A, B]$ is completely stabilizable.

(d) The proof of this statement is analogous to (c). □

3.2 Solutions, relations and decompositions

In this section we give the definitions for system and feedback equivalence of DAE control systems (see [103, 211, 241]) and state a decomposition under system and feedback equivalence (see [167]).

3.2.1 System and feedback equivalence

We define the essential concepts of system and feedback equivalence. System equivalence was first studied by Rosenbrock [211] (called restricted system equivalence in his work, see also [241]) and later became a crucial concept in the control theory of DAEs [34, 35, 103, 104, 113]. Feedback equivalence for DAEs seems to have been first considered in [103] to derive a feedback canonical form for regular systems, little later also in [167] (for general DAEs) where additionally also derivative feedback was investigated and respective canonical forms derived, see also Subsection 3.2.3.

Definition 3.2.1 (System and feedback equivalence)**.**
Two systems $[E_i, A_i, B_i] \in \Sigma_{l,n,m}$, $i = 1, 2$, are called

- *system equivalent* if, and only if,

$$\exists\, W \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}) : \\ \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} = W \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix};$$

  we write

$$[\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T}{\sim_{se}} [\,E_2\,,\,A_2\,,\,B_2\,]\,.$$

- *feedback equivalent* if, and only if,

$$\exists\, W \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), V \in \mathbf{Gl}_m(\mathbb{R}), F \in \mathbb{R}^{m\times n} : \\ \begin{bmatrix} sE_1 - A_1 & B_1 \end{bmatrix} = W \begin{bmatrix} sE_2 - A_2 & B_2 \end{bmatrix} \begin{bmatrix} T & 0 \\ -F & V \end{bmatrix}; \tag{3.2.1}$$

  we write

$$[\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T,V,F}{\sim_{fe}} [\,E_2\,,\,A_2\,,\,B_2\,]\,.$$

It is easy to observe that both system and feedback equivalence are equivalence relations on $\Sigma_{l,n,m}$. To see the latter, note that if $[\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T,V,F}{\sim_{fe}} [\,E_2\,,\,A_2\,,\,B_2\,]$, then

$$[\,E_2\,,\,A_2\,,\,B_2\,] \overset{W^{-1},T^{-1},V^{-1},-V^{-1}FT^{-1}}{\sim_{fe}} [\,E_1\,,\,A_1\,,\,B_1\,].$$

The behaviors of system and feedback equivalent systems are connected via

$$\begin{aligned}
&\text{If } [\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T}{\sim_{se}} [\,E_2\,,\,A_2\,,\,B_2\,]\,,\ \text{then}\\
&(x,u)\in\mathfrak{B}_{[E_1,A_1,B_1]} \Leftrightarrow (Tx,u)\in\mathfrak{B}_{[E_2,A_2,B_2]}\\
&\text{If } [\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T,V,F}{\sim_{fe}} [\,E_2\,,\,A_2\,,\,B_2\,]\,,\ \text{then}\\
&(x,u)\in\mathfrak{B}_{[E_1,A_1,B_1]} \Leftrightarrow (Tx,Fx+Vu)\in\mathfrak{B}_{[E_2,A_2,B_2]}.
\end{aligned}\tag{3.2.2}$$

In particular, if $[\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T}{\sim_{se}} [\,E_2\,,\,A_2\,,\,B_2\,]$, then

$$\mathcal{V}_{[E_1,A_1,B_1]} = T^{-1}\cdot\mathcal{V}_{[E_2,A_2,B_2]},\quad \mathcal{R}^t_{[E_1,A_1,B_1]} = T^{-1}\cdot\mathcal{R}^t_{[E_2,A_2,B_2]}.$$

Further, if $[\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T,V,F}{\sim_{fe}} [\,E_2\,,\,A_2\,,\,B_2\,]$, then

$$\mathcal{V}_{[E_1,A_1,B_1]} = T^{-1}\cdot\mathcal{V}_{[E_2,A_2,B_2]},\quad \mathcal{R}^t_{[E_1,A_1,B_1]} = T^{-1}\cdot\mathcal{R}^t_{[E_2,A_2,B_2]},$$

and properties of controllability at infinity, impulse controllability, R-controllability, behavioral controllability, behavioral stabilizability, complete controllability, complete stabilizability, strong controllability and strong stabilizability are invariant under system and feedback equivalence.

Remark 3.2.2 (Equivalence and minimality in the behavioral sense)**.**

(i) Another equivalence concept has been introduced by Willems in [246] (see also [198, Def. 2.5.2]): Two systems are called *equivalent in the behavioral sense*, if their behaviors coincide. This definition however, would be a bit too restrictive in the framework of this thesis, since the two systems

$$[[0],[1],[0]],\qquad \left[\begin{bmatrix}0\\1\end{bmatrix},\begin{bmatrix}1\\0\end{bmatrix},\begin{bmatrix}0\\0\end{bmatrix}\right]$$

would not be equivalent in the behavioral sense as in the first system any function which vanishes almost everywhere belongs to the behavior, while the behavior of the second system consists only of the zero function. This is due to the requirement that

$Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l)$ in the definition of the behavior. To resolve this problem, we say that $[E_i, A_i, B_i] \in \Sigma_{l_i,n,m}$, $i = 1, 2$, are *equivalent in the behavioral sense* if, and only if,

$$\mathfrak{B}_{[E_1,A_1,B_1]} \cap \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m) = \mathfrak{B}_{[E_2,A_2,B_2]} \cap \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m).$$

(ii) It is shown in [198, Thm. 2.5.4] that for a unimodular matrix $U(s) \in \mathbb{R}[s]^{l\times l}$ and $[E, A, B] \in \Sigma_{l,n,m}$ it holds that $(x, u) \in \mathfrak{B}_{[E,A,B]} \cap \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m)$ if, and only if,

$$U(\tfrac{\mathrm{d}}{\mathrm{d}t})\tfrac{\mathrm{d}}{\mathrm{d}t}Ex = U(\tfrac{\mathrm{d}}{\mathrm{d}t})Ax + U(\tfrac{\mathrm{d}}{\mathrm{d}t})Bu.$$

The unimodular matrix $U(s)$ can be chosen such that

$$U(s)\ \begin{bmatrix} sE - A, & -B \end{bmatrix} = \begin{bmatrix} R_x(s) & R_u(s) \\ 0 & 0 \end{bmatrix},$$

where $\begin{bmatrix} R_x(s) & R_u(s) \end{bmatrix} \in \mathbb{R}[s]^{k\times(n+m)}$ has full row rank over $\mathbb{R}(s)$ [198, Thm. 3.6.2]. It is shown that $R_x(\frac{\mathrm{d}}{\mathrm{d}t})x + R_u(\frac{\mathrm{d}}{\mathrm{d}t})u = 0$ is *minimal in the behavioral sense*, i.e., it describes the behavior by a minimal number of k differential equations among all behavioral descriptions of $\mathfrak{B}_{[E,A,B]}$. By using a special decomposition, we will later remark that for any $[E, A, B] \in \Sigma_{l,n,m}$ there exists a unimodular transformation from the left such that the resulting differential-algebraic system is minimal in the behavioral sense.

(iii) Conversely, if two systems $[E_i, A_i, B_i] \in \Sigma_{l_i,n,m}$, $i = 1, 2$ are equivalent in the behavioral sense, and, moreover, $l_1 = l_2$, then there exists some unimodular $U(s) \in \mathbb{R}[s]^{l_1\times l_1}$, such that

$$U(s)\ \begin{bmatrix} sE_1 - A_1, & -B_1 \end{bmatrix} = \begin{bmatrix} sE_2 - A_2, & -B_2 \end{bmatrix}.$$

If $[E_i, A_i, B_i]$ $i = 1, 2$, contain different numbers of equations (such as, e.g. $l_1 > l_2$), then one can first add $l_1 - l_2$ equations of type '$0 = 0$' to the second system and, thereafter, perform a unimodular transformation leading from one system to the other.

(iv) Provided that a unimodular transformation of $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t)$ again leads to a differential-algebraic system (that is, neither a derivative of the input nor a higher derivative of the state occurs), the properties of controllability at infinity, R-controllability, behavioral controllability, behavioral stabilizability, complete controllability, complete stabilizability are invariant under this transformation. However, since the differential variables may be changed under a transformation of this kind, the properties of impulse controllability, strong controllability and strong stabilizability are not invariant. We will see in Remark 3.2.12 that any $[E, A, B] \in \Sigma_{l,n,m}$ is, in the behavioral sense, equivalent to a system that is controllable at infinity.

For the purpose of this chapter we need a version of the quasi-Kronecker form from Theorem 2.3.3 where the nilpotent, the underdetermined and the overdetermined part are in Kronecker canonical form and the transformation matrices are real-valued. Due to this requirement we cannot use the KCF of the whole pencil; we use the QKF and transform all parts except for the ODE part into KCF as carried out in Corollary 2.4.11; we state this result again here. For the notation used in the following proposition see the List of Symbols.

Proposition 3.2.3 (Quasi-Kronecker form).
For any matrix pencil $sE - A \in \mathbb{R}[s]^{l\times n}$ there exist $W \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$W(sE-A)T \;=\; \begin{bmatrix} sI_{n_s}-A_s & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix} \tag{3.2.3}$$

for some $A_s \in \mathbb{R}^{n_s\times n_s}$ and multi-indices $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\beta}$, $\gamma \in \mathbb{N}^{n_\gamma}$. The multi-indices α, β, γ are uniquely determined by $sE - A$. Further, the matrix A_s is unique up to similarity.

The components of α are the orders of the infinite elementary divisors, the components of β are the column minimal indices and the components of γ are the row minimal indices of the matrix pencil $sE-A$, see Subsection 2.3.4. The number of column (row) minimal indices equal to one corresponds to the dimension of $\ker_{\mathbb{R}} E \cap \ker_{\mathbb{R}} A$ ($\ker_{\mathbb{R}} E^\top \cap \ker_{\mathbb{R}} A^\top$) or, equivalently, the number of zero columns (rows) in a QKF of $sE-A$. Further, note that $sI_{n_s} - A_s$ may be further transformed into Jordan canonical form to obtain the finite elementary divisors.

Since the multi-indices $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\gamma}$, $\gamma \in \mathbb{N}^{n_\gamma}$ are well-defined by means of the pencil $sE - A$ and, furthermore, the matrix A_s is unique up to similarity, this justifies the introduction of the following quantities.

Definition 3.2.4 (Index of $sE - A$)**.**
Let the matrix pencil $sE - A \in \mathbb{R}[s]^{l \times n}$ be given with QKF (3.2.3). Then the *index* $\nu \in \mathbb{N}_0$ of $sE - A$ is defined as

$$\nu = \max\{\alpha_1, \ldots, \alpha_{\ell(\alpha)}, \gamma_1, \ldots, \gamma_{\ell(\gamma)}, 0\}.$$

The index is larger or equal to the index of nilpotency ζ of N_α, i.e., $\zeta \leq \nu$, $N_\alpha^\zeta = 0$ and $N_\alpha^{\zeta-1} \neq 0$. By means of the QKF (3.2.3) it can be seen that the index of $sE - A$ does not exceed one if, and only if,

$$\operatorname{im}_{\mathbb{R}} A \subseteq \operatorname{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E. \tag{3.2.4}$$

This is moreover equivalent to the fact that for some (and hence any) real matrix Z with $\operatorname{im}_{\mathbb{R}} Z = \ker_{\mathbb{R}} E$, we have

$$\operatorname{im}_{\mathbb{R}}[E, AZ] = \operatorname{im}_{\mathbb{R}}[E, A]. \tag{3.2.5}$$

Since each block in $sK_\beta - L_\beta$ ($sK_\gamma^\top - L_\gamma^\top$) causes a single drop of the column (row) rank of $sE - A$, we have

$$\ell(\beta) = n - \operatorname{rk}_{\mathbb{R}(s)}(sE - A), \quad \ell(\gamma) = l - \operatorname{rk}_{\mathbb{R}(s)}(sE - A). \tag{3.2.6}$$

Further, $\lambda \in \mathbb{C}$ is an *eigenvalue* of $sE - A$ if, and only if,

$$\operatorname{rk}_{\mathbb{C}}(\lambda E - A) < \operatorname{rk}_{\mathbb{R}(s)}(sE - A).$$

3.2.2 A decomposition under system equivalence

Using Proposition 3.2.3 it is easy to determine a decomposition under system equivalence. For regular systems this decomposition was first discovered by Rosenbrock [211].

Corollary 3.2.5 (Decoupled DAE)**.**
Let $[E, A, B] \in \Sigma_{l,n,m}$*. Then there exist* $W \in \mathbf{Gl}_l(\mathbb{R})$*,* $T \in \mathbf{Gl}_n(\mathbb{R})$ *such*

that

$$[E, A, B] \overset{W,T}{\sim}_{se} \left[\begin{bmatrix} I_{n_s} & 0 & 0 & 0 \\ 0 & N_\alpha & 0 & 0 \\ 0 & 0 & K_\beta & 0 \\ 0 & 0 & 0 & K_\gamma^\top \end{bmatrix}, \begin{bmatrix} A_s & 0 & 0 & 0 \\ 0 & I_{|\alpha|} & 0 & 0 \\ 0 & 0 & L_\beta & 0 \\ 0 & 0 & 0 & L_\gamma^\top \end{bmatrix}, \begin{bmatrix} B_s \\ B_f \\ B_u \\ B_o \end{bmatrix} \right], \tag{3.2.7}$$

for some $B_s \in \mathbb{R}^{n_s \times m}$, $B_f \in \mathbb{R}^{|\alpha| \times m}$, $B_o \in \mathbb{R}^{(|\beta| - \ell(\beta)) \times m}$, $B_u \in \mathbb{R}^{|\gamma| \times m}$, $A_s \in \mathbb{R}^{n_s \times n_s}$ *and multi-indices* $\alpha \in \mathbb{N}^{n_\alpha}$, $\beta \in \mathbb{N}^{n_\beta}$, $\gamma \in \mathbb{N}^{n_\gamma}$. *This is interpreted, in terms of the DAE* (3.1.1)*, as follows:* $(x, u) \in \mathfrak{B}_{[E,A,B]}$ *if, and only if,*

$$\left(x_s^\top, x_f^\top, x_u^\top, x_o^\top\right)^\top := Tx$$

with

$$x_f = \begin{pmatrix} x_{f[1]} \\ \vdots \\ x_{f[\ell(\alpha)]} \end{pmatrix}, \quad x_u = \begin{pmatrix} x_{u[1]} \\ \vdots \\ x_{u[\ell(\beta)]} \end{pmatrix}, \quad x_o = \begin{pmatrix} x_{o[1]} \\ \vdots \\ x_{o[\ell(\gamma)]} \end{pmatrix}$$

solves the decoupled DAEs

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} x_s(t) &= A_s\, x_s(t) + B_s\, u(t), \\ \tfrac{\mathrm{d}}{\mathrm{d}t} N_{\alpha_i} x_{f[i]}(t) &= x_{f[i]}(t) + B_{f[i]}\, u(t) && \text{for } i = 1, \ldots, \ell(\alpha), \\ \tfrac{\mathrm{d}}{\mathrm{d}t} K_{\beta_i} x_{u[i]}(t) &= L_{\beta_i} x_{u[i]}(t) + B_{u[i]}\, u(t) && \text{for } i = 1, \ldots, \ell(\beta), \\ \tfrac{\mathrm{d}}{\mathrm{d}t} K_{\gamma_i}^\top x_{o[i]}(t) &= L_{\gamma_i}^\top x_{o[i]}(t) + B_{o[i]}\, u(t) && \text{for } i = 1, \ldots, \ell(\gamma) \end{aligned}$$

with suitably labeled partitions of B_f, B_u *and* B_o.

Note that the form (3.2.7) is not a canonical form in the sense of Definition 2.2.20 since it is not a unique representative within its equivalence class. Compared to the above derived decomposition under system equivalence, more information can be gathered from the decomposition under feedback equivalence obtained in the next subsection.

3.2.3 A decomposition under feedback equivalence

A decomposition under feedback transformation (3.2.1) was first studied for systems governed by ordinary differential equations by

Brunovský [48]. In this subsection we present a generalization of the Brunovský form for general DAE systems $[E, A, B] \in \Sigma_{l,n,m}$ from [167]. For more details on the feedback form and a more geometric point of view on feedback invariants and feedback canonical forms see [143,167].

Theorem 3.2.6 (Decomposition under feedback equivalence [167]). *Let* $[E, A, B] \in \Sigma_{l,n,m}$. *Then there exist* $W \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), V \in \mathbf{Gl}_m(\mathbb{R}), F \in \mathbb{R}^{m\times n}$ *such that*

$$[E, A, B] \overset{W,T,V,F}{\sim_{fe}} \left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & N_\kappa & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{c}}} \end{bmatrix}, \begin{bmatrix} N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & L_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \right], \tag{3.2.9}$$

for some multi-indices $\alpha \in \mathbb{N}^{n_\alpha}, \beta \in \mathbb{N}^{n_\beta}, \gamma \in \mathbb{N}^{n_\gamma}, \delta \in \mathbb{N}^{n_\delta}, \kappa \in \mathbb{N}^{n_\kappa}$ *and a matrix* $A_{\overline{c}} \in \mathbb{R}^{n_{\overline{c}}\times n_{\overline{c}}}$. *This is interpreted, in terms of the DAE* (3.1.1), *as follows:* $(x, u) \in \mathfrak{B}_{[E,A,B]}$ *if, and only if,*

$$\begin{aligned} \left(x_c^\top, x_u^\top, x_{ob}^\top, x_o^\top, x_f^\top, x_{\overline{c}}^\top\right)^\top &:= Tx, \\ \left(u_c^\top, u_{ob}^\top, u_s^\top\right)^\top &:= V(u - Fx), \end{aligned}$$

with

$$x_c = \begin{pmatrix} x_{c[1]} \\ \vdots \\ x_{c[\ell(\alpha)]} \end{pmatrix}, \quad u_c = \begin{pmatrix} u_{c[1]} \\ \vdots \\ u_{c[\ell(\alpha)]} \end{pmatrix}, \quad x_u = \begin{pmatrix} x_{u[1]} \\ \vdots \\ x_{u[\ell(\beta)]} \end{pmatrix},$$

$$x_{ob} = \begin{pmatrix} x_{ob[1]} \\ \vdots \\ x_{ob[\ell(\gamma)]} \end{pmatrix}, \quad u_{ob} = \begin{pmatrix} u_{ob[1]} \\ \vdots \\ u_{ob[\ell(\gamma)]} \end{pmatrix}, \quad x_o = \begin{pmatrix} x_{o[1]} \\ \vdots \\ x_{o[\ell(\delta)]} \end{pmatrix},$$

$$x_f = \begin{pmatrix} x_{f[1]} \\ \vdots \\ x_{f[\ell(\kappa)]} \end{pmatrix}$$

solves the decoupled DAEs

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t} x_{c[i]}(t) &= N_{\alpha_i}^\top x_c(t) + e_{\alpha_i}^{[\alpha_i]} u_{c[i]}(t) && \textit{for } i = 1,\dots,\ell(\alpha) && \text{(3.2.10a)}\\
\tfrac{\mathrm{d}}{\mathrm{d}t} K_{\beta_i} x_{u[i]}(t) &= L_{\beta_i} x_{u[i]}(t) && \textit{for } i = 1,\dots,\ell(\beta), && \text{(3.2.10b)}\\
\tfrac{\mathrm{d}}{\mathrm{d}t} L_{\gamma_i}^\top x_{ob[i]}(t) &= K_{\gamma_i}^\top x_{ob[i]}(t) + e_{\gamma_i}^{[\gamma_i]} u_{ob[i]} && \textit{for } i = 1,\dots,\ell(\gamma), && \text{(3.2.10c)}\\
\tfrac{\mathrm{d}}{\mathrm{d}t} K_{\delta_i}^\top x_{o[i]}(t) &= L_{\delta_i}^\top x_{o[i]}(t) && \textit{for } i = 1,\dots,\ell(\delta), && \text{(3.2.10d)}\\
\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\kappa_i} x_{f[i]}(t) &= x_c(t) && \textit{for } i = 1,\dots,\ell(\kappa) && \text{(3.2.10e)}\\
\tfrac{\mathrm{d}}{\mathrm{d}t} x_{\overline{c}}(t) &= A_{\overline{c}}\, x_{\overline{c}}(t). && && \text{(3.2.10f)}
\end{aligned}$$

Note that the decomposition (3.2.9) is not a canonical form in the sense of Definition 2.2.20. However, if we apply an additional state space transformation to the block $[I_{n_{\overline{c}}}, A_{\overline{c}}, 0]$ which puts $A_{\overline{c}}$ into Jordan canonical form, and then prescribe the order of the blocks of each type, e.g. from largest dimension to lowest (what would mean $\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{\ell(\alpha)}$ for α for instance), then (3.2.9) becomes a canonical form.

Remark 3.2.7 (DAEs corresponding to the blocks in the feedback form)**.**
The form in Theorem 3.2.6 again leads to the separate consideration of the differential-algebraic equations (3.2.10a)-(3.2.10f):

(i) (3.2.10a) is given by $[I_{\alpha_i}, N_{\alpha_i}^\top, e_{\alpha_i}^{[\alpha_i]}]$, and is completely controllable by the classical results for ODE systems (see e.g. [230, Sec. 3.2]). This system has furthermore the properties of being R-controllable, and both controllable and stabilizable in the behavioral sense.

(ii) (3.2.10b) corresponds to an under-determined system with zero dimensional input space. From Subsection 2.4.2 we know that with

$$z_{u[i]} := \begin{pmatrix} x_{u[i],1} \\ \vdots \\ x_{u[i],\beta_i-1} \end{pmatrix},$$

we may rewrite this equation as

$$\tfrac{\mathrm{d}}{\mathrm{d}t} z_{u[i]}(t) = N_{\beta_i-1}^\top z_{u[i]}(t) + e_{\beta_i}^{[\beta_i]} x_{u[i],\beta_i}(t),$$

where we can treat the free variable $x_{u[i],\beta_i}$ as an input. Then system (3.2.10b) has the same properties as (3.2.10a).

(iii) Denoting

$$z_{ob[i]} = \begin{pmatrix} x_{ob[i]} \\ u_{ob[i]} \end{pmatrix},$$

(3.2.10c) can be rewritten as

$$N_{\gamma_i} \dot{z}_{ob[i]}(t) = z_{ob[i]}(t),$$

from which it follows that $z_{ob[i]} \overset{\text{a.e.}}{=} 0$ and $N_{\gamma_i} z_{ob[i]} \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^{\gamma_i})$ as it can be deduced from Subsection 2.4.2. Hence,

$$\mathfrak{B}_{[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]} = \left\{ (x,u) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{\gamma_i - 1} \times \mathbb{R}) \;\middle|\; x = 0 \;\wedge\; u \overset{\text{a.e.}}{=} 0 \right\}.$$

The system $[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$ is therefore completely controllable if, and only if, $\gamma_i = 1$. In the case where $\gamma_i > 1$, this system is not even impulse controllable. However, independent of γ_i, $[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$ is R-controllable, and both controllable and stabilizable in the behavioral sense.

(iv) Similarly, we have

$$\mathfrak{B}_{[K_{\delta_i}^\top, L_{\delta_i}^\top, 0_{\delta_i,0}]} = \{0\},$$

whence, in dependence on δ_i, we can infer the same properties as in (iii).

(v) Similar to (iii) we find that

$$\mathfrak{B}_{[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]} = \left\{ x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{\kappa_i}) \;\middle|\; \begin{array}{r} N_{\kappa_i} x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^{\kappa_i}) \\ \wedge\; x \overset{\text{a.e.}}{=} 0 \end{array} \right\},$$

and hence the system $[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]$ is never controllable at infinity, but always R-controllable and both controllable and stabilizable in the behavioral sense. $[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]$ is strongly controllable if, and only if, $\kappa_i = 1$.

(vi) The system $[I_{n_{\overline{c}}}, A_{\overline{c}}, 0_{\overline{c},0}]$ satisfies

$$\mathfrak{B}_{[I_{n_{\overline{c}}}, A_{\overline{c}}, 0_{n_{\overline{c}},0}]} = \left\{ e^{A_{\overline{c}} \cdot} x^0 \;\middle|\; x^0 \in \mathbb{R}^{n_{\overline{c}}} \right\},$$

whence it is controllable at infinity, but neither strongly controllable nor controllable in the behavioral sense nor R-controllable. The properties of complete and strong stabilizability and stabilizability in the behavioral sense are attained if, and only if, $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$.

By using the implications shown in Proposition 3.1.7, we can deduce the properties of the systems arising in the feedback form, see Figure 3.1.

Corollary 3.2.8.
A system $[E, A, B] \in \Sigma_{l,n,m}$ *with feedback form* (3.2.9) *is*

(a) controllable at infinity if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$ *and* $\ell(\kappa) = 0$;

(b) impulse controllable if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$ *and* $\kappa = (1, \dots, 1)$;

(c) strongly controllable (and thus also strongly reachable) if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$, $\kappa = (1, \dots, 1)$ *and* $n_{\overline{c}} = 0$;

(d) completely controllable (and thus also completely reachable) if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$ *and* $\ell(\kappa) = n_{\overline{c}} = 0$;

(e) R-controllable if, and only if, $n_{\overline{c}} = 0$;

(f) controllable in the behavioral sense if, and only if, $n_{\overline{c}} = 0$;

(g) strongly stabilizable if, and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$, $\ell(\kappa) = 0$, *and* $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$;

(h) completely stabilizable if and only if, $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$, $\kappa = (1, \dots, 1)$, *and* $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$;

(i) stabilizable in the behavioral sense if, and only if, $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$.

Remark 3.2.9 (Parametrization of the behavior of systems in feedback form)**.**
With the findings in Remark 3.2.7, we may explicitly characterize the behavior of systems in feedback form. Define

$$V_q(s) = [1, s, \dots, s^q]^\top \in \mathbb{R}[s]^{q \times 1}, \quad q \in \mathbb{N},$$

	$[I_{\alpha_i}, N_{\alpha_i}^\top, e_{\alpha_i}^{[\alpha_i]}]$	$[K_{\beta_i}, L_{\beta_i}, 0_{\beta_i-1,0}]$	$[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$	$[K_{\delta_i}^\top, L_{\delta_i}^\top, 0_{\delta_i,0}]$	$[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]$	$[I_{n_{\overline{c}}}, A_{\overline{c}}, 0_{\overline{c},0}]$
controllable at infinity	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	✗	✓
impulse controllable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	$\Leftrightarrow \kappa_i = 1$	✓
completely controllable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	✗	✗
completely reachable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	✗	✗
strongly controllable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	$\Leftrightarrow \kappa_i = 1$	✗
strongly reachable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	$\Leftrightarrow \kappa_i = 1$	✗
completely stabilizable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	✗	$\Leftrightarrow \sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$
strongly stabilizable	✓	✓	$\Leftrightarrow \gamma_i = 1$	$\Leftrightarrow \delta_i = 1$	$\Leftrightarrow \kappa_i = 1$	$\Leftrightarrow \sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$
R - controllable	✓	✓	✓	✓	✓	✗
controllable in the behavioral sense	✓	✓	✓	✓	✓	✗
stabilizable in the behavioral sense	✓	✓	✓	✓	✓	$\Leftrightarrow \sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$

Figure 3.1: *Properties of the subsystems arising in the feedback form* (3.2.9).

and, for some multi-index $\mu = (\mu_1, \ldots, \mu_q) \in \mathbb{N}^q$,

$$\begin{aligned} V_\mu(s) &= \mathrm{diag}\,(V_{\mu_1}(s), \ldots, V_{\mu_q}(s)) \in \mathbb{R}[s]^{|\mu|\times\ell(\mu)}, \\ W_\mu(s) &= \mathrm{diag}\,(s^{\mu_1}, \ldots, s^{\mu_q}) \in \mathbb{R}[s]^{\ell(\mu)\times\ell(\mu)}. \end{aligned}$$

Further let $\mu + k := (\mu_1 + k, \ldots, \mu_q + k)$ for $k \in \mathbb{Z}$, and

$$\mathcal{W}^{\mu,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{\ell(\mu)}) := \mathcal{W}^{\mu_1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}) \times \cdots \times \mathcal{W}^{\mu_{\ell(\mu)},1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$$

and define

$$\mathcal{Z}(\mathbb{R};\mathbb{R}^q) := \left\{ f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^q) \;\middle|\; f \overset{\text{a.e.}}{=} 0 \right\}.$$

Then the behavior of a system in feedback form can, formally, be written as

$$\mathfrak{B}_{[E,A,B]} = \left[\begin{array}{cccccc} V_{\alpha-1}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & 0 & 0 & 0 & 0 \\ 0 & V_{\beta-1}(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_\kappa & 0 & 0 & 0 \\ 0 & 0 & 0 & e^{A_{\overline{c}}\cdot} & 0 & 0 \\ \hline W_\alpha(\frac{\mathrm{d}}{\mathrm{d}t}) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{\ell(\gamma)} & 0 \\ 0 & 0 & 0 & 0 & 0 & I_\xi \end{array}\right] \cdot \left[\begin{array}{c} \mathcal{W}^{\alpha,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{\ell(\alpha)}) \\ \mathcal{W}^{\beta-1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{\ell(\beta)}) \\ \mathcal{Z}(\mathbb{R};\mathbb{R}^{\ell(\kappa)}) \\ \mathbb{R}^{n_{\overline{c}}} \\ \mathcal{Z}(\mathbb{R};\mathbb{R}^{\ell(\gamma)}) \\ \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^\xi) \end{array}\right],$$

where the sizes of the blocks are according to the block structure in the feedback form (3.2.9), $\xi = m - \ell(\alpha) - \ell(\gamma)$, and the horizontal line is the dividing line between x- and u-variables. If the system $[E, A, B] \in \Sigma_{l,n,m}$ is not in feedback form, then a parametrization of the behavior can be found by using the above representation and relation (3.2.2) expressing the connection between behaviors of feedback equivalent systems.

For general differential behaviors, a parametrization of the above kind is called *image representation* [198, Sec. 6.6].

Remark 3.2.10 (Derivative feedback)**.**
A canonical form under proportional and derivative feedback (PD feedback) was derived in [167] as well (note that PD feedback defines an

equivalence relation on $\Sigma_{l,n,m}$). The main tool for doing this is the restriction pencil (see Section 3.7 (xi)): Clearly, the system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} NEx &= NAx, \\ u &= B^{\dagger}(\tfrac{\mathrm{d}}{\mathrm{d}t} Ex - Ax) \end{aligned}$$

is equivalent under PD feedback to the system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} NEx &= NAx, \\ u &= 0. \end{aligned}$$

Then putting $sNE - NA$ into KCF yields a PD canonical form for the DAE system with a 5×4-block structure.

We may, however, directly derive this PD canonical form from the decomposition (3.2.9). To this end we may observe that the system $[I_{\alpha_i}, N_{\alpha_i}^\top, e_{\alpha_i}^{[\alpha_i]}]$ can be written as

$$\tfrac{\mathrm{d}}{\mathrm{d}t} K_{\alpha_i} x_{c[i]}(t) = L_{\alpha_i} x_{c[i]}(t), \quad \tfrac{\mathrm{d}}{\mathrm{d}t} x_{c[i],\alpha_i}(t) = u_{c[i]}(t),$$

and hence is, via PD feedback, equivalent to the system

$$\left[\begin{bmatrix} K_{\alpha_i} \\ 0 \end{bmatrix}, \begin{bmatrix} L_{\alpha_i} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right].$$

On the other hand, the system $[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$ can be written as

$$\tfrac{\mathrm{d}}{\mathrm{d}t} N_{\gamma_i - 1} x_{ob[i]}(t) = x_{ob[i]}(t), \quad \tfrac{\mathrm{d}}{\mathrm{d}t} x_{ob[i],\gamma_{i-1}}(t) = u_{ob[i]}(t),$$

and hence is, via PD feedback, equivalent to the system

$$\left[\begin{bmatrix} N_{\gamma_i - 1} \\ 0 \end{bmatrix}, \begin{bmatrix} I_{\gamma_i - 1} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right].$$

A canonical form for $[E, A, B] \in \Sigma_{l,n,m}$ under PD feedback (for the proof see [167]) is therefore given by

$$[E, A, B] \ \sim_{PD} \ \left[\begin{bmatrix} K_\beta & 0 & 0 & 0 \\ 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & N_\kappa & 0 \\ 0 & 0 & 0 & I_{n_{\overline{c}}} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} L_\beta & 0 & 0 & 0 \\ 0 & L_\delta^\top & 0 & 0 \\ 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & A_{\overline{c}} \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ I_\zeta & 0 \end{bmatrix} \right],$$

where $A_{\overline{c}}$ is in Jordan canonical form, and the blocks of each type are ordered from largest dimension to lowest.

Note that the properties of complete controllability, controllability at infinity and controllability in the behavioral sense are invariant under PD feedback. However, since derivative feedback changes the set of differential variables, the properties of strong controllability as well as impulse controllability may be lost/gained after PD feedback.

Remark 3.2.11 (Connection to Kronecker form)**.**
We may observe from (3.2.1) that feedback transformation may be alternatively considered as a transformation of the extended pencil

$$s\mathcal{E} - \mathcal{A} = \left[sE - A, \;\; -B\right], \tag{3.2.11}$$

that is based on a multiplication from the left by $\mathcal{W} = W \in \mathbf{Gl}_l(\mathbb{R})$, and from the right by

$$\mathcal{T} = \begin{bmatrix} T & 0 \\ -F & V \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}).$$

This equivalence is therefore a subclass of the class which is induced by the pre- and post-multiplication of $s\mathcal{E} - \mathcal{A}$ by arbitrary invertible matrices. Loosely speaking, one can hence expect a decomposition under feedback equivalence which specializes the QKF of $s\mathcal{E} - \mathcal{A}$. Indeed, the latter form may be obtained from the feedback form of $[E, A, B]$ by several simple row transformations of $s\mathcal{E} - \mathcal{A}$ which are not interpretable as feedback group actions anymore. More precisely, simple permutations of columns lead to the separate consideration of the extended pencils corresponding to the systems (3.2.10a)-(3.2.10f): The extended pencils corresponding to $[I_{\alpha_i}, N_{\alpha_i}^\top, e_{\alpha_i}^{[\alpha_i]}]$ and $[K_{\beta_i}, L_{\beta_i}, 0_{\alpha_i,0}]$ are $sK_{\alpha_i} - L_{\alpha_i}$ and $sK_{\beta_i} - L_{\beta_i}$, resp. The extended matrix pencil corresponding to the system $[L_{\gamma_i}^\top, K_{\gamma_i}^\top, e_{\gamma_i}^{[\gamma_i]}]$ is given by $sN_{\gamma_i} - I_{\gamma_i}$. The extended matrix pencils corresponding to the systems $[K_{\delta_i}^\top, L_{\delta_i}^\top, 0_{\delta_i,0}]$, $[N_{\kappa_i}, I_{\kappa_i}, 0_{\kappa_i,0}]$ and $[I_{n_{\overline{c}}}, A_{\overline{c}}, 0_{\overline{c},0}]$ are obviously given by $sK_{\delta_i}^\top - L_{\delta_i}^\top$, $sN_{\kappa_i} - I_{\kappa_i}$ and $sI_{n_{\overline{c}}} - A_{\overline{c}}$, respectively. In particular, $\lambda \in \mathbb{C}$ is an eigenvalue of $s\mathcal{E} - \mathcal{A}$, if, and only if, $\lambda \in \sigma(A_{\overline{c}})$.

Remark 3.2.12 (Minimality in the behavioral sense)**.**

(i) According to Remark 3.2.2, a differential-algebraic system $[E, A, B] \in \Sigma_{l,n,m}$ is minimal in the behavioral sense if, and only if, the extended pencil $s\mathcal{E} - \mathcal{A}$ as in (3.2.11) has full row rank over $\mathbb{R}(s)$. On the other hand, a system $[E, A, B] \in \Sigma_{l,n,m}$ with feedback form (3.2.9) satisfies

$$\mathrm{rk}_{\mathbb{R}(s)}(s\mathcal{E} - \mathcal{A}) = l - \ell(\delta).$$

Using that $\mathrm{rk}_{\mathbb{R}(s)}(s\mathcal{E} - \mathcal{A})$ is invariant under feedback transformation (3.2.1), we can conclude that minimality of $[E, A, B] \in \Sigma_{l,n,m}$ in the behavioral sense corresponds to the absence of blocks of type (3.2.10d) in its feedback form.

(ii) The findings in Remark 3.2.7 imply that a system in feedback form is, in the behavioral sense, equivalent to

$$\left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{c}}} \end{bmatrix}, \begin{bmatrix} N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{|\delta|-\ell(\delta)} & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{\overline{c}} \end{bmatrix}, \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right].$$

This system can alternatively be achieved by multiplying the extended pencil (3.2.11) in feedback form (3.2.9) from the left with the polynomial matrix

$$Z(s) = \mathrm{diag}\left(I_{|\alpha|},\ I_{|\beta|-\ell(\beta)},\ -\sum_{k=0}^{\nu_\gamma - 1} s^k N_\gamma^k,\ P_\delta(s),\ -\sum_{k=0}^{\nu_\kappa - 1} s^k N_\kappa^k,\ I_{n_{\overline{c}}} \right),$$

where $\nu_\gamma = \max\{\gamma_1, \ldots, \gamma_{\ell(\gamma)}\}$, $\nu_\kappa = \max\{\kappa_1, \ldots, \kappa_{\ell(\kappa)}\}$, and

$$P_\delta(s) = \mathrm{diag}\left(\left[0_{\delta_i - 1,1},\ \ -\sum_{k=0}^{\delta_i - 2} s^k \left(N_{\delta_i - 1}^\top\right)^k \right]\right)_{j=1,\ldots,\ell(\delta)}.$$

(iii) Let a differential-algebraic system $[E, A, B] \in \Sigma_{l,n,m}$ be given. Using the notation from (3.2.9) and the previous item, a behaviorally equivalent and minimal system $[E_M, A_M, B_M] \in \Sigma_{l-\ell(\delta),n,m}$ can be constructed by

$$\left[sE_M - A_M,\ \ -B_M \right] = Z(s) W \left[sE - A,\ \ -B \right].$$

It can be seen that this representation is furthermore controllable at infinity. Moreover, it minimizes, among all differential-algebraic equations representing the same behavior, the index and the rank of the matrix in front of the state derivative (that is, loosely speaking, the number of differential variables). This procedure is very much related to *index reduction* [152, Sec. 6.1].

3.3 Criteria of Hautus type

In this section we derive equivalent criteria on the matrices $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$ for the controllability and stabilizability concepts of Definition 3.1.5. The criteria are generalizations of the Hautus test (also called Popov-Belevitch-Hautus test, since independently developed by Popov [199], Belevitch [18] and Hautus [112]) in terms of rank criteria on the involved matrices. Note that these conditions are not new – we refer to the relevant literature. However, we provide proofs using only the feedback form (3.2.9).

First we show that certain rank criteria on the matrices involved in control systems are invariant under feedback equivalence. After that, we relate these rank criteria to the feedback form (3.2.9).

Lemma 3.3.1.
Let $[E_1, A_1, B_1], [E_2, A_2, B_2] \in \Sigma_{l,n,m}$ *be given such that for* $W \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$ *and* $F \in \mathbb{R}^{m\times n}$ *we have*

$$[\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T,V,F}{\sim_{fe}} [\,E_2\,,\,A_2\,,\,B_2\,]\,.$$

Then, for all $\lambda \in \mathbb{C}$,

$$\begin{aligned}
\operatorname{im}_{\mathbb{R}} E_1 + \operatorname{im}_{\mathbb{R}} A_1 + \operatorname{im}_{\mathbb{R}} B_1 &= W\left(\operatorname{im}_{\mathbb{R}} E_2 + \operatorname{im}_{\mathbb{R}} A_2 + \operatorname{im}_{\mathbb{R}} B_2\right),\\
\operatorname{im}_{\mathbb{R}} E_1 + A_1 \ker_{\mathbb{R}} E_1 + \operatorname{im}_{\mathbb{R}} B_1 &= W\left(\operatorname{im}_{\mathbb{R}} E_2 + A_2 \ker_{\mathbb{R}} E_2 + \operatorname{im}_{\mathbb{R}} B_2\right),\\
\operatorname{im}_{\mathbb{R}} E_1 + \operatorname{im}_{\mathbb{R}} B_1 &= W\left(\operatorname{im}_{\mathbb{R}} E_2 + \operatorname{im}_{\mathbb{R}} B_2\right),\\
\operatorname{im}_{\mathbb{C}}(\lambda E_1 - A_1) + \operatorname{im}_{\mathbb{C}} B_1 &= W\left(\operatorname{im}_{\mathbb{C}}(\lambda E_2 - A_2) + \operatorname{im}_{\mathbb{C}} B_2\right),\\
\operatorname{im}_{\mathbb{R}(s)}(sE_1 - A_1) + \operatorname{im}_{\mathbb{R}(s)} B_1 &= W\left(\operatorname{im}_{\mathbb{R}(s)}(sE_2 - A_2) + \operatorname{im}_{\mathbb{R}(s)} B_2\right).
\end{aligned}$$

Proof: Immediate from (3.2.1). □

Lemma 3.3.2 (Algebraic criteria via feedback form)**.**
For a system $[E, A, B] \in \Sigma_{l,n,m}$ *with feedback form* (3.2.9) *the following statements hold true:*

(a)

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B \\ \Longleftrightarrow\ &\gamma = (1, \dots, 1),\ \delta = (1, \dots, 1),\ \ell(\kappa) = 0. \end{aligned}$$

(b)

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B \\ \Longleftrightarrow\ &\gamma = (1, \dots, 1),\ \delta = (1, \dots, 1),\ \kappa = (1, \dots, 1). \end{aligned}$$

(c)

$$\begin{aligned} &\operatorname{im}_{\mathbb{C}} E + \operatorname{im}_{\mathbb{C}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{C}}(\lambda E - A) + \operatorname{im}_{\mathbb{C}} B \\ \Longleftrightarrow\ &\delta = (1, \dots, 1),\ \lambda \notin \sigma(A_{\overline{c}}). \end{aligned}$$

(d) For $\lambda \in \mathbb{C}$ we have

$$\begin{aligned} &\dim\left(\operatorname{im}_{\mathbb{R}(s)}(sE - A) + \operatorname{im}_{\mathbb{R}(s)} B\right) \\ &\qquad = \dim\left(\operatorname{im}_{\mathbb{C}}(\lambda E - A) + \operatorname{im}_{\mathbb{C}} B\right) \\ \Longleftrightarrow\ &\lambda \notin \sigma(A_{\overline{c}}). \end{aligned}$$

Proof: It is, by Lemma 3.3.1, no loss of generality to assume that $[E, A, B]$ is already in feedback form. The results then follow by a simple verification of the above statements by means of the feedback form. □

Combining Lemmas 3.3.1 and 3.3.2 with Corollary 3.2.8, we may deduce the following criteria for the controllability and stabilizability concepts introduced in Definition 3.1.5.

Proposition 3.3.3 (Algebraic criteria for controllability/stabilizability)**.**
Let a system $[E, A, B] \in \Sigma_{l,n,m}$ be given. Then the following holds:

$[E, A, B]$ *is*	*if, and only if,*
controllable at infinity	$\operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} A + \operatorname{im}_{\mathbb{R}} B = \operatorname{im}_{\mathbb{R}} E + \operatorname{im}_{\mathbb{R}} B.$

impulse controllable	$\mathrm{im}_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} A + \mathrm{im}_{\mathbb{R}} B = \mathrm{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} B.$
completely controllable	$\mathrm{im}_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} A + \mathrm{im}_{\mathbb{R}} B = \mathrm{im}_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} B$ $\wedge\ \mathrm{im}_{\mathbb{C}} E + \mathrm{im}_{\mathbb{C}} A + \mathrm{im}_{\mathbb{C}} B = \mathrm{im}_{\mathbb{C}}(\lambda E - A) + \mathrm{im}_{\mathbb{C}} B\ \ \forall \lambda \in \mathbb{C}.$
strongly controllable	$\mathrm{im}_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} A + \mathrm{im}_{\mathbb{R}} B = A \cdot \ker_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} B$ $\wedge\ \mathrm{im}_{\mathbb{C}} E + \mathrm{im}_{\mathbb{C}} A + \mathrm{im}_{\mathbb{C}} B = \mathrm{im}_{\mathbb{C}}(\lambda E - A) + \mathrm{im}_{\mathbb{C}} B\ \ \forall \lambda \in \mathbb{C}.$
completely stabilizable	$\mathrm{im}_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} A + \mathrm{im}_{\mathbb{R}} B = \mathrm{im}_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} B$ $\wedge\ \mathrm{im}_{\mathbb{C}} E + \mathrm{im}_{\mathbb{C}} A + \mathrm{im}_{\mathbb{C}} B = \mathrm{im}_{\mathbb{C}}(\lambda E - A) + \mathrm{im}_{\mathbb{C}} B\ \ \forall \lambda \in \overline{\mathbb{C}}_+.$
strongly stabilizable	$\mathrm{im}_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} A + \mathrm{im}_{\mathbb{R}} B = \mathrm{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} B$ $\wedge\ \mathrm{im}_{\mathbb{C}} E + \mathrm{im}_{\mathbb{C}} A + \mathrm{im}_{\mathbb{C}} B = \mathrm{im}_{\mathbb{C}}(\lambda E - A) + \mathrm{im}_{\mathbb{C}} B\ \ \forall \lambda \in \overline{\mathbb{C}}_+.$
controllable in the behavioral sense	$\mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] = \mathrm{rk}_{\mathbb{C}}[\lambda E - A, B]\ \ \forall \lambda \in \mathbb{C}.$
stabilizable in the behavioral sense	$\mathrm{rk}_{\mathbb{R}(s)}[sE - A, B] = \mathrm{rk}_{\mathbb{C}}[\lambda E - A, B]\ \ \forall \lambda \in \overline{\mathbb{C}}_+.$

The above result leads to the an extension of the diagram in Proposition 3.1.7, which is shown in Figure 3.2. Note that the equivalence of R-controllability and controllability in the behavioral sense was already shown in Corollary 3.2.8.

In the following we consider further criteria for the concepts introduced in Definition 3.1.5.

Remark 3.3.4 (Controllability at infinity)**.**
Proposition 3.3.3 immediately implies that controllability at infinity is equivalent to

$$\mathrm{im}_{\mathbb{R}} A \subseteq \mathrm{im}_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}} B.$$

In terms of a rank criterion, this is the same as

$$\mathrm{rk}_{\mathbb{R}}[E, A, B] = \mathrm{rk}_{\mathbb{R}}[E, B]. \tag{3.3.1}$$

Criterion (3.3.1) has first been derived by Geerts [101, Thm. 4.5] for the case $\mathrm{rk}[E, A, B] = l$, although he does not use the name ‘controlla-

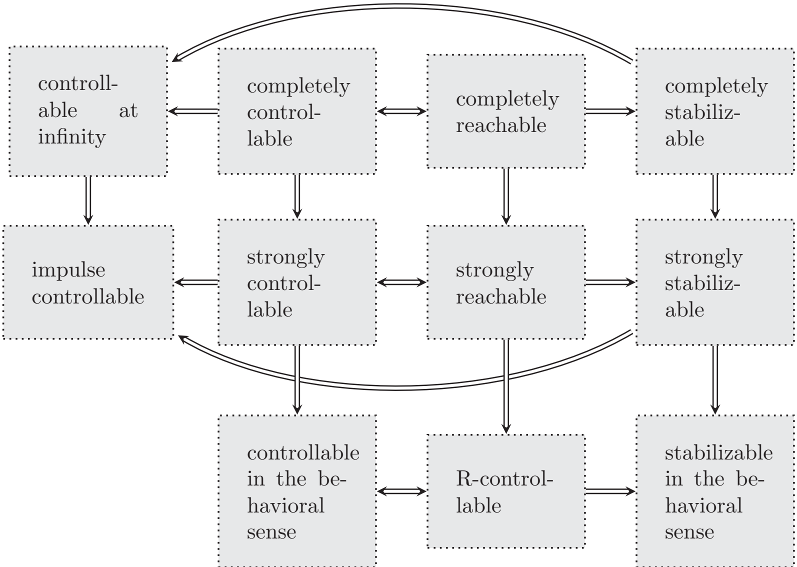


Figure 3.2: *Relations between the controllability concepts.*

bility at infinity'.

In the case of regular $sE - A \in \mathbb{R}[s]^{n \times n}$, condition (3.3.1) reduces to

$$\mathrm{rk}_{\mathbb{R}}[E, B] = n.$$

Remark 3.3.5 (Impulse controllability)**.**
By Proposition 3.3.3, impulse controllability of $[E, A, B] \in \Sigma_{l,n,m}$ is equivalent to

$$\mathrm{im}_{\mathbb{R}}\, A \subseteq \mathrm{im}_{\mathbb{R}}\, E + A \cdot \ker_{\mathbb{R}} E + \mathrm{im}_{\mathbb{R}}\, B.$$

Another equivalent characterization is that, for one (and hence any) matrix Z with $\mathrm{im}_{\mathbb{R}}\, Z = \ker_{\mathbb{R}} E$, we have

$$\mathrm{rk}_{\mathbb{R}}[E, A, B] = \mathrm{rk}_{\mathbb{R}}[E, AZ, B]. \tag{3.3.2}$$

This has first been derived by Geerts [101, Rem. 4.9], again for the case $\mathrm{rk}[E, A, B] = l$. In [130, Thm. 3] and [117] it has been obtained that impulse controllability is equivalent to

$$\mathrm{rk}_{\mathbb{R}} \begin{bmatrix} E & 0 & 0 \\ A & E & B \end{bmatrix} = \mathrm{rk}_{\mathbb{R}}[E, A, B] + \mathrm{rk}_{\mathbb{R}} E,$$

which is in fact equivalent to (3.3.2). It has also been shown in [130, p. 1] that impulse controllability is equivalent to

$$\mathrm{rk}_{\mathbb{R}(s)}(s\mathcal{E} - \mathcal{A}) = \mathrm{rk}_{\mathbb{R}}[E, A, B].$$

This criterion can be alternatively shown by using the feedback form (3.2.9). Using condition (3.2.5) we may also infer that this is equivalent to the index of the extended pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{R}[s]^{l\times(n+m)}$ being at most one.
If the pencil $sE - A$ is regular, then condition (3.3.2) reduces to

$$\mathrm{rk}_{\mathbb{R}}[E, AZ, B] = n.$$

This condition can be also inferred from [80, Th. 2-2.3].

Remark 3.3.6 (Controllability in the behavioral sense and R-controllability)**.**
The concepts of controllability in the behavioral sense and R-controllability are equivalent by Corollary 3.2.8. The algebraic criterion for behavioral controllability in Proposition 3.3.3 is equivalent to the fact that the extended matrix pencil $s\mathcal{E} - \mathcal{A} \in \mathbb{R}[s]^{l\times(n+m)}$ has no eigenvalues, or, equivalently, in the feedback form (3.2.9) we have $n_{\overline{c}} = 0$.
The criterion for controllability in the behavioral sense is shown in [198, Thm. 5.2.10] for the larger class of linear differential behaviors. R-controllability for systems with regular $sE - A$ was considered in [80, Thm. 2-2.2], where the condition

$$\mathrm{rk}_{\mathbb{C}}[\lambda E - A, B] = n \;\; \forall\, \lambda \in \mathbb{C}$$

was derived. This is, for regular $sE - A$, in fact equivalent to the criterion for behavioral stabilizability in Proposition 3.3.3.

Remark 3.3.7 (Complete controllability and strong controllability)**.**
By Proposition 3.3.3, complete controllability of $[E, A, B] \in \Sigma_{l,n,m}$ is equivalent to $[E, A, B]$ being R-controllable and controllable at infinity, whereas strong controllability of $[E, A, B] \in \Sigma_{l,n,m}$ is equivalent to

$[E, A, B]$ being R-controllable and impulse controllable.
BANASZUK et al. [12] already obtained the condition in Proposition 3.3.3 for complete controllability considering discrete systems. Complete controllability is called $\mathcal{H}$-controllability in [12]. Recently, ZUBOVA [260] considered full controllability, which is just complete controllability with the additional assumption that solutions have to be unique, and obtained three equivalent criteria [260, Sec. 7], where the first one characterizes the uniqueness and the other two are equivalent to the condition for complete controllability in Proposition 3.3.3. For regular systems, the conditions in Proposition 3.3.3 for complete and strong controllability are also derived in [80, Thm. 2-2.1 & Thm. 2-2.3].

Remark 3.3.8 (Stabilizability)**.**
By Proposition 3.3.3, complete stabilizability of $[E, A, B] \in \Sigma_{l,n,m}$ is equivalent to $[E, A, B]$ being stabilizable in the behavioral sense and controllable at infinity, whereas strong stabilizability of $[E, A, B] \in \Sigma_{l,n,m}$ is equivalent to $[E, A, B]$ being stabilizable in the behavioral sense and impulse controllable.
The criterion for stabilizability in the behavioral sense is shown in [198, Thm. 5.2.30] for the class of linear differential behaviors.

3.4 Feedback, stability and autonomous systems

State feedback is, roughly speaking, the special choice of the input as a function of the state. Due to the mutual dependence of state and input in a feedback system, this is often referred to as *closed-loop control*. In the linear case, feedback is the imposition of the additional relation $u(t) = Fx(t)$ for some $F \in \mathbb{R}^{m\times n}$. This results in the system

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = (A + BF)x(t).$$

Feedback for linear ODE systems was studied by WONHAM [251], where it is shown that controllability of $[I, A, B] \in \Sigma_{n,n,m}$ is equivalent to any set $\Lambda \subseteq \mathbb{C}$ which has at most n elements and is symmetric with respect to the imaginary axis (that is, $\lambda \in \Lambda \Leftrightarrow \overline{\lambda} \in \Lambda$) being achievable by a suitable feedback, i.e., there exists some $F \in \mathbb{R}^{m\times n}$ with the property that $\sigma(A+BF)=\Lambda$. In particular, the input may be chosen in a way that the closed-loop system is stable, i.e., any state trajectory tends to zero. Using the *Kalman decomposition* [138] (see also Section 3.6), it can be shown for ODE systems that stabilizability is equivalent to the existence of a feedback such that the resulting system is stable.

These results have been generalized to regular DAE systems by Cobb [74], see also [80, 96, 162, 163, 189, 191]. Note that, for DAE systems, not only the problem of assignment of eigenvalues occurs, but also the index may be changed by imposing feedback.

The crucial ingredient for the treatment of DAE systems with non-regular pencil $sE-A$ will be the feedback form by Loiseau et al. [167] (see Thm. 3.2.6).

3.4.1 Stabilizability, autonomy and stability

The feedback law $u(t)=Fx(t)$ applied to (3.1.1) results in a DAE in which the input is completely eliminated. We now focus on DAEs without input, and we introduce several properties and concepts. Consider, for matrices $E, A \in \mathbb{R}^{l\times n}$, the DAE

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t)=Ax(t); \tag{3.4.1}$$

the set of these systems is denoted by $\Sigma_{l,n}$. Its *behavior* is given by

$$\mathfrak{B}_{[E,A]} := \left\{ x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \;\middle|\; \begin{array}{l} Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l) \text{ and } x \text{ satisfies (3.4.1)} \\ \text{for almost all } t\in\mathbb{R} \end{array} \right\}.$$

Definition 3.4.1 (Stability/Stabilizability concepts for DAEs)**.** A linear time-invariant DAE $[E,A]\in\Sigma_{l,n}$ is called

(a) *completely stabilizable*

$$:\Leftrightarrow\ \forall\, x^0\in\mathbb{R}^n\ \exists\, x\in\mathfrak{B}_{[E,A]}:\quad x\in\mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^)\ \wedge\ x(0)=x^0\ \wedge\ \lim_{t\to\infty}x(t)=0.$$

(b) *strongly stabilizable*

$$:\Leftrightarrow\ \forall\, x^0\in\mathbb{R}^n\ \exists\, x\in\mathfrak{B}_{[E,A]}:\ \ Ex(0)=Ex^0\ \wedge\ \lim_{t\to\infty}Ex(t)=0.$$

(c) *stabilizable in the behavioral sense*

$$:\Leftrightarrow\ \forall\, x \in \mathfrak{B}_{[E,A]}\ \exists\, x_0 \in \mathfrak{B}_{[E,A]}:\quad x|_{(-\infty,0)} \overset{\text{a.e.}}{=} x_0|_{(-\infty,0)}\ \wedge\ \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|x_0\| = 0.$$

(d) *autonomous*

$$:\Leftrightarrow\ \forall\, x_1, x_2 \in \mathfrak{B}_{[E,A]}:\ x_1|_{(-\infty,0)} \overset{\text{a.e.}}{=} x_2|_{(-\infty,0)} \implies x_1 \overset{\text{a.e.}}{=} x_2.$$

(e) *completely stable*

$$:\Leftrightarrow\ \left\{\ x(0)\ \middle|\ x \in \mathfrak{B}_{[E,A]} \cap \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)\ \right\} = \mathbb{R}^n \quad \wedge\ \forall\, x \in \mathfrak{B}_{[E,A]} \cap \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n):\ \lim_{t\to\infty} x(t) = 0.$$

(f) *strongly stable*

$$:\Leftrightarrow\ \left\{Ex(0)\ \middle|\ x \in \mathfrak{B}_{[E,A]}\right\} = \text{im}_{\mathbb{R}}\, E\ \wedge\ \forall\, x \in \mathfrak{B}_{[E,A]}:\ \lim_{t\to\infty} Ex(t) = 0.$$

(g) *stable in the behavioral sense*

$$:\Leftrightarrow\ \forall\, x \in \mathfrak{B}_{[E,A]}:\ \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|x\| = 0.$$

Remark 3.4.2 (Stabilizable and autonomous DAEs are stable)**.**
The notion of autonomy is introduced by Polderman and Willems in [198, Sec. 3.2] for general behaviors. For DAE systems $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)$ we can further conclude that autonomy is equivalent to any $x \in \mathfrak{B}_{[E,A]}$ being uniquely determined (up to equality almost everywhere) by $x|_I$ for any open interval $I \subseteq \mathbb{R}$. This gives also rise to the fact that, for

$$\overline{\mathfrak{B}}_{[E,A]} := \left\{\ [z] = \left\{\ x \in \mathfrak{B}_{[E,A]}\ \middle|\ z \overset{\text{a.e.}}{=} x\ \right\}\ \middle|\ z \in \mathfrak{B}_{[E,A]}\ \right\},$$

autonomy is equivalent to $\dim_{\mathbb{R}} \overline{\mathfrak{B}}_{[E,A]} \leq n$, which is, on the other hand, equivalent to $\dim_{\mathbb{R}} \overline{\mathfrak{B}}_{[E,A]} < \infty$. Autonomy indeed means that the DAE is not underdetermined.

Moreover, due to possible underdetermined blocks of type $[K_\beta, L_\beta, 0_{|\beta|-\ell(\beta),0}]$, in general there are solutions $x \in \mathfrak{B}_{[E,A]}$ which grow unboundedly. As a consequence, for a QKF of any completely stable, strongly stable or behavioral stable DAE, it holds $\ell(\beta) = 0$. Hence, systems of this type are autonomous. In fact, complete, strong and behavioral stability are equivalent to the respective stabilizability notion together with autonomy, cf. also Proposition 3.4.3.

In regard of the considerations in Subsection 2.4.2 we can infer the following.

Proposition 3.4.3 (Stability/Stabilizability criteria and QKF). *Let $[E, A] \in \Sigma_{l,n}$ and assume that the QKF of $sE - A$ is given by (3.2.3). Then the following holds true:*

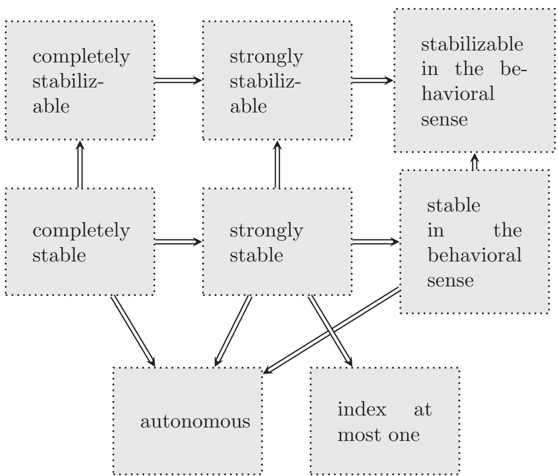
$[E, A]$ *is*	*if, and only if,*
completely stabilizable	$\ell(\alpha) = 0$, $\gamma = (1, \dots, 1)$ *and* $\sigma(A_s) \subseteq \mathbb{C}_-$.
strongly stabilizable	$\alpha = (1, \dots, 1)$, $\gamma = (1, \dots, 1)$ *and* $\sigma(A_s) \subseteq \mathbb{C}_-$.
stabilizable in the behavioral sense	$\sigma(A_s) \subseteq \mathbb{C}_-$.
autonomous	$\ell(\beta) = 0$.
completely stable	$\ell(\alpha) = 0$, $\ell(\beta) = 0$, $\gamma = (1, \dots, 1)$ *and* $\sigma(A_s) \subseteq \mathbb{C}_-$.
strongly stable	$\alpha = (1, \dots, 1)$, $\ell(\beta) = 0$, $\gamma = (1, \dots, 1)$ *and* $\sigma(A_s) \subseteq \mathbb{C}_-$.
stable in the behavioral sense	$\ell(\beta) = 0$, $\sigma(A_s) \subseteq \mathbb{C}_-$.

The subsequent algebraic criteria for the previously defined notions of stabilizability and autonomy follow from Proposition 3.4.3 by using the invariance under transformation and the direct verification of the criteria by means of the QKF – similar to the argumentation in Section 3.3.

Corollary 3.4.4 (Algebraic criteria for stabilizability)**.**
Let $[E, A] \in \Sigma_{l,n}$*. Then the following holds true:*

$[E, A]$ *is*	*if, and only if,*
completely stabilizable	$\operatorname{im}_{\mathbb{R}} A \subseteq \operatorname{im}_{\mathbb{R}} E$ *and* $\operatorname{rk}_{\mathbb{R}(s)}(sE - A) = \operatorname{rk}_{\mathbb{C}}(\lambda E - A)$ *for all* $\lambda \in \overline{\mathbb{C}}_+$.
strongly stabilizable	$\operatorname{im}_{\mathbb{R}} A \subseteq \operatorname{im}_{\mathbb{R}} E + A \cdot \ker_{\mathbb{R}} E$ *and* $\operatorname{rk}_{\mathbb{R}(s)}(sE - A) = \operatorname{rk}_{\mathbb{C}}(\lambda E - A)$ *for all* $\lambda \in \overline{\mathbb{C}}_+$.
stabilizable in the behavioral sense	$\operatorname{rk}_{\mathbb{R}(s)}(sE - A) = \operatorname{rk}_{\mathbb{C}}(\lambda E - A)$ *for all* $\lambda \in \overline{\mathbb{C}}_+$.
autonomous	$\ker_{\mathbb{R}(s)}(sE - A) = \{0\}$.

Corollary 3.4.4 leads to the following implications:



Remark 3.4.5.

(i) Strong stabilizability implies that the index of $sE - A$ is at most one. In the case where the matrix $[E, A] \in \mathbb{R}^{l\times 2n}$ has full row rank, complete stabilizability is sufficient for the index of $sE - A$ being zero.
On the other hand, behavioral stabilizability of $[E, A]$ together with the index of $sE-A$ being not greater than one implies strong stabilizability of $[E, A]$. Furthermore, for systems $[E, A] \in \Sigma_{l,n}$ with $\mathrm{rk}_{\mathbb{R}}[E, A] = l$, complete stabilizability is equivalent to behavioral stabilizability together with the property that the index of $sE - A$ is zero.
For ODEs the notions of complete stabilizability, strong stabilizability, stabilizability in the behavioral sense, complete stability, strong stability and stability in the behavioral sense are equivalent.

(ii) The behavior of an autonomous system $[E, A]$ satisfies $\dim_{\mathbb{R}} \overline{\mathfrak{B}}_{[E,A]} = n_s$, where n_s denotes the number of rows of the matrix A_s in the QKF (3.2.3) of $sE - A$. Note that regularity of $sE - A$ is sufficient for autonomy of $[E, A]$.

(iii) Autonomy has been algebraically characterized for linear differential behaviors in [198, Sec. 3.2]. The characterization of autonomy in Corollary 3.4.4 can indeed be generalized to a larger class of linear differential equations.

3.4.2 Stabilization by feedback

A system $[E, A, B] \in \Sigma_{l,n,m}$ can, via state feedback with some $F \in \mathbb{R}^{m\times n}$, be turned into a DAE $[E, A+BF] \in \Sigma_{l,n}$. We now present some properties of $[E, A + BF] \in \Sigma_{l,n}$ that can be achieved by a suitable feedback matrix $F \in \mathbb{R}^{m\times n}$. Recall that the stabilizability concepts for a system $[E, A, B] \in \Sigma_{l,n,m}$ have been defined in Definition 3.1.5.

Theorem 3.4.6 (Stabilizing feedback)**.**
For a system $[E, A, B] \in \Sigma_{l,n,m}$ the following holds true:

(a) $[E, A, B]$ is impulse controllable if, and only if, there exists $F \in \mathbb{R}^{m\times n}$ such that the index of $sE - (A + BF)$ is at most one.

(b) $[E, A, B]$ *is completely stabilizable if, and only if, there exists* $F \in \mathbb{R}^{m \times n}$ *such that* $[E, A + BF]$ *is completely stabilizable.*

(c) $[E, A, B]$ *is strongly stabilizable if, and only if, there exists* $F \in \mathbb{R}^{m \times n}$ *such that* $[E, A + BF]$ *is strongly stabilizable.*

Proof:

(a) Let $[E, A, B]$ be impulse controllable. Then $[E, A, B]$ can be put into feedback form (3.2.9), i.e., there exist $W \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R})$ and $\tilde{F} \in \mathbb{R}^{m \times n}$ such that

$$\begin{aligned} &W(sE - (A + B\tilde{F}T^{-1})T \\ &= \begin{bmatrix} sI_{|\alpha|}-N_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & sK_\beta-L_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & sL_\gamma^\top-K_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & sK_\delta^\top-L_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & sN_\kappa-I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}}-A_{\overline{c}} \end{bmatrix}. \end{aligned} \tag{3.4.2}$$

By Corollary 3.2.8 (b) the impulse controllability of $[E, A, B]$ implies that $\gamma = (1, \ldots, 1)$, $\delta = (1, \ldots, 1)$ and $\kappa = (1, \ldots, 1)$. Therefore, we have that, with $F = \tilde{F}T^{-1}$, the pencil $sE - (A + BF)$ has index at most one as the index is preserved under system equivalence.

Conversely, assume that $[E, A, B]$ is not impulse controllable. We show that for all $F \in \mathbb{R}^{m \times n}$ the index of $sE - (A + BF)$ is greater than one. To this end, let $F \in \mathbb{R}^{m \times n}$ and choose $W \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R})$ and $\tilde{F} \in \mathbb{R}^{m \times n}$ such that (3.2.9) holds. Then, partitioning $V^{-1}FT = [F_{ij}]_{i=1,\ldots,3,j=1,\ldots,6}$ accordingly, we obtain

$$\begin{aligned} s\tilde{E} - \tilde{A} &:= W(sE - (A + BF + B\tilde{F}T^{-1}))T \\ &= W(sE - (A + B\tilde{F}T^{-1}))T - WBVV^{-1}FT \\ &= \begin{bmatrix} sI_{|\alpha|}-(N_\alpha^\top+E_\alpha F_{11}) & -E_\alpha F_{12} & -E_\alpha F_{13} & -E_\alpha F_{14} & -E_\alpha F_{15} & -E_\alpha F_{16} \\ 0 & sK_\beta-L_\beta & 0 & 0 & 0 & 0 \\ -E_\gamma F_{21} & -E_\gamma F_{22} & sL_\gamma^\top-(K_\gamma^\top+E_\gamma F_{23}) & -E_\gamma F_{24} & -E_\gamma F_{25} & -E_\gamma F_{26} \\ 0 & 0 & 0 & sK_\delta^\top-L_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & sN_\kappa-I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}}-A_{\overline{c}} \end{bmatrix}. \end{aligned} \tag{3.4.3}$$

Now the assumption that $[E, A, B]$ is not impulse controllable leads to $\gamma \neq (1, \ldots, 1)$, $\delta \neq (1, \ldots, 1)$ or $\kappa \neq (1, \ldots, 1)$. We will now show that the index of $sE - (A + BF + B\tilde{F}T^{-1})$ is greater than one by showing this for the equivalent pencil in (3.4.3) via applying the condition in (3.2.5): Let Z be a real matrix with $\operatorname{im}_{\mathbb{R}} Z = \ker_{\mathbb{R}} \tilde{E}$. Then

$$Z = \begin{bmatrix} 0 & Z_1^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & Z_2^\top & 0 \end{bmatrix}^\top,$$

where $\operatorname{im} Z_1 = \ker K_\beta = \operatorname{im} E_\beta$ and $\operatorname{im} Z_2 = \ker N_\kappa = \operatorname{im} E_\kappa$. Taking into account that $\operatorname{im}_{\mathbb{R}} E_\gamma \subseteq \operatorname{im}_{\mathbb{R}} L_\gamma^\top$, we obtain that

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} \begin{bmatrix} 0_{|\alpha|-\ell(\alpha)+|\beta|-\ell(\beta),l} & I_{|\gamma|+|\delta|+|\kappa|} & 0_{l,n_{\overline{c}}} \end{bmatrix} \begin{bmatrix} \tilde{E} & \tilde{A}Z \end{bmatrix} \\ &\qquad = \operatorname{im}_{\mathbb{R}} \begin{bmatrix} L_\gamma^\top & 0 & 0 & E_\gamma F_{25} Z_2 \\ 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & N_\kappa & Z_2 \end{bmatrix}. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\operatorname{im}_{\mathbb{R}} \begin{bmatrix} 0_{|\alpha|-\ell(\alpha)+|\beta|-\ell(\beta),l} & I_{|\gamma|+|\delta|+|\kappa|} & 0_{l,n_{\overline{c}}} \end{bmatrix} \begin{bmatrix} \tilde{E} & \tilde{A} \end{bmatrix} \\ &\qquad = \operatorname{im}_{\mathbb{R}} \begin{bmatrix} L_\gamma^\top & 0 & 0 & K_\gamma^\top + E_\gamma F_{23} & E_\gamma F_{24} & E_\gamma F_{25} \\ 0 & K_\delta^\top & 0 & 0 & L_\delta^\top & 0 \\ 0 & 0 & N_\kappa & 0 & 0 & I_{|\kappa|} \end{bmatrix}. \end{aligned}$$

Since the assumption that at least one of the multi-indices satisfies $\gamma \neq (1, \dots, 1)$, $\delta \neq (1, \dots, 1)$, or $\kappa \neq (1, \dots, 1)$ and the fact that $\operatorname{im} Z_2 = \operatorname{im} E_\kappa$ lead to

$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} L_\gamma^\top & 0 & 0 & E_\gamma F_{25} Z_2 \\ 0 & K_\delta^\top & 0 & 0 \\ 0 & 0 & N_\kappa & Z_2 \end{bmatrix} \subsetneq \operatorname{im}_{\mathbb{R}} \begin{bmatrix} L_\gamma^\top & 0 & 0 & K_\gamma^\top + E_\gamma F_{23} & E_\gamma F_{24} & E_\gamma F_{25} \\ 0 & K_\delta^\top & 0 & 0 & L_\delta^\top & 0 \\ 0 & 0 & N_\kappa & 0 & 0 & I_{|\kappa|} \end{bmatrix},$$

and thus

$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} \tilde{E} & \tilde{A}Z \end{bmatrix} \subsetneq \operatorname{im}_{\mathbb{R}} \begin{bmatrix} \tilde{E} & \tilde{A} \end{bmatrix},$$

we find that, by condition (3.2.5), the index of $sE - (A + BF + B\tilde{F}T^{-1})$ has to be greater than one. Since F was chosen arbitrarily we may conclude that $sE - (A + BF)$ has index greater than one for all $F \in \mathbb{R}^{m \times n}$, which completes the proof of (a).

(b) If $[E, A, B]$ is completely stabilizable, then we may transform the system into feedback form (3.4.2). Corollary 3.2.8 (h) implies $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$, $\ell(\kappa) = 0$, and $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$. Further, by [230, Thm. 4.20], there exists some $F_{11} \in \mathbb{R}^{|\alpha|\times\ell(\alpha)}$ such that $\sigma(N_\alpha + E_\alpha F_{11}) \subseteq \mathbb{C}_-$. Setting $\hat{F} := [F_{ij}]_{i=1,\dots,3,j=1,\dots,6}$ with $F_{ij} = 0$ for $i \neq 1$ or $j \neq 1$, we obtain that with $F = \tilde{F}T^{-1} + V\hat{F}T^{-1}$ the system $[E, A+BF]$ is completely stabilizable by Proposition 3.4.3 as complete stabilizability is preserved under system equivalence.

On the other hand, assume that $[E, A, B]$ is not completely stabilizable. We show that for all $F \in \mathbb{R}^{m\times n}$ the system $[E, A+BF]$ is not completely stabilizable. To this end, let $F \in \mathbb{R}^{m\times n}$ and observe that we may apply a transformation as in (3.4.3). Then the assumption that $[E, A, B]$ is not completely stabilizable yields $\gamma \neq (1, \dots, 1)$, $\delta \neq (1, \dots, 1)$, $\ell(\kappa) > 0$, or $\sigma(A_{\overline{c}}) \not\subseteq \mathbb{C}_-$. If $\gamma \neq (1, \dots, 1)$, $\delta \neq (1, \dots, 1)$ or $\ell(\kappa) > 0$, then $\operatorname{im}_{\mathbb{R}} \tilde{A} \not\subseteq \operatorname{im}_{\mathbb{R}} \tilde{E}$, and by Corollary 3.4.4 the system $[\tilde{E}, \tilde{A}]$ is not completely stabilizable. On the other hand, if $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$, $\ell(\kappa) = 0$, and $\lambda \in \sigma(A_{\overline{c}}) \cap \overline{\mathbb{C}}_+$, we find $\operatorname{im}_{\mathbb{C}} \left(\lambda\tilde{E} - \tilde{A}\right) \subsetneq \operatorname{im}_{\mathbb{C}} \tilde{E}$, which implies

$$\operatorname{rk}_{\mathbb{C}}(\lambda\tilde{E}-\tilde{A}) < \operatorname{rk}_{\mathbb{C}}\tilde{E} = n-\ell(\beta)-\ell(\kappa) = n-\ell(\beta) \overset{(3.2.6)}{=} \operatorname{rk}_{\mathbb{R}(s)}(s\tilde{E}-\tilde{A}).$$

Hence, applying Corollary 3.4.4 again, the system $[\tilde{E}, \tilde{A}]$ is not completely stabilizable. As complete stabilizability is invariant under system equivalence it follows that $[E, A+BF+B\tilde{F}T^{-1}]$ is not completely stabilizable. Since F was chosen arbitrarily we may conclude that $[E, A+BF]$ is not completely stabilizable for all $F \in \mathbb{R}^{m\times n}$, which completes the proof of (b).

(c) The proof is analogous to (b). □

Remark 3.4.7 (State feedback)**.**

(i) If the pencil $sE-A$ is regular and $[E, A, B]$ is impulse controllable, then a feedback $F \in \mathbb{R}^{m\times n}$ can be constructed such that the pencil $sE - (A + BF)$ is regular and its index does not exceed one: First we choose $W, T, \tilde{F}$ such that we can put the system into the form (3.4.2). Now, impulse controllability implies that $\gamma = (1, \dots, 1)$, $\delta = (1, \dots, 1)$ and $\kappa = (1, \dots, 1)$. Assuming $\ell(\delta) > 0$ implies that for all $\hat{F} \in \mathbb{R}^{m\times n}$ any QKF of the pencil $sE - (A + B\tilde{F}T^{-1} + B\hat{F})$ satisfies $\ell(\gamma) > 0$ (in the form (3.2.3)), as the feedback $\hat{F}$ cannot act on this block, which contradicts regularity of $sE - A$. Hence it holds $\ell(\delta) = 0$ and from $l = n$ we further obtain that $\ell(\gamma) = \ell(\beta)$. Now applying another feedback as in (3.4.3), where we choose $F_{22} = E_\beta^\top \in \mathbb{R}^{\ell(\beta)\times|\beta|}$ and $F_{ij} = 0$ otherwise, we obtain, taking into account that $E_\gamma = I_{\ell(\gamma)}$ and that the pencil $\begin{bmatrix} sK_\beta - L_\beta \\ -E_\beta^\top \end{bmatrix}$ is regular, that $sE - (A + BF)$ is indeed regular with index at most one.

(ii) The matrix F_{11} in the proof of Theorem 3.4.6 (b) can be constructed as follows: For $j = 1, \ldots, \ell(\alpha)$, consider vectors

$$a_j = -[a_{j\alpha_j-1}, \ldots, a_{j0}] \in \mathbb{R}^{1\times\alpha_j}.$$

Then, for

$$F_{11} = \operatorname{diag}(a_1, \ldots, a_{\ell(\alpha)}) \in \mathbb{R}^{\ell(\alpha)\times|\alpha|}$$

the matrix $N_\alpha + E_\alpha F_{11}$ is diagonally composed of companion matrices, whence, for

$$p_j(s) = s^{\alpha_j} + a_{j\alpha_j-1}s^{\alpha_j-1} + \ldots + a_{j0} \in \mathbb{R}[s]$$

the characteristic polynomial of $N_\alpha + E_\alpha F_{11}$ is given by

$$\det(sI_{|\alpha|} - (N_\alpha + E_\alpha F_{11})) = \prod_{j=1}^{\ell(\alpha)} p_j(s).$$

Hence, choosing the coefficients a_{ji}, $j = 1, \ldots, \ell(\alpha)$, $i = 0, \ldots, \alpha_j$ such that the polynomials $p_1(s), \ldots, p_{\ell(\alpha)}(s) \in \mathbb{R}[s]$ are all Hurwitz, i.e., all roots of $p_1(s), \ldots, p_{\ell(\alpha)}(s)$ are in $\mathbb{C}_-$, we obtain stability.

3.4.3 Control in the behavioral sense

The hitherto presented feedback concept consists of the additional application of the relation $u(t) = Fx(t)$ to the system $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t)$. Feedback can therefore be seen as an additional algebraic constraint that can be resolved for the input. Control in the behavioral sense or, also called, *control via interconnection* [248] generalizes this approach by also allowing further algebraic relations in which the state does not necessarily uniquely determine the input. That is, for given (or to be determined) $K = [K_x, K_u]$ with $K_x \in \mathbb{R}^{q\times n}$, $K_u \in \mathbb{R}^{q\times m}$, we consider the interconnected behavior

$$\begin{aligned}\mathfrak{B}^K_{[E,A,B]} &:= \left\{ (x,u) \in \mathfrak{B}_{[E,A,B]} \;\middle|\; \begin{array}{l}(x(t)^\top, u(t)^\top)^\top \in \ker_{\mathbb{R}} K \\ \text{for almost all } t \in \mathbb{R}\end{array} \right\} \\ &= \mathfrak{B}_{[E,A,B]} \cap \mathfrak{B}_{[0_{q,n},K_x,K_u]}.\end{aligned}$$

We can alternatively write

$$\mathfrak{B}^K_{[E,A,B]} = \mathfrak{B}_{[E^K,A^K]},$$

where

$$[E^K, A^K] = \left[\begin{bmatrix} E & 0 \\ 0 & 0\end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u\end{bmatrix}\right].$$

The concept of control in the behavioral sense has its origin in the works by Willems, Polderman and Trentelman [19, 198, 231, 247, 248], where differential behaviors and their stabilization via *control by interconnection* is considered. The latter means a systematic addition of some further (differential) equations such that a desired behavior is achieved. In contrast to these works we only add equations which are purely algebraic. This justifies to speak of *control by interconnection using static control laws.* We will give equivalent conditions for this type of generalized feedback stabilizing the system. Note that, in principle, one could make the extreme choice $K = I_{n+m}$ to end up with a behavior

$$\mathfrak{B}^K_{[E,A,B]} \subseteq \left\{ (x,u) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m) \;\middle|\; (x,u) \overset{\text{a.e.}}{=} 0 \right\},$$

which is obviously autonomous and stable in the behavioral sense. This, however, is not suitable from a practical point of view, since in this interconnection, the space of consistent initial differential variables is a proper subset of the initial differential variables which are consistent with the original system $[E, A, B]$. Consequently, the interconnected system does not have the causality property - that is, the implementation of the controller at a certain time $t \in \mathbb{R}$ is not possible, since this causes jumps in the differential variables. To avoid this, we introduce the concept of *compatibility.*

Definition 3.4.8 (Compatible and stabilizing control)**.**
Let $[E, A, B] \in \Sigma_{l,n,m}$. The control matrix $K = [K_x, K_u] \in \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ is called

(a) *compatible* for $[E, A, B]$ if, and only if, for any $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B]}$, there exists some $(x, u) \in \mathfrak{B}^K_{[E,A,B]}$ with $Ex(0) = Ex^0$.

(b) *stabilizing* for $[E, A, B]$ if, and only if, $[E^K, A^K] \in \Sigma_{l+q,n+m}$ is stabilizable in the behavioral sense.

Remark 3.4.9 (Compatible control)**.**
Our definition of compatible control is a slight modification of the concept introduced by Julius and van der Schaft in [134] where an interconnection is called compatible, if any trajectory of the system without control law can be concatenated with a trajectory of the interconnected system. This certainly implies that the space of initial differential variables of the interconnected system cannot be smaller than the corresponding set for the nominal system.

Theorem 3.4.10 (Stabilizing control in the behavioral sense)**.**
Let $[E, A, B] \in \Sigma_{l,n,m}$. Then there exists a compatible and stabilizing control $K = [K_x, K_u] \in \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ for $[E, A, B]$, if, and only if, $[E, A, B]$ is stabilizable in the behavioral sense. If $[E, A, B]$ is stabilizable in the behavioral sense, then the compatible and stabilizing control K can moreover be chosen such that $[E^K, A^K]$ is autonomous, i.e., the interconnected system $[E^K, A^K]$ is stable in the behavioral sense.

Proof: Since, by definition, $[E, A, B] \in \Sigma_{l,n,m}$ is stabilizable in the behavioral sense if, and only if, for $s\mathcal{E} - \mathcal{A} = [sE - A, -B]$, the DAE $[\mathcal{E}, \mathcal{A}] \in \Sigma_{l,n+m}$ is stabilizable in the behavioral sense, necessity follows from setting $q = 0$.

In order to show sufficiency, let $K = [K_x, K_u] \in \mathbb{R}^{q\times n} \times \mathbb{R}^{q\times m}$ be a compatible and stabilizing control for $[E, A, B]$. Now the system can be put into feedback form, i.e., there exist $W \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$ and $F \in \mathbb{R}^{m\times n}$ such that

$$\begin{bmatrix} s\tilde{E} - \tilde{A} & \tilde{B} \\ -\tilde{K}_x & \tilde{K}_u \end{bmatrix} = \begin{bmatrix} W & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} sE - A & B \\ -K_x & K_u \end{bmatrix} \begin{bmatrix} T & 0 \\ -F & V \end{bmatrix},$$

where $[\tilde{E}, \tilde{A}, \tilde{B}]$ is in the form (3.2.9). Now the behavioral stabilizability of $[E^K, A^K]$ implies that the system $[\tilde{E}^K, \tilde{A}^K] := \left[\left[\begin{smallmatrix} \tilde{E} & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} \tilde{A} & \tilde{B} \\ \tilde{K}_x & \tilde{K}_u \end{smallmatrix}\right] \right]$ is stabilizable in the behavioral sense as well. Assume that $[E, A, B]$ is not stabilizable in the behavioral sense, that is, by Corollary 3.2.8 (i), there exists $\lambda \in \sigma(A_{\overline{c}}) \cap \overline{\mathbb{C}}_+$. Hence we find $x_6^0 \in \mathbb{R}^{n_{\overline{c}}} \setminus \{0\}$ such that $A_{\overline{c}} x_6^0 = \lambda x_6^0$. Then, with $x(\cdot) := \left(0, \ldots, 0, (e^{\lambda \cdot} x_6^0)^\top\right)^\top$, we have that $(x, 0) \in \mathcal{B}_{[\tilde{E}, \tilde{A}, \tilde{B}]}$. As $x(0) \in \mathcal{V}^{\text{diff}}_{[\tilde{E}, \tilde{A}, \tilde{B}]} = T^{-1} \cdot \mathcal{V}^{\text{diff}}_{[E,A,B]}$, the compatibility of the control K implies that there exists $(\tilde{x}, \tilde{u}) \in \mathfrak{B}^K_{[E,A,B]}$ with $E\tilde{x}(0) = ETx(0)$. This gives $(WET)T^{-1}\tilde{x}(0) = WETx(0)$ and writing $T^{-1}\tilde{x}(t) = (\tilde{x}_1(t)^\top, \ldots, \tilde{x}_6(t)^\top)^\top$ with vectors of appropriate size, we obtain $\tilde{x}_6(0) = x_6^0$. Since the solution of the initial value problem $\dot{y} = A_{\overline{c}} y$, $y(0) = x_6^0$, is unique, we find $\tilde{x}_6(t) = e^{\lambda t} x_6^0$ for all $t \in \mathbb{R}$. Now $(T^{-1}\tilde{x}, -V^{-1}FT^{-1}\tilde{x} + V^{-1}\tilde{u}) \in \mathcal{B}_{[\tilde{E}^K, \tilde{A}^K]}$ and as for all $(\hat{x}, \hat{u}) \in \mathcal{B}_{[\tilde{E}^K, \tilde{A}^K]}$ with $(\hat{x}(t), \hat{u}(t)) = (T^{-1}\tilde{x}(t), -V^{-1}FT^{-1}\tilde{x} + V^{-1}\tilde{u}(t))$ for almost all $t < 0$ we have $\hat{x}_6(t) = \tilde{x}_6(t)$ for all $t \in \mathbb{R}$, and $\tilde{x}_6(t) \not\to 0$ as $t \to \infty$ since $\lambda \in \overline{\mathbb{C}}_+$, this contradicts that $[\tilde{E}^K, \tilde{A}^K]$ is stabilizable in the behavioral sense.

It remains to show the second assertion, that is, for a system $[E, A, B] \in \Sigma_{l,n,m}$ that is stabilizable in the behavioral sense, there exists some compatible and stabilizing control K such that $[E^K, A^K]$ is autonomous: Since, for $[E_1, A_1, B_1], [E_2, A_2, B_2] \in \Sigma_{l,n,m}$ with

$$[\,E_1\,,\ A_1\,,\ B_1\,] \overset{W,T,V,F}{\sim_{fe}} [\,E_2\,,\ A_2\,,\ B_2\,],$$

$$K_2 \in \mathbb{R}^{q \times (n+m)} \quad \text{and} \quad K_1 = K_2 \begin{bmatrix} T & 0 \\ F & V \end{bmatrix},$$

the behaviors of the interconnected systems are related by

$$\begin{bmatrix} T & 0 \\ F & V \end{bmatrix} \mathfrak{B}^{K_1}_{[E_1, A_1, B_1]} = \mathfrak{B}^{K_2}_{[E_2, A_2, B_2]},$$

it is no loss of generality to assume that $[E, A, B]$ is in feedback form (3.2.9), i.e.,

$$sE - A = \begin{bmatrix} sI_{|\alpha|} - N_\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & sK_\beta - L_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & sK_\gamma^\top - L_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & sK_\delta^\top - L_\delta^\top & 0 & 0 \\ 0 & 0 & 0 & 0 & sN_\kappa - I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & sI_{n_{\overline{c}}} - A_{\overline{c}} \end{bmatrix},$$

$$B = \begin{bmatrix} E_\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & E_\gamma & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Let $F_{11} \in \mathbb{R}^{\ell(\alpha)\times|\alpha|}$ be such that $\det(sI_{|\alpha|} - (N_\alpha + E_\alpha F_{11}))$ is Hurwitz. Then the DAE

$$\begin{bmatrix} I_{|\alpha|} & 0 \\ 0 & 0 \end{bmatrix} \dot{z}(t) = \begin{bmatrix} N_\alpha & E_\alpha \\ F_{11} & -I_{\ell(\alpha)} \end{bmatrix} z(t)$$

is stable in the behavioral sense. Furthermore, by reasoning as in Remark 3.4.7 (ii), for

$$a_j = [a_{j\beta_j-2}, \ldots, a_{j0}, 1] \in \mathbb{R}^{1\times\beta_j}$$

with the property that the polynomials

$$p_j(s) = s^{\beta_j} + a_{j\beta_j-1}s^{\beta_j-1} + \ldots + a_{j0} \in \mathbb{R}[s]$$

are Hurwitz for $j = 1, \ldots, \ell(\alpha)$, the choice

$$K_x = \operatorname{diag}(a_1, \ldots, a_{\ell(\beta)}) \in \mathbb{R}^{\ell(\beta)\times|\beta|}$$

leads to an autonomous system

$$\begin{bmatrix} K_\beta \\ 0 \end{bmatrix} \dot{z}(t) = \begin{bmatrix} L_\beta \\ K_x \end{bmatrix} z(t),$$

which is also stable in the behavioral sense. Since, moreover, by Corollary 3.2.8 (i), we have $\sigma(A_{\overline{c}}) \subseteq \mathbb{C}_-$, the choice

$$K = \left[\begin{array}{cccccc|ccc} F_{11} & 0 & 0 & 0 & 0 & 0 & -I_{\ell(\alpha)} & 0 & 0 \\ 0 & K_x & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{m-\ell(\alpha)-\ell(\gamma)} \end{array}\right]$$

leads to a behavioral stable (in particular autonomous) system. Since the differential variables can be arbitrarily initialized in any of the previously discussed subsystems, the constructed control is also compatible. □

3.5 Invariant subspaces

This section is dedicated to some selected results of the geometric theory of differential-algebraic control systems. Geometric theory plays a fundamental role in standard ODE system theory and has been introduced independently by Wonham and Morse and Basile and Marro, see the famous books [16, 252] and also [230], which are the three standard textbooks on geometric control theory. In [159] Lewis gave an overview of the to date geometric theory of DAEs. As we will do here he put special emphasis on the two fundamental sequences $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ and $(\mathcal{W}_i)_{i\in\mathbb{N}_0}$ of subspaces defined as follows:

$$\begin{aligned} \mathcal{V}_0 &:= \mathbb{R}^n, & \mathcal{V}_{i+1} &:= A^{-1}(E\mathcal{V}_i + \operatorname{im}_{\mathbb{R}} B) \subseteq \mathbb{R}^n, & \mathcal{V}^* &:= \bigcap_{i\in\mathbb{N}_0} \mathcal{V}_i, \\ \mathcal{W}_0 &:= \{0\}, & \mathcal{W}_{i+1} &:= E^{-1}(A\mathcal{W}_i + \operatorname{im}_{\mathbb{R}} B) \subseteq \mathbb{R}^n, & \mathcal{W}^* &:= \bigcup_{i\in\mathbb{N}_0} \mathcal{W}_i. \end{aligned}$$

Compared to the Wong sequences for matrix pencils in Definition 2.2.1, in the definition of the sequences $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ and $(\mathcal{W}_i)_{i\in\mathbb{N}_0}$ the image of B is added before taking the pre-image. If $B = 0$, then we end up with the Wong sequences. However, if $B \neq 0$, then $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ and $(\mathcal{W}_i)_{i\in\mathbb{N}_0}$ are in general not Wong sequences corresponding to any matrix pencil. This justifies to call $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ and $(\mathcal{W}_i)_{i\in\mathbb{N}_0}$ *augmented Wong sequences* with respect to the control system (3.1.1).

The reachable space of a system $[E, A, B]$ can be represented by the intersection of the limits of the augmented Wong sequences.

Proposition 3.5.1 (Reachable space [200, Sec. 4])**.**
For $[E, A, B] \in \Sigma_{l,n,m}$ and limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the augmented Wong sequences we have

$$\mathcal{R}_{[E,A,B]} = \mathcal{V}^* \cap \mathcal{W}^*.$$

It has been shown in [13] (for discrete systems), see also [11, 14, 45, 188], that the limit $\mathcal{V}^*$ of the first augmented Wong sequence is the space of consistent initial states. For regular systems this was proved in [158].

Proposition 3.5.2 (Consistent initial states [13])**.**
For $[E, A, B] \in \Sigma_{l,n,m}$ and limit $\mathcal{V}^*$ of the first augmented Wong sequence we have

$$\mathcal{V}_{[E,A,B]} = \mathcal{V}^*.$$

Various other properties of $\mathcal{V}^*$ and $\mathcal{W}^*$ have been derived in [13] in the context of discrete systems.

For regular systems Özçaldiran [187] showed that $\mathcal{V}^*$ is the supremal (A, E, B)-invariant subspace of $\mathbb{R}^n$ and $\mathcal{W}^*$ is the infimal restricted (E, A, B)-invariant subspace of $\mathbb{R}^n$. These concepts, which have also been used in [3, 13, 158, 175] are defined as follows.

Definition 3.5.3 ((A, E, B)- and (E, A, B)-invariance [187])**.**
Let $[E, A, B] \in \Sigma_{l,n,m}$. A subspace $\mathcal{V} \subseteq \mathbb{R}^n$ is called (A, E, B)*-invariant* if, and only if,

$$A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im}_{\mathbb{R}} B.$$

A subspace $\mathcal{W} \subseteq \mathbb{R}^n$ is called *restricted* (E, A, B)*-invariant* if, and only if,

$$\mathcal{W} = E^{-1}(A\mathcal{W} + \operatorname{im}_{\mathbb{R}} B).$$

It is easy to verify that the proofs given in [187, Lems. 2.1 & 2.2] remain the same for general $E, A \in \mathbb{R}^{l\times n}$ and $B \in \mathbb{R}^{l\times m}$ - this was shown in [13] as well. For $\mathcal{V}^*$ this can be found in [3], see also [175]. So we have the following proposition.

Proposition 3.5.4 (Augmented Wong sequences as invariant subspaces)**.**
Consider $[E, A, B] \in \Sigma_{l,n,m}$ *and the limits* $\mathcal{V}^*$ *and* $\mathcal{W}^*$ *of the augmented Wong sequences. Then the following statements hold true.*

(a) $\mathcal{V}^*$ *is* (A, E, B)*-invariant and for any* $\mathcal{V} \subseteq \mathbb{R}^n$ *which is* (A, E, B)*-invariant it holds* $\mathcal{V} \subseteq \mathcal{V}^*$*;*

(b) $\mathcal{W}^*$ *is restricted* (E, A, B)*-invariant and for any* $\mathcal{W} \subseteq \mathbb{R}^n$ *which is restricted* (E, A, B)*-invariant it holds* $\mathcal{W}^* \subseteq \mathcal{W}$*.*

It is now clear how the controllability concepts can be characterized in terms of the invariant subspaces $\mathcal{V}^*$ and $\mathcal{W}^*$. However, the statement about R-controllability (behavioral controllability) seems to be new. The only other appearance of a subspace inclusion as a characterization of R-controllability that the author is aware of occurs in [70] for regular systems: if $A = I$, then the system is R-controllable if, and only if, $\operatorname{im}_{\mathbb{R}} E^D \subseteq \langle E^D | B\rangle$, where E^D is the Drazin inverse of E, cf. Section 3.7 (iv).

Theorem 3.5.5 (Geometric criteria for controllability)**.**
Consider $[E, A, B] \in \Sigma_{l,n,m}$ *and the limits* $\mathcal{V}^*$ *and* $\mathcal{W}^*$ *of the augmented Wong sequences. Then* $[E, A, B]$ *is*

(a) *controllable at infinity if, and only if,* $\mathcal{V}^* = \mathbb{R}^n$*;*

(b) *impulse controllable if, and only if,* $\mathcal{V}^* + \ker_{\mathbb{R}} E = \mathbb{R}^n$ *or, equivalently,* $E\mathcal{V}^* = \operatorname{im}_{\mathbb{R}} E$*;*

(c) *controllable in the behavioral sense if, and only if,* $\mathcal{V}^* \subseteq \mathcal{W}^*$*;*

(d) *completely controllable if, and only if,* $\mathcal{V}^* \cap \mathcal{W}^* = \mathbb{R}^n$*;*

(e) *strongly controllable if, and only if,* $(\mathcal{V}^* \cap \mathcal{W}^*) + \ker_{\mathbb{R}} E = \mathbb{R}^n$ *or, equivalently,* $E(\mathcal{V}^* \cap \mathcal{W}^*) = \operatorname{im}_{\mathbb{R}} E$*.*

Proof: By Propositions 3.5.1 and 3.5.2 it is clear that it only remains to prove (c). We proceed in several steps.

Step 1: Let $[E_1, A_1, B_1], [E_2, A_2, B_2] \in \Sigma_{l,n,m}$ such that for some $W \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$ and $F \in \mathbb{R}^{m\times n}$ it holds

$$[\,E_1\,,\,A_1\,,\,B_1\,] \overset{W,T,V,F}{\sim_{fe}} [\,E_2\,,\,A_2\,,\,B_2\,]\,.$$

We show that the augmented Wong sequences $\mathcal{V}_i^1$, $\mathcal{W}_i^1$ of $[E_1, A_1, B_1]$ and the augmented Wong sequences $\mathcal{V}_i^2$, $\mathcal{W}_i^2$ of $[E_2, A_2, B_2]$ are related by

$$\forall\, i \in \mathbb{N}_0 :\ \mathcal{V}_i^1 = T^{-1}\mathcal{V}_i^2 \ \wedge\ \mathcal{W}_i^1 = T^{-1}\mathcal{W}_i^2.$$

We prove the statement by induction. It is clear that $\mathcal{V}_0^1 = T^{-1}\mathcal{V}_0^2$. Assuming that $\mathcal{V}_i^1 = T^{-1}\mathcal{V}_i^2$ for some $i \geq 0$ we find that, by (3.2.1),

$$\begin{aligned}
\mathcal{V}_{i+1}^1 &= A_1^{-1}(E_1\mathcal{V}_i^1 + \operatorname{im}_{\mathbb{R}} B_1)\\
&= \left\{\ x \in \mathbb{R}^n \ \middle|\ \begin{array}{l} \exists\, y \in \mathcal{V}_i^1\ \exists\, u \in \mathbb{R}^m :\\ W(A_2T + B_2T)x = WE_2Ty + WB_2Vu \end{array} \right\}\\
&= \left\{\ x \in \mathbb{R}^n \ \middle|\ \exists\, z \in \mathcal{V}_i^2\ \exists\, v \in \mathbb{R}^m :\ A_2Tx = E_2z + B_2v\ \right\}\\
&= T^{-1}\left(A_2^{-1}(E_2\mathcal{V}_i^1 + \operatorname{im}_{\mathbb{R}} B_2)\right) = T^{-1}\mathcal{V}_{i+1}^2.
\end{aligned}$$

The statement about $\mathcal{W}_i^1$ and $\mathcal{W}_i^2$ can be proved analogous.

Step 2: By Step 1 we may without loss of generality assume that $[E, A, B]$ is given in feedback form (3.2.9). We make the convention that if $\alpha \in \mathbb{N}^k$ is some multi-index, then $\alpha - 1 := (\alpha_1 - 1, \ldots, \alpha_k - 1)$. It not follows that

$$\forall\, i \in \mathbb{N}_0 :\ \mathcal{V}_i = \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \operatorname{im}_{\mathbb{R}} N_{\gamma-1}^i \times \operatorname{im}_{\mathbb{R}}(N_{\delta-1}^\top)^i \times \operatorname{im}_{\mathbb{R}} N_\kappa^i \times \mathbb{R}^{n_{\overline{c}}}, \tag{3.5.1}$$

which is immediate from observing that $K_\gamma^\top x = L_\gamma^\top y + E_\gamma u$ for some x, y, u of appropriate dimension yields $x = N_{\gamma-1} y$ and $L_\delta^\top x = K_\delta^\top y$ for some x, y yields $x = N_{\delta-1}^\top y$. Note that in the case $\gamma_i = 1$ or $\delta_i = 1$, i.e., we have a 1×0 block, we find that N_{γ_i-1} and N_{δ_i-1} are absent, so these relations are consistent.
On the other hand we find that

$$\forall\, i \in \mathbb{N}_0 : \\ \mathcal{W}_i = \ker_{\mathbb{R}} N_\alpha^i \times \ker_{\mathbb{R}} N_\beta^i \times \ker_{\mathbb{R}} N_{\gamma-1}^i \times \{0\}^{|\delta|-\ell(\delta)} \times \ker_{\mathbb{R}} N_\kappa^i \times \{0\}^{n_{\overline{c}}}, \tag{3.5.2}$$

which indeed needs some more rigorous proof. First observe that $\operatorname{im}_{\mathbb{R}} E_\alpha = \ker_{\mathbb{R}} N_\alpha$, $\ker_{\mathbb{R}} K_\beta = \ker_{\mathbb{R}} N_\beta$ and $(L_\gamma^\top)^{-1}(\operatorname{im}_{\mathbb{R}} E_\gamma) = \operatorname{im}_{\mathbb{R}} E_{\gamma-1} = \ker_{\mathbb{R}} N_{\gamma-1}$. Therefore we have

$$\begin{aligned} \mathcal{W}_1 &= E^{-1}(\operatorname{im}_{\mathbb{R}} B) \\ &= \ker_{\mathbb{R}} N_\alpha \times \ker_{\mathbb{R}} N_\beta \times \ker_{\mathbb{R}} N_{\gamma-1} \times \{0\}^{|\delta|-\ell(\delta)} \times \ker_{\mathbb{R}} N_\kappa \times \{0\}^{n_{\overline{c}}}. \end{aligned}$$

Further observe that $N_\alpha^i N_\alpha^\top = N_\alpha N_\alpha^\top N_\alpha^{i-1}$ for all $i \in \mathbb{N}$ and, hence, if $x = N_\alpha^\top y + E_\alpha u$ for some x, u and $y \in \ker_{\mathbb{R}} N_\alpha^{i-1}$ it follows $x \in \ker_{\mathbb{R}} N_\alpha^i$. Likewise, if $L_\gamma^\top x = K_\gamma^\top y + E_\gamma u$ for some x, u and $y \in \ker_{\mathbb{R}} N_{\gamma-1}^{i-1}$ we find $x = N_{\gamma-1}^\top y + E_{\gamma-1}^\top u$ and hence $x \in \ker_{\mathbb{R}} N_{\gamma-1}^i$. Finally, if $K_\beta x = L_\beta y$ for some x and some $y \in \ker_{\mathbb{R}} N_\beta^{i-1}$ it follows that by adding some zero rows we obtain $N_\beta x = N_\beta N_\beta^\top y$ and hence, as above, $x \in \ker_{\mathbb{R}} N_\beta^i$. This proves (3.5.2).

Step 3: From (3.5.1) and (3.5.2) it follows that

$$\begin{aligned} \mathcal{V}^* &= \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \operatorname{im}_{\mathbb{R}} \{0\}^{|\gamma|-\ell(\gamma)} \times \{0\}^{|\delta|-\ell(\delta)} \times \{0\}^{|\kappa|} \times \mathbb{R}^{n_{\overline{c}}}, \\ \mathcal{W}^* &= \mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \operatorname{im}_{\mathbb{R}} \mathbb{R}^{|\gamma|-\ell(\gamma)} \times \{0\}^{|\delta|-\ell(\delta)} \times \mathbb{R}^{|\kappa|} \times \{0\}^{n_{\overline{c}}}. \end{aligned}$$

As by Corollary 3.2.8 (f) the system $[E, A, B]$ is controllable in the behavioral sense if, and only if, $n_{\overline{c}} = 0$ we may immediately deduce that this is the case if, and only if, $\mathcal{V}^* \subseteq \mathcal{W}^*$. This proves the theorem. ☐

Remark 3.5.6 (Representation of the reachable space)**.**
From Proposition 3.5.1 and the proof of Theorem 3.5.5 we may immediately observe that, using the notation from Theorem 3.2.6, we have

$$\begin{aligned} &\mathcal{R}_{[E,A,B]} \\ &\quad = T^{-1}\left(\mathbb{R}^{|\alpha|} \times \mathbb{R}^{|\beta|} \times \operatorname{im}_{\mathbb{R}} \{0\}^{|\gamma|-\ell(\gamma)} \times \{0\}^{|\delta|-\ell(\delta)} \times \{0\}^{|\kappa|} \times \{0\}^{n_{\overline{c}}}\right). \end{aligned}$$

3.6 Kalman decomposition

Nearly fifty years ago Kalman [138] derived his famous decomposition of linear ODE control systems. This decomposition has later been generalized to regular DAEs by Verghese at al. [241], see also [80]. A Kalman decomposition of general discrete-time DAE systems has been provided by Banaszuk et al. [14] (later generalized to systems with output equation in [11]) in a very nice way using the augmented Wong sequences (cf. Section 3.5). They derive a system

$$\left[\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right], \tag{3.6.1}$$

which is system equivalent to given $[E, A, B] \in \Sigma_{l,n,m}$ with the properties that the system $[E_{11}, A_{11}, B_1]$ is completely controllable and the matrix $[E_{11}, A_{11}, B_1]$ has full row rank (strongly $\mathcal{H}$-controllable in the notation of [14]) and, furthermore, $\mathcal{R}_{[E_{22},A_{22},0]} = \{0\}$.

This last condition is reasonable, as one should wonder what properties a Kalman decomposition of a DAE system should have. In the case of ODEs the decomposition simply is

$$\left[\begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix}\right], \quad \text{where } [A_{11}, B_1] \text{ is controllable.}$$

Therefore, an ODE system is decomposed into a controllable and an uncontrollable part, since clearly $[A_{22}, 0]$ is not controllable at all. For DAEs however, the situation is more subtle, since in a decomposition (3.6.1) with $[E_{11}, A_{11}, B_1]$ completely controllable (and $[E_{11}, A_{11}, B_1]$ full row rank) the conjectural 'uncontrollable' part $[E_{22}, A_{22}, 0]$ may still have a controllable subsystem, since systems of the type $[K_\beta, L_\beta, 0]$ are always controllable. To exclude this and ensure that all controllable parts are included in $[E_{11}, A_{11}, B_1]$ we may state the additional condition (as in [14]) that

$$\mathcal{R}_{[E_{22},A_{22},0]} = \{0\}.$$

This then also guarantees certain uniqueness properties of the Kalman decomposition. Hence, any system (3.6.1) with the above properties we may call a Kalman decomposition.

Definition 3.6.1 (Kalman decomposition)**.**
A system $[E, A, B] \in \Sigma_{l,n,m}$ is said to be a *Kalman decomposition* if, and only if,

$$[E, A, B] = \left[\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right], \qquad (3.6.2)$$

where $E_{11}, A_{11} \in \mathbb{R}^{l_1 \times n_1}$, $E_{12}, A_{12} \in \mathbb{R}^{l_1 \times n_2}$, $E_{22}, A_{22} \in \mathbb{R}^{l_2 \times n_2}$ and $B_1 \in \mathbb{R}^{l_1 \times m}$, such that $[E_{11}, A_{11}, B_1] \in \Sigma_{l_1,n_1,m}$ is completely controllable, $\mathrm{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = l_1$ and $\mathcal{R}_{[E_{22},A_{22},0_{l_2,m}]} = \{0\}$.

We cite the result of [14] and sketch the proof.

Theorem 3.6.2 (Kalman decomposition)**.**
For $[E, A, B] \in \Sigma_{l,n,m}$, there exist $W \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$ such that $[WET, WAT, WB]$ is a Kalman decomposition 3.6.2.

Proof: The Kalman decomposition (3.6.2) can be derived using the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the augmented Wong sequences presented in Section 3.5. It is clear that these spaces satisfy the following subspace relations:

$$\begin{aligned} E(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq (E\mathcal{V}^* + \mathrm{im}_{\mathbb{R}} B) \cap (A\mathcal{W}^* + \mathrm{im}_{\mathbb{R}} B), \\ A(\mathcal{V}^* \cap \mathcal{W}^*) &\subseteq (E\mathcal{V}^* + \mathrm{im}_{\mathbb{R}} B) \cap (A\mathcal{W}^* + \mathrm{im}_{\mathbb{R}} B). \end{aligned}$$

Therefore, if we choose any full rank matrices $R_1 \in \mathbb{R}^{n \times n_1}, P_1 \in \mathbb{R}^{n \times n_2}$, $R_2 \in \mathbb{R}^{l \times l_1}, P_2 \in \mathbb{R}^{l \times l_2}$ such that

$$\begin{aligned} &\mathrm{im}_{\mathbb{R}} R_1 = \mathcal{V}^* \cap \mathcal{W}^*, \quad \mathrm{im}_{\mathbb{R}} R_1 \oplus \mathrm{im}_{\mathbb{R}} P_1 = \mathbb{R}^n, \quad \mathrm{im}_{\mathbb{R}} R_2 \oplus \mathrm{im}_{\mathbb{R}} P_2 = \mathbb{R}^l, \\ &\mathrm{im}_{\mathbb{R}} R_2 = (E\mathcal{V}^* + \mathrm{im}_{\mathbb{R}} B) \cap (A\mathcal{W}^* + \mathrm{im}_{\mathbb{R}} B), \end{aligned}$$

then $[R_1, P_1] \in \mathbf{Gl}_n(\mathbb{R})$ and $[R_2, P_2] \in \mathbf{Gl}_l(\mathbb{R})$, and, furthermore, there exist matrices $E_{11}, A_{11} \in \mathbb{R}^{l_1 \times n_1}$, $E_{12}, A_{12} \in \mathbb{R}^{l_1 \times n_2}$, $E_{22}, A_{22} \in \mathbb{R}^{l_2 \times n_2}$ such that

$$\begin{aligned} ER_1 &= R_2 E_{11}, & AR_1 &= R_2 A_{11}, \\ EP_1 &= R_2 E_{12} + P_2 E_{22}, & AP_1 &= R_2 A_{12} + P_2 A_{22}. \end{aligned}$$

Since $\mathrm{im}_{\mathbb{R}} B \subseteq (E\mathcal{V}^* + \mathrm{im}_{\mathbb{R}} B) \cap (A\mathcal{W}^* + \mathrm{im}_{\mathbb{R}} B) = \mathrm{im}_{\mathbb{R}} R_2$, there exists $B_1 \in \mathbb{R}^{l_1 \times m}$ such that $B = R_2 B_1$. All these relations together yield the decomposition (3.6.2) with $W = [R_2, P_2]$ and $T = [R_1, P_1]^{-1}$. The properties of the entries in (3.6.2) essentially rely on the observation that by Proposition 3.5.1

$$\mathcal{R}_{[E,A,B]} = \mathcal{V}^* \cap \mathcal{W}^* = \operatorname{im}_{\mathbb{R}} R_1 = T^{-1}(\mathbb{R}^{n_1} \times \{0\}^{n_2}).$$

We omit the proof of this and refer to [14]. □

Remark 3.6.3 (Kalman decomposition)**.**
It is important to note that a trivial reachable space does not necessarily imply that $B = 0$. An intriguing example which illustrates this is the system

$$[E, A, B] = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right]. \tag{3.6.3}$$

Another important fact we like to stress by means of this example is that $B \neq 0$ does no necessarily imply $n_1 \neq 0$ in the Kalman decomposition (3.6.2). In fact, the above system $[E, A, B]$ is already in Kalman decomposition with $l_1 = l_2 = 1, n_1 = 0, n_2 = 1, m = 1$ and $E_{12} = 1$, $A_{12} = 0$, $B_1 = 1$ as well as $E_{22} = 0$, $A_{22} = 1$. Then all the required properties are satisfied, in particular $\operatorname{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = \operatorname{rk}_{\mathbb{R}}[1] = 1$ and the system $[E_{11}, A_{11}, B_1]$ is completely controllable as it is in feedback form (3.2.9) with $\gamma = 1$; complete controllability then follows from Corollary 3.2.8. However, $[E_{11}, A_{11}, B_1]$ is hard to view as a control system as no equation can be written down. Nevertheless, the space $\mathcal{R}_{[E_{11},A_{11},B_1]}$ has dimension zero and obviously every state can be steered to every other state.

We now analyze how two forms of type (3.6.2) of one system $[E, A, B] \in \Sigma_{l,n,m}$ differ.

Proposition 3.6.4 (Uniqueness of the Kalman decomposition)**.**
Let $[E, A, B] \in \Sigma_{l,n,m}$ be given and assume that, for all $i \in \{1, 2\}$, the systems $[E_i, A_i, B_i] \overset{W_i,T_i}{\sim}_{se} [E, A, B]$ with

$$sE_i - A_i = \begin{bmatrix} sE_{11,i} - A_{11,i} & sE_{12,i} - A_{12,i} \\ 0 & sE_{22,i} - A_{22,i} \end{bmatrix}, \quad B_i = \begin{bmatrix} B_{1,i} \\ 0 \end{bmatrix}$$

where $E_{11,i}, A_{11,i} \in \mathbb{R}^{l_{1,i} \times n_{1,i}}$, $E_{12,i}, A_{12,i} \in \mathbb{R}^{l_{1,i} \times n_{2,i}}$, $E_{22,i}, A_{22,i} \in \mathbb{R}^{l_{2,i} \times n_{2,i}}$, $B_{1,i} \in \mathbb{R}^{l_{1,i} \times m}$ are Kalman decompositions.

Then $l_{1,1} = l_{1,2}$, $l_{2,1} = l_{2,2}$, $n_{1,1} = n_{1,2}$, $n_{2,1} = n_{2,2}$. *Moreover, for some* $W_{11} \in \mathbf{Gl}_{l_{1,1}}(\mathbb{R})$, $W_{12} \in \mathbb{R}^{l_{1,1}\times l_{2,1}}$, $W_{22} \in \mathbf{Gl}_{l_{2,1}}(\mathbb{R})$, $T_{11} \in \mathbf{Gl}_{n_{1,1}}(\mathbb{R})$, $T_{12} \in \mathbb{R}^{n_{1,1}\times n_{2,1}}$, $T_{22} \in \mathbf{Gl}_{n_{2,1}}(\mathbb{R})$, *we have*

$$W_2W_1^{-1} = \begin{bmatrix} W_{11} & W_{12} \\ 0 & W_{22} \end{bmatrix}, \quad T_1^{-1}T_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix}.$$

In particular, the systems $[E_{11,1}, A_{11,1}, B_{1,1}]$, $[E_{11,2}, A_{11,2}, B_{1,2}]$ *and, respectively,*
$[E_{22,1}, A_{22,1}, 0]$, $[E_{22,2}, A_{22,2}, 0]$ *are system equivalent.*

Proof: It is no loss of generality to assume that $W_1 = I_l$, $T_1 = I_n$. Then we obtain

$$\mathbb{R}^{n_{1,1}} \times \{0\} = \mathcal{R}_{[E_1,A_1,B_1]} = T_2\mathcal{R}_{[E_2,A_2,B_2]} = T_2\big(\mathbb{R}^{n_{1,2}} \times \{0\}\big).$$

This implies $n_{1,1} = n_{1,2}$ and

$$T_2 = \begin{bmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{bmatrix} \text{ for some } T_{11} \in \mathbf{Gl}_{n_{1,1}},\ T_{12} \in \mathbb{R}^{n_{1,1}\times\times n_{2,1}},\ T_{22} \in \mathbf{Gl}_{n_{2,1}}.$$

Now partitioning

$$W_2 = \begin{bmatrix} W_{11} & W_{12} \\ W_{21} & W_{22} \end{bmatrix}, \qquad \begin{array}{l} W_{11} \in \mathbb{R}^{l_{1,1}\times l_{1,2}},\ W_{12} \in \mathbb{R}^{l_{1,1}\times l_{2,2}}, \\ W_{21} \in \mathbb{R}^{l_{2,1}\times l_{1,2}},\ W_{22} \in \mathbb{R}^{l_{2,1}\times l_{2,2}}, \end{array}$$

the block $(2,1)$ of the equations $W_1E_1T_1 = E_2$, $W_1A_1T_1 = A_2$ and $W_1B_1 = B_2$ give rise to

$$0 = W_{21}\begin{bmatrix} E_{11,2} & A_{11,2} & B_{1,2} \end{bmatrix}.$$

Since the latter matrix is supposed to have full row rank, we obtain $W_{21} = 0$. The assumption of W_2 being invertible then leads to $l_{1,1} \le l_{1,2}$. Reversing the roles of $[E_1, A_1, B_1]$ and $[E_2, A_2, B_2]$, we further obtain $l_{1,2} \le l_{1,1}$, whence $l_{1,2} = l_{1,1}$. Using again the invertibility of W, we obtain that both W_{11} and W_{22} are invertible. □

It is immediate from the form (3.6.2) that $[E, A, B]$ is completely controllable if, and only if, $n_1 = n$. The following result characterizes the further controllability and stabilizability notions in terms of properties of the submatrices in (3.6.2).

Corollary 3.6.5 (Properties induced from the Kalman decomposition)**.**
Consider $[E, A, B] \in \Sigma_{l,n,m}$ *with*

$$[E, A, B] \overset{W,T}{\sim}_{se} [\tilde{E}, \tilde{A}, \tilde{B}] = \left[\begin{bmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{bmatrix}, \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix}, \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \right]$$

such that $[\tilde{E}, \tilde{A}, \tilde{B}]$ *is a Kalman decomposition. Then the following statements hold true:*

(a) $\mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = n_2$.

(b) If $sE - A$ *is regular, then both pencils* $sE_{11} - A_{11}$ *and* $sE_{22} - A_{22}$ *are regular. In particular, it holds* $l_1 = n_1$ *and* $l_2 = n_2$.

(c) If $[E, A, B]$ *is impulse controllable, then the index of the pencil* $sE_{22} - A_{22}$ *is at most one.*

(d) $[E, A, B]$ *is controllable at infinity if, and only if,* $\mathrm{im}_{\mathbb{R}} A_{22} \subseteq \mathrm{im}_{\mathbb{R}} E_{22}$.

(e) $[E, A, B]$ *is controllable in the behavioral sense if, and only if,* $\mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = \mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22})$ *for all* $\lambda \in \mathbb{C}$.

(f) $[E, A, B]$ *is stabilizable in the behavioral sense if, and only if,* $\mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = \mathrm{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22})$ *for all* $\lambda \in \overline{\mathbb{C}}_+$.

Proof:

(a) Assuming that $\mathrm{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) < n_2$, then, in a QKF (3.2.3) of $sE_{22} - A_{22}$, it holds $\ell(\beta) > 0$ by (3.2.6). By the findings of Remark 3.2.7 (ii), we can conclude $\mathcal{R}_{[E_{22}, A_{22}, 0_{l_2,m}]} \neq \{0\}$, a contradiction.

(b) We can infer from (a) that $n_2 \le l_2$. We can further deduce from the regularity of $sE - A$ that $n_2 \ge l_2$. The regularity of $sE_{11} - A_{11}$ and $sE_{22} - A_{22}$ then follows immediately from $\det(sE - A) = \det(W \cdot T) \cdot \det(sE_{11} - A_{11}) \cdot \det(sE_{22} - A_{22})$.

(c) Assume that $[E, A, B]$ is impulse controllable. By Proposition 3.3.3 and the invariance of impulse controllability under system equivalence this implies that

$$\mathrm{im}_{\mathbb{R}} \begin{bmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{bmatrix} \subseteq \mathrm{im}_{\mathbb{R}} \begin{bmatrix} E_{11} & E_{12} & B_1 & A_{11}Z_1 + A_{12}Z_2 \\ 0 & E_{22} & 0 & A_{22}Z_2 \end{bmatrix},$$

where $Z = [Z_1^\top, Z_2^\top]^\top$ is a real matrix such that $\operatorname{im}_{\mathbb{R}} Z = \ker_{\mathbb{R}} \left[\begin{smallmatrix} E_{11} & E_{12} \\ 0 & E_{22} \end{smallmatrix}\right]$. The last condition in particular implies that $\operatorname{im}_{\mathbb{R}} Z_2 \subseteq \ker_{\mathbb{R}} E_{22}$ and therefore we obtain

$$\operatorname{im}_{\mathbb{R}} A_{22} \subseteq \operatorname{im}_{\mathbb{R}} E_{22} + A_{22} \ \ker_{\mathbb{R}} E_{22},$$

which is, by (3.2.4), equivalent to the index of $sE_{22} - A_{22}$ being at most one.

(d) Since $\operatorname{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = l_1$ and the system $[E_{11}, A_{11}, B_1]$ is controllable at infinity, Proposition 3.3.3 leads to $\operatorname{rk}_{\mathbb{R}}[E_{11}, B_1] = l_1$. Therefore, we have

$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} E_{11} & E_{12} & B_1 \\ 0 & E_{22} & 0 \end{bmatrix} = \mathbb{R}^{l_1} \times \operatorname{im}_{\mathbb{R}} E_{22}.$$

Analogously, we obtain

$$\operatorname{im}_{\mathbb{R}} \begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_1 \\ 0 & E_{22} & 0 & A_{22} & 0 \end{bmatrix} = \mathbb{R}^{l_1} \times (\operatorname{im}_{\mathbb{R}} E_{22} + \operatorname{im}_{\mathbb{R}} A_{22}).$$

Again using Proposition 3.3.3 and the invariance of controllability at infinity under system equivalence, we see that $[E, A, B]$ is controllable at infinity if, and only if,

$$\mathbb{R}^{l_1} \times (\operatorname{im}_{\mathbb{R}} E_{22} + \operatorname{im}_{\mathbb{R}} A_{22}) = \mathbb{R}^{l_1} \times \operatorname{im}_{\mathbb{R}} E_{22},$$

which is equivalent to $\operatorname{im}_{\mathbb{R}} A_{22} \subseteq \operatorname{im}_{\mathbb{R}} E_{22}$.

(e) Since $\operatorname{rk}_{\mathbb{R}}[E_{11}, A_{11}, B_1] = l_1$ and $[E_{11}, A_{11}, B_1] \in \Sigma_{l_1, n_1, m}$ is completely controllable it holds

$$\operatorname{rk}_{\mathbb{C}}[\lambda E_{11} - A_{11}, B_1] = l_1 \quad \text{for all } \lambda \in \mathbb{C}.$$

Therefore, we have

$$\operatorname{rk}_{\mathbb{C}}[\lambda E - A, B] = \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E_{11} - A_{11} & \lambda E_{12} - A_{12} & B_1 \\ 0 & \lambda E_{22} - A_{22} & 0 \end{bmatrix} = l_1 + \operatorname{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22}),$$

and, analogously, $\operatorname{rk}_{\mathbb{R}(s)}[sE - A, B] = l_1 + \operatorname{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22})$. Now applying Proposition 3.3.3 we find that $[E, A, B]$ is controllable in the behavioral sense if, and only if, $\operatorname{rk}_{\mathbb{R}(s)}(sE_{22} - A_{22}) = \operatorname{rk}_{\mathbb{C}}(\lambda E_{22} - A_{22})$ for all $\lambda \in \mathbb{C}$.

(f) The proof of this statement is analogous to (e). □

Remark 3.6.6 (Kalman decomposition and controllability).
Note that the condition of the index of $sE_{22} - A_{22}$ being at most one in Corollary 3.6.5(c) is equivalent to the system $[E_{22}, A_{22}, 0_{l_2,m}]$ being impulse controllable. Likewise, the condition $\operatorname{im}_{\mathbb{R}} A_{22} \subseteq \operatorname{im}_{\mathbb{R}} E_{22}$ in (d) is equivalent to $[E_{22}, A_{22}, 0_{l_2,m}]$ being controllable at infinity. Obviously, the conditions in (e) and (f) are equivalent to behavioral controllability and stabilizability of $[E_{22}, A_{22}, 0_{l_2,m}]$, resp.
Furthermore, the converse implication in (b) does not hold true. That is, the index of $sE_{22} - A_{22}$ being at most one is in general not sufficient for $[E, A, B]$ being impulse controllable. For instance, reconsider system (3.6.3) which is not impulse controllable, but $sE_{22} - A_{22} = -1$ is of index one. Even in the case where $sE - A$ is regular, the property of the index of $sE_{22} - A_{22}$ being zero or one is not enough to infer impulse controllability of $sE - A$. As a counterexample, consider

$$[E, A, B] = \left[\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right].$$

3.7 Notes and References

(i) The controllability concepts are not consistently treated in the literature. For instance, one has to pay attention if it is (tacitly) claimed that $[E, B] \in \mathbb{R}^{l\times(n+m)}$ or $[E, A, B] \in \mathbb{R}^{l\times(2n+m)}$ have full rank.
For regular systems we have the following:

concept	coincides with notion in	called [...] in
controllability at infinity	see item (v)	reachability at ∞ in [158]
impulse controllability	[77] and [121, Rem. 2]	controllability at ∞ in [158]; controllability at infinity in [6, 7, 241]
R-controllability	[70, 80, 254] and [121, Rem. 2]	–

complete controllability	[70, 80, 254]	controllability in [77]
strong controllability	[241] and [121, Rem. 2]	impulse controllability in [103]

Some of these aforementioned articles introduce the controllability by means of certain rank criteria for the matrix triple $[E, A, B]$. The connection of the concepts introduced in Definition 3.1.5 to linear algebraic properties of E, A and B are highlighted in Section 3.3.

For general DAE systems we have:

concept	coincides with notion in	called [...] in
controllability at infinity	–	–
impulse controllability	[101, 117, 130]	controllability at infinity in [52]
R-controllability	–	–
complete controllability	[188]	controllability in [98]
strong controllability	–	controllability in [188]

Our behavioral controllability coincides with the framework which is introduced in [198, Definition 5.2.2] for so-called *differential behaviors*, which are general (possibly higher order) DAE systems with constant coefficients. Note that the concept of behavioral controllability does not require a distinction between input and state. The concepts of reachability and controllability in [12–15] coincide with our behavioral and complete controllability, resp. (see Sec. 3.3). Full controllability of [260] is our complete controllability together with the additional assumption that solutions have to be unique.

(ii) Stabilizability in the behavioral sense is introduced in [198, Definition 5.2.2]. For regular systems, stabilizability is usually defined either via linear algebraic properties of E, A and B, or by the existence of a stabilizing state feedback, see [50, 51, 96] and [80, Definition 3-1.2.]. Our concepts of behavioral stabilizability and stabilizability coincide with the notions of internal stability and complete stabilizability, resp., defined in [177] for the system $\mathcal{E}\dot{z}(t) = \mathcal{A}z(t)$ with $\mathcal{E} = [\,E\,,\,0\,]$, $\mathcal{A} = [\,A\,,\,B\,]$, $z(t) = [\,x^\top(t)\,,\,u^\top(t)\,]^\top$.

(iii) Other concepts, not related to the ones considered in this chapter, are e.g. the instantaneous controllability (reachability) of order k in [188] or the impulsive mode controllability in [117]. Furthermore, the concept of strong controllability introduced in [230, Exercise 8.5] for ODE systems differs from the concepts considered in this thesis.

(iv) The reachable and controllable spaces are some of the most important notions for (DAE) control systems and have been considered in [158] for regular systems. They are the fundamental subspaces considered in the geometric theory, see Section 3.5. Further usage of these concepts can be found in the following: in [190] generalized reachable and controllable subspaces of regular systems are considered; Eliopoulou and Karcanias [92] consider reachable and almost reachable subspaces of general DAE systems; Frankowska [98] considers the reachable subspace in terms of differential inclusions.
A nice formula for the reachable space of a regular system has been derived by Yip et al. [254] (and later been adopted by Cobb [77], however called controllable subspace): Consider a regular system $[E, A, B] \in \Sigma_{n,n,m}$ with (E, A) in QWF (2.1.1) (in fact, the WCF is considered in [254], but the calculation is the same) and B partitioned accordingly, that is

$$E = \begin{bmatrix} I_{n_1} & 0 \\ 0 & N \end{bmatrix}, \quad A = \begin{bmatrix} J & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix},$$

where N is nilpotent. Then [254, Thm. 2]

$$\mathcal{R}_{[E,A,B]} = \langle J|B_1\rangle \times \langle N|B_2\rangle,$$

where $\langle K|L\rangle := \operatorname{im}_{\mathbb{R}}[L, KL, \ldots, K^{n-1}L]$ for some matrices $K \in \mathbb{R}^{n\times n}$, $L \in \mathbb{R}^{n\times m}$. Furthermore, we have [254, Thm. 3]

$$\mathcal{V}_{[E,A,B]} = \mathbb{R}^{n_1} \times \langle N|B_2\rangle.$$

This result has been improved later in [70] so that the QWF is no longer needed. Denoting by E^D the Drazin inverse of a given matrix $E \in \mathbb{R}^{n\times n}$ (see [66]), it is shown [70, Thm. 3.1] that, for $A = I$,

$$\mathcal{R}_{[E,A,B]} = E^D \langle E^D | B \rangle \oplus (I - EE^D)\langle E|B\rangle,$$

where the consideration of $A = I$ is justified by a certain (time-varying) transformation of the system [192]. We further have [70, Thm. 3.2]

$$\mathcal{V}_{[E,A,B]} = \operatorname{im}_{\mathbb{R}} E^D \oplus (I - EE^D)\langle E|B\rangle.$$

Yet another approach was followed by Cobb [73] who obtains that

$$\mathcal{R}_{[E,A,B]} = \langle (\alpha E - A)^{-1} E | (\alpha E - A)^{-1} B \rangle$$

for some $\alpha \in \mathbb{R}$ with $\det(\alpha E - A) \neq 0$. A simple proof of this result can also be found in [259].

(v) Impulse controllability and controllability at infinity are usually defined by considering distributional solutions of (3.1.1), see e.g. [77, 101, 130], sometimes called impulsive modes, see [20, 117, 241]. For regular systems, impulse controllability has been introduced by Verghese et al. [241] (called controllability at infinity in this work) as controllability of the impulsive modes of the system, and later made more precise by Cobb [77], see also Armentano [6, 7] (who also calls it controllability at infinity) for a more geometric point of view. In [241] the authors do also develop the notion of strong controllability as impulse controllability with, additionally, controllability in the regular sense. Cobb [74] showed that under the condition of impulse controllability, the infinite eigenvalues of regular $sE - A$ can be assigned via a state feedback $u = Fx$ to arbitrary finite positions. Armentano [6] later showed how to calculate F. This topic has been further pursued in [150] in the form of invariant polynomial assignment. The name 'controllability at infinity' comes from the claim that the system has no infinite uncontrollable modes: Speaking in terms of rank criteria (see also Section 3.3) the system $[E, A, B] \in \Sigma_{l,n,m}$ is said to have an uncontrollable mode at $\frac{\alpha}{\beta}$ if, and only if, rk$[\alpha E + \beta A, B] <$ rk$[E, A, B]$ for some $\alpha, \beta \in \mathbb{C}$. If $\beta = 0$, then the uncontrollable mode is infinite. Controllability at infinity has been introduced by ROSENBROCK [211] - although he does not use this phrase - as controllability of the infinite frequency zeros. Later COBB [77] compared the concepts of impulse controllability and controllability at infinity, see [77, Thm. 5]; the notions we use in the present thesis go back to the distinction in this work. The concepts have later been generalized by GEERTS [101] (see [101, Thm. 4.5 & Rem. 4.9], however he does not use the name 'controllability at infinity'). Controllability at infinity of (3.1.1) is equivalent to the strictness of the corresponding differential inclusion [98, Prop. 2.6]. The concept of impulsive mode controllability in [117] is even weaker than impulse controllability.

(vi) For a discussion of distributional solutions for DAEs see Section 2.5. We briefly refer to [227, 228], where the notions of impulse controllability and jump controllability are introduced, which coincide with our impulse controllability and behavioral controllability, resp.

(vii) R-controllability has been first defined in [254] for regular DAEs. Roughly speaking, R-controllability is the property that any consistent initial state x_0 can be steered to any reachable state x_f, where here x_f is reachable if, and only if, there exist $t > 0$ and $(x, u) \in \mathfrak{B}_{[E,A,B]}$ such that $x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$, $x(0) = 0$ and $x(t) = x_f$; by (3.1.2) the latter is equivalent to $x_f \in \mathcal{V}_{[E,A,B]}$, as stated in Definition 3.1.5.

(viii) The concept of behavioral controllability has been introduced by WILLEMS [245], see also [198]. This concept is suitable for generalizations in various directions, see e.g. [53, 67, 120, 153, 203, 248, 253]. Having found the behavior of the considered control system one can take over the definition of behavioral controllability without the need for any further changes. From this point of view this appears to be the most natural of the controllability concepts. However, this concept also seems to be the least regarded in the DAE literature.

(ix) The controllability theory of DAE systems can also be treated with the theory of differential inclusions [9, 10] as showed by FRANKOWSKA [98].

(x) KARCANIAS and HAYTON [141] pursued a special ansatz to simplify the system (3.1.1): provided that B has full column rank, take a left annihilator N and the left inverse $B^\dagger = (B^\top B)^{-1}B^\top$ of B (i.e., $NB = 0$ and $B^\dagger B = I$) such that $W = \left[\begin{smallmatrix} N \\ B^\dagger \end{smallmatrix}\right]$ is invertible and then pre-multiply (3.1.1) by W, thus obtaining the equivalent system

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} NEx &= NAx, \\ u &= B^\dagger(\tfrac{\mathrm{d}}{\mathrm{d}t}Ex - Ax). \end{aligned}$$

The reachability (controllability) properties of (3.1.1) may now be studied in terms of the pencil $sNE - NA$, which is called the restriction pencil [133], first introduced as zero pencil for the investigation of system zeros of ODEs in [147, 148], see also [144]. For a comprehensive study of the properties of the pencil $sNE - NA$ see e.g. [140–143].

(xi) BANASZUK and PRZYŁUSKI [15] have considered perturbations of DAE control systems and obtained conditions under which the sets of all completely controllable systems (systems controllable in the behavioral sense) within the set of all systems $\Sigma_{l,n,m}$ contain an open and dense subset, or its complement contains an open and dense subset.

(xii) For regular systems which are completely controllable, two canonical forms of $[E, A, B] \in \Sigma_{n,n,m}$ under system equivalence have been obtained: the Jordan control canonical form in [104] and, later, the simpler canonical form in [113] based on the Hermite canonical form for controllable ODEs $[I, A, B]$.

(xiii) It is known [12, 103] that the class of regular DAE systems is not closed under the action of state feedback. Therefore, in [219] the class of regular systems is divided into the families

$$\Sigma_\theta := \{\ (E, A, B) \in \Sigma_{n,n,m} \ \mid\ \det(\cos\theta\ E - \sin\theta\ A) \neq 0\ \},$$

for $\theta \in [0, \pi)$, and it is shown that any of these families is dense in the set of regular systems and the union of these families is exactly the set of regular systems. The authors of [219] then introduce the 'constant-ratio proportional and derivative' feedback on Σ_θ, i.e.

$$u = F(\cos\theta \; x - \sin\theta \; \dot{x}) + v.$$

This feedback leads to a group action and enables them to obtain a generalization of Brunovský's theorem [48] on each of the subsets of completely controllable systems in Σ_θ, see [219, Thm. 6].

(xiv) Glüsing-Lüerssen [103] derived a canonical form under the unchanged feedback equivalence (3.2.1) on the set of strongly controllable (called impulse controllability in [103]) regular systems, see [103, Thm. 4.7]. In particular it is shown that this set is closed under the action of a feedback group.

(xv) For regular systems $[E, A, B] \in \Sigma_{n,n,m}$ with $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$ the usual Hautus and Kalman criteria can be found in a summarized form e.g. in [80]. Other approaches to derive controllability criteria rely on the expansion of $(sE - A)^{-1}$ as a power series in s at $s_0 = 0$, which is only feasible in the regular case. For instance, in [178] the numerator matrices of this expansion, i.e., the coefficients of the polynomial $\operatorname{adj}(sE - A)$, are used to derive a rank criterion for complete controllability. Then again, in [146] Kalman rank criteria for complete controllability, R-controllability and controllability at infinity are derived in terms of the coefficients of the power series expansion of $(sE - A)^{-1}$. The advantage of these criteria, especially the last one, is that no transformation of the system needs to be performed as it is usually necessary in order to derive Kalman rank criteria for DAEs, see e.g. [80].
However, simple criteria can be obtained using only a left transformation of little impact: if $\alpha \in \mathbb{R}$ is chosen such that $\det(\alpha E - A) \neq 0$, then the system is complete controllable if, and only if, [259, Cor. 1]

$$\begin{aligned} \operatorname{rk}_{\mathbb{R}} \big[(\alpha E - A)^{-1}B, \big((\alpha E - A)^{-1}E\big)(\alpha E - A)^{-1}B, \ldots \\ \ldots, \big((\alpha E - A)^{-1}E\big)^{n-1}(\alpha E - A)^{-1}B \Big] = n, \end{aligned}$$

and it is impulse controllable if, and only if, [259, Thm. 2]

$$\operatorname{im}_{\mathbb{R}}(\alpha E - A)^{-1}E + \ker(\alpha E - A)^{-1}E + \operatorname{im}_{\mathbb{R}}(\alpha E - A)^{-1}B = \mathbb{R}^n.$$

The result concerning complete controllability has also been obtained in [70, Thm. 4.1] for the case $A = I$ and $\alpha = 0$.
Yet another approach was followed by Kučera and Zagalak [150] who introduced controllability indices and characterized strong controllability in terms of an equation for these indices.

(xvi) The augmented Wong sequences (for $B \neq 0$) have been extensively studied by several authors, see e.g. [158, 174, 175, 186, 187, 189, 190, 238] for regular systems and [3, 11, 13, 14, 44, 45, 92, 159, 167, 188, 200] for general DAE systems. Frankowska [98] did a nice investigation of systems (3.1.1) in terms of differential inclusions [9, 10], however requiring controllability at infinity (see [98, Prop. 2.6]). Nevertheless, she is the first to derive a formula for the reachable space [98, Thm. 3.1], which was later generalized by Przyłuski and Sosnowski [200, Sec. 4] (in fact, the same generalization has been announced in [167, p. 296], [159, Sec. 5] and [11, p. 1510], however without proof); it also occurred in [92, Thm. 2.5].

(xvii) A characterization of the limits $\mathcal{V}^*$ and $\mathcal{W}^*$ of the augmented Wong sequences in terms of distributions is given in [200]: $\mathcal{V}^* + \ker_{\mathbb{R}} E$ is the set of all initial values such that the distributional initial value problem [200, (3)] has a smooth solution (x, u); $\mathcal{W}^*$ is the set of all initial values such that [200, (3)] has an impulsive solution (x, u); $\mathcal{V}^* + \mathcal{W}^*$ is the set of all initial values such that [200, (3)] has an impulsive-smooth solution (x, u).

4 Zero dynamics

In this chapter we study linear differential-algebraic control systems of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\,, \end{aligned}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$. The set of these systems is denoted by $\Sigma_{l,n,m,p}$ and we write $[E,A,B,C] \in \Sigma_{l,n,m,p}$. We put special emphasis on the non-regular case, i.e., we do not assume that $sE - A$ is regular.

Compared to the class of systems studied in Chapter 3, here we allow for an additional output equation $y(t) = Cx(t)$. The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. By virtue of Definition 2.4.1, a trajectory $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of $[E,A,B,C]$ if, and only if, it belongs to the *behavior* of $[E,A,B,C]$:

$$\begin{aligned} \mathfrak{B}_{[E,A,B,C]} := \big\{ (x,u,y) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \;\big|\; Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^l) \\ \text{and } (x,u) \text{ satisfies } \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t), \\ y(t) = Cx(t) \text{ for almost all } t \in \mathbb{R} \big\}. \end{aligned}$$

Particular emphasis is placed on the *zero dynamics* of $[E,A,B,C]$. These are, for $[E,A,B,C] \in \Sigma_{l,n,m,p}$, defined by

$$\mathcal{ZD}_{[E,A,B,C]} := \left\{ \; (x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \;\middle|\; y \overset{\text{a.e.}}{=} 0 \; \right\}.$$

By linearity of $[E,A,B,C]$, $\mathcal{ZD}_{[E,A,B,C]}$ is a real vector space.

The zero dynamics of $[E,A,B,C]$ are called *autonomous* if, and only if,

$$\forall\, w_1, w_2 \in \mathcal{ZD}_{[E,A,B,C]} \;\forall\, I \subseteq \mathbb{R} \text{ open nonempty interval}: \quad w_1|_I \overset{\text{a.e.}}{=} w_2|_I \implies w_1 \overset{\text{a.e.}}{=} w_2\,;$$

and *asymptotically stable* if, and only if,

$$\forall\,(x,u,y) \in \mathcal{ZD}_{[E,A,B,C]}: \quad \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|(x,u)\| = 0.$$

Note that the above definitions are within the spirit of the *behavioral approach* [198] and take into account that the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are a linear behavior. In this framework the definition for autonomy of a general behavior is given in [198, Sec. 3.2] and the definition of asymptotic stability in [198, Def. 7.2.1].

(Asymptotically stable) zero dynamics are the vector space of those trajectories of the system which are, loosely speaking, not visible at the output (and tend to zero).

For convenience we call the extended matrix pencil $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ the *system pencil* of $[E,A,B,C] \in \Sigma_{l,n,m,p}$. In Section 4.1 we will derive a so called 'zero dynamics form' of $[E,A,B,C]$ within the equivalence class defined by *system equivalence* (in generalization of the concept introduced in Definition 3.2.1): two systems $[E_i,A_i,B_i,C_i] \in \Sigma_{l,n,m,p}$, $i=1,2$, are called *system equivalent* if, and only if,

$$\exists\, S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}): \\ \begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE_1 - A_1 & B_1 \\ C_1 & 0 \end{bmatrix} \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} sE_2 - A_2 & B_2 \\ C_2 & 0 \end{bmatrix};$$

we write

$$[\,E_1\,,\,A_1\,,\,B_1\,,\,C_1\,] \overset{S,T}{\sim} [\,E_2\,,\,A_2\,,\,B_2\,,\,C_2\,]\,.$$

It is easy to see that system equivalence is an equivalence relation on $\Sigma_{l,n,m,p} \times \Sigma_{l,n,m,p}$. The notion of system equivalence goes back to Rosenbrock [211], cf. also Section 3.2.

This chapter is organized as follows: In Section 4.1 we study autonomous zero dynamics and derive several important characterizations for it as well as one of the main results of this chapter, the zero dynamics form (ZDF) in Theorem 4.1.7; this form is obtained by using the largest (A,E,B)-invariant subspace included in $\ker C$. The ZDF is then refined in Theorem 4.2.7 in Section 4.2, where concepts of system invertibility are introduced and studied in relation to the zero dynamics. We also show how the inverse system can be calculated and under which condition a system with autonomous zero dynamics is right-invertible. This class of systems is then considered in Section 4.3 and it is shown in Theorem 4.3.12 that its elements have asymptotically stable zero dynamics if, and only if, the conditions stabilizability in the behavioral sense, detectability in the behavioral sense and the absence of transmission zero in the closed right complex half-plane are satisfied; these concepts are rigorously introduced and characterized. The result of Theorem 4.3.12 is then exploited for stabilization via compatible control in the behavioral sense in Theorem 4.4.3 in Section 4.4. It is shown that the Lyapunov exponent of the interconnected system can be assigned to the Lyapunov exponent of the zero dynamics.

The results of Sections 4.1, 4.2 and 4.3 are partly contained in the submitted manuscript [28]. The submitted work [29] contains (parts of) Sections 4.3 and 4.4. Lemma 4.3.11 has already been published in a joint work with Achim Ilchmann and Timo Reis [35].

4.1 Autonomous zero dynamics

In this section we investigate autonomy of the zero dynamics of control systems

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\,, \end{aligned} \tag{4.1.1}$$

where $[E, A, B, C] \in \Sigma_{l,n,m,p}$. We derive several important characterizations of autonomous zero dynamics and, as the main result of this section, the so called zero dynamics form in Theorem 4.1.7. The following observation is immediate from the definition of autonomous zero dynamics.

Remark 4.1.1 (Autonomous zero dynamics)**.**
Since $\mathcal{ZD}_{[E,A,B,C]}$ is a real vector space, the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are autonomous if, and only if, for any $w \in \mathcal{ZD}_{[E,A,B,C]}$ which satisfies $w|_I \overset{\text{a.e.}}{=} 0$ on some open interval $I \subseteq \mathbb{R}$, it follows that $w \overset{\text{a.e.}}{=} 0$.

In order to characterize (autonomous) zero dynamics we use the concept of (A, E, B)-invariance from Definition 3.5.3. For a system $[E, A, B, C] \in \Sigma_{l,n,m,p}$, we define the set of all (A, E, B)-invariant subspaces included in $\ker C$ by

$$\mathcal{L}(E, A, B; \ker C) := \left\{ \mathcal{V} \subseteq \mathbb{R}^n \;\middle|\; \begin{array}{l} \mathcal{V} \text{ is } (A, E, B)\text{-invariant subspace} \\ \text{of } \mathbb{R}^n \text{ and } \mathcal{V} \subseteq \ker C \end{array} \right\}.$$

It can easily be verified that $\mathcal{L}(E,A,B;\ker C)$ is closed under subspace addition and thus $\mathcal{L}(E,A,B;\ker C)$ is an upper semi-lattice relative to subspace inclusion and addition. Hence, by [252, Lem. 4.4], there exists a supremal element of $\mathcal{L}(E,A,B;\ker C)$, namely

$$\max(E,A,B;\ker C) := \sup \mathcal{L}(E,A,B;\ker C) = \max \mathcal{L}(E,A,B;\ker C).$$

We show that $\max(E,A,B;\ker C)$ can be derived from a sequence of subspaces which terminates after finitely many steps. These sequences can be viewed as a modification of the (augmented) Wong sequences from Definition 2.2.1 (see also Section 3.5).

Lemma 4.1.2 (Subspace sequences leading to $\max(E,A,B;\ker C)$)**.** *Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ and define $\mathcal{V}_0 := \ker C$ and*

$$\forall\, i \in \mathbb{N}:\ \mathcal{V}_i := A^{-1}(E\mathcal{V}_{i-1} + \operatorname{im} B) \cap \ker C.$$

Then the sequence $(\mathcal{V}_i)_{i\in\mathbb{N}_0}$ satisfies

$$\exists\, k^* \in \mathbb{N}\ \forall\, j \in \mathbb{N}: \quad \mathcal{V}_0 \supsetneq \mathcal{V}_1 \supsetneq \cdots \supsetneq \mathcal{V}_{k^*} = \mathcal{V}_{k^*+j} = A^{-1}(E\mathcal{V}_{k^*} + \operatorname{im} B) \cap \ker C. \tag{4.1.2}$$

Furthermore,

$$\mathcal{V}_{k^*} = \max(E,A,B;\ker C) \tag{4.1.3}$$

and, if $(x,u,y) \in \mathcal{ZD}_{[E,A,B,C]}$, then

$$\text{for almost all } t \in \mathbb{R}:\ x(t) \in \mathcal{V}_{k^*}.$$

Proof: It is easy to see that (4.1.2) holds true and (4.1.3) follows from [187, Lem. 2.1]. For the last statement let $(x,u,y) \in \mathcal{ZD}_{[E,A,B,C]}$. Then we have

$$Ax(t) = \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) - Bu(t) \quad \text{and} \quad x(t) \in \ker C$$

for almost all $t \in \mathbb{R}$. Since, for any subspace $\mathcal{S} \subseteq \mathbb{R}^n$, if $x(t) \in \mathcal{S}$ for almost all $t \in \mathbb{R}$, then $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) \in E\mathcal{S}$ for almost all $t \in \mathbb{R}$, we conclude

$$x(t) \in A^{-1}(\{\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t)\} + \operatorname{im} B) \cap \ker C \subseteq \mathcal{V}_1 \text{ for almost all } t \in \mathbb{R}.$$

Inductively, we obtain $x(t) \in \mathcal{V}_{k^*}$ for almost all $t \in \mathbb{R}$. □

For later use we collect the following lemma about the quasi-Kronecker form from Proposition 3.2.3.

Lemma 4.1.3 (Full column rank and quasi-Kronecker form).
Let $s\widehat{E} - \widehat{A} \in \mathbb{R}[s]^{\widehat{l} \times \widehat{n}}$ *and consider any QKF* (3.2.3) *of* $s\widehat{E} - \widehat{A}$. *Then* $\ell(\beta) = 0$ *if, and only if,* $\operatorname{rk}_{\mathbb{R}[s]} s\widehat{E} - \widehat{A} = \widehat{n}$.

Proof: The assertion is immediate from (3.2.6) and $\operatorname{rk}_{\mathbb{R}[s]} s\widehat{E} - \widehat{A} = \operatorname{rk}_{\mathbb{R}(s)} s\widehat{E} - \widehat{A}$. □

As an immediate consequence of Lemma 4.1.2 we infer that the state x of any trajectory $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$ evolves in $\max(E, A, B; \ker C)$.

Proposition 4.1.4 (Characterization of zero dynamics).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. *If* $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]}$, *then*

$$(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]} \iff x(t) \in \max(E, A, B; \ker C) \quad \text{for almost all } t \in \mathbb{R}\,.$$

Next, we state some important characterizations of autonomous zero dynamics in terms of a pencil rank condition (exploiting the QKF) and some conditions involving the largest (A, E, B)-invariant subspace included in $\ker C$.

Proposition 4.1.5 (Characterization of autonomous zero dynamics).

Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. *Then the following three statements are equivalent:*

(i) $\mathcal{Z}D_{[E,A,B,C]}$ *is autonomous.*

(ii) $\operatorname{rk}_{\mathbb{R}[s]} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = n + m$.

(iii) **(A1)** $\operatorname{rk} B = m$,

(A2) $\ker E \cap \max(E, A, B; \ker C) = \{0\}$,

(A3) $\operatorname{im} B \cap E \max(E, A, B; \ker C) = \{0\}$.

Proof: In view of Proposition 3.2.3, there exist $S \in \mathbf{Gl}_{l+p}(\mathbb{R}), T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ such that

$$S \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} T = \begin{bmatrix} sI_{n_s} - A_s & 0 & 0 & 0 \\ 0 & sN_\alpha - I_{|\alpha|} & 0 & 0 \\ 0 & 0 & sK_\beta - L_\beta & 0 \\ 0 & 0 & 0 & sK_\gamma^\top - L_\gamma^\top \end{bmatrix} \tag{4.1.4}$$

(i)⇒(ii): Suppose that (ii) does not hold. Then Lemma 4.1.3 yields $\ell(\beta) > 0$. Therefore, we find $z \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R}^{|\beta|}) \setminus \{0\}$ and $I \subseteq \mathbb{R}$ open interval such that $z|_I = 0$ and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\beta - L_\beta) z = 0$. This implies that

$$\begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} E - A & -B \\ -C & 0 \end{bmatrix} T\,(0, z^\top, 0, 0)^\top = 0,$$

which contradicts autonomous zero dynamics.

(ii)⇒(i): By (ii) and Lemma 4.1.3 it follows that $\ell(\beta) = 0$ in (4.1.4). Let $w \in \mathcal{ZD}_{[E,A,B,C]}$ and $I \subseteq \mathbb{R}$ be an open interval such that $w|_I \overset{\text{a.e.}}{=} 0$. Then, with $(v_1^\top, v_2^\top, v_3^\top)^\top := T^{-1} w$, we have

$$S^{-1} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_{n_s} - A_s & 0 & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|} & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top \end{bmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \overset{\text{a.e.}}{=} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} E - A & -B \\ -C & 0 \end{bmatrix} w \overset{\text{a.e.}}{=} 0,$$

and thus $(\frac{\mathrm{d}}{\mathrm{d}t} I_{n_s} - A_s) v_1 \overset{\text{a.e.}}{=} 0$, $(\frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|}) v_2 \overset{\text{a.e.}}{=} 0$, and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top) v_3 \overset{\text{a.e.}}{=} 0$. Then, successively solving each block in $(\frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|}) v_2 \overset{\text{a.e.}}{=} 0$ and $(\frac{\mathrm{d}}{\mathrm{d}t} K_\gamma^\top - L_\gamma^\top) v_3 \overset{\text{a.e.}}{=} 0$ (cf. also Subsection 2.4.2) gives $v_2 \overset{\text{a.e.}}{=} 0$ and $v_3 \overset{\text{a.e.}}{=} 0$. Since $v_1|_I \overset{\text{a.e.}}{=} 0$ it follows that $v_1 = 0$. So we may conclude that $w \overset{\text{a.e.}}{=} 0$, by which the zero dynamics are autonomous.

(i)⇒(iii): *Step 1*: (A1) follows from (ii).

Step 2: We show (A2). Let $V \in \mathbb{R}^{n \times k}$ with full column rank such that $\operatorname{im} V = \max(E, A, B; \ker C)$. By definition of $\max(E, A, B; \ker C)$ there exist $N \in \mathbb{R}^{k \times k}, M \in \mathbb{R}^{m \times k}$ such that $AV = EVN + BM$ and $CV = 0$. Therefore, we have

$$\left(s \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right) \begin{bmatrix} V \\ -M \end{bmatrix} = \begin{bmatrix} EV \\ 0 \end{bmatrix} (sI_k - N)$$

By (ii) we find $s_0 \in \mathbb{C}$ such that $\left[\begin{smallmatrix} s_0E-A & -B \\ -C & 0 \end{smallmatrix}\right]$ has full column rank and $s_0 I_k - N$ is invertible. Let $y \in \ker E \cap \max(E, A, B; \ker C)$. Then there exists $x \in \mathbb{R}^k$ such that $y = Vx$ and $EVx = 0$. Therefore,

$$\begin{bmatrix} s_0E - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} V \\ -M \end{bmatrix} (s_0 I_k - N)^{-1} x = \begin{bmatrix} EV \\ 0 \end{bmatrix} x = 0.$$

This implies that $\left[\begin{smallmatrix} V \\ -M \end{smallmatrix}\right](s_0 I_k - N)^{-1} x = 0$ and since V has full column rank we find $x = 0$.

Step 3: We show (A3). Choose $W \in \mathbb{R}^{n\times(n-k)}$ such that $[V, W] \in \mathbf{Gl}_n(\mathbb{R})$. Then

$$\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} [V, W] & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} s[EV, EW] - [AV, AW] & -B \\ [0, C_2] & 0 \end{bmatrix}$$

and since EV has full column rank by Step 2, there exists $S \in \mathbf{Gl}_l(\mathbb{R})$ such that $SEV = \left[\begin{smallmatrix} I \\ 0 \end{smallmatrix}\right]$, thus

$$\begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} [V, W] & 0 \\ 0 & I_m \end{bmatrix} = \begin{bmatrix} s\begin{bmatrix} I & E_2 \\ 0 & E_4 \end{bmatrix} - \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ [0, C_2] & 0 \end{bmatrix}. \tag{4.1.5}$$

Since $AV = EVN + BM$, we obtain $SAV = SEVN + SBM$, whence

$$\begin{bmatrix} A_1 \\ A_3 \end{bmatrix} = \begin{bmatrix} N \\ 0 \end{bmatrix} + \begin{bmatrix} B_1 M \\ B_2 M \end{bmatrix}.$$

Therefore,

$$\begin{aligned} &\begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} [V, W] & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ [M, 0] & I_m \end{bmatrix} \\ &\qquad = \begin{bmatrix} s\begin{bmatrix} I & E_2 \\ 0 & E_4 \end{bmatrix} - \begin{bmatrix} N & A_2 \\ 0 & A_4 \end{bmatrix} & \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \\ [0, C_2] & 0 \end{bmatrix}, \end{aligned}$$

Now, let $v \in \mathbb{R}^k$ and $w \in \mathbb{R}^m$ be such that $EVv = Bw \in \operatorname{im} B \cap E \max(E, A, B; \ker C)$, hence

$$\begin{pmatrix} v \\ 0 \end{pmatrix} = SEVv = SBw = \begin{pmatrix} B_1 w \\ B_2 w \end{pmatrix}.$$

For s_0 as in Step 2 we find

$$\begin{bmatrix} s_0I - N & s_0E_2 - A_2 & B_1 \\ 0 & s_0E_4 - A_4 & B_2 \\ 0 & C_2 & 0 \end{bmatrix} \begin{pmatrix} -(s_0I - N)^{-1}v \\ 0 \\ w \end{pmatrix} = 0,$$

and so $v = 0$ and $w = 0$.

(iii)⇒(i): By (A2) we obtain that (4.1.5) holds. Incorporating (A3) gives

$$\begin{aligned}\{0\} &= \operatorname{im} B \cap E \max(E, A, B; \ker C) \\ &= \operatorname{im} SB \cap \operatorname{im} SEV = \operatorname{im} \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} \cap \operatorname{im} \begin{bmatrix} I_k \\ 0 \end{bmatrix},\end{aligned}$$

by which $B_1 = 0$. From (A1) it follows that B_2 has full column rank. Now, let $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$ and $I \subseteq \mathbb{R}$ an open interval such that $(x, u)|_I \overset{\text{a.e.}}{=} 0$. Applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ and observing that, by Proposition 4.1.4, $x(t) \in \operatorname{im} V$ for almost all $t \in \mathbb{R}$, it follows $Wz_2(t) = x(t) - Vz_1(t) \in \operatorname{im} W \cap \operatorname{im} V = \{0\}$ for almost all $t \in \mathbb{R}$. Therefore, $z_2 \overset{\text{a.e.}}{=} 0$ and $w := z_1 + E_2z_2 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^k)$ satisfies $w \overset{\text{a.e.}}{=} z_1$. Furthermore, $A_1z_1 \overset{\text{a.e.}}{=} A_1w$, $A_3z_1 \overset{\text{a.e.}}{=} A_3w$ and since $E_4z_2 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^{n-k})$ we find $\frac{\mathrm{d}}{\mathrm{d}t}E_4z_2 \overset{\text{a.e.}}{=} 0$. Then (4.1.5) implies that

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}w &\overset{\text{a.e.}}{=} A_1w, \\ 0 &\overset{\text{a.e.}}{=} A_3w + B_2u, \end{aligned}$$

hence

$$w|_I \overset{\text{a.e.}}{=} z_1|_I \overset{\text{a.e.}}{=} Vx|_I \overset{\text{a.e.}}{=} 0$$

gives $w = 0$ and therefore $u \overset{\text{a.e.}}{=} 0$. □

The characterization in Proposition 4.1.5 was observed for ODE systems
$(I, A, B, C) \in \Sigma_{n,n,m,m}$ by Ilchmann and Wirth (personal communication, 2012). The following zero dynamics form in Definition 4.1.6 was derived for ODE systems (I, A, B, C) by Isidori [131, Rem. 6.1.3]. In Theorem 4.1.7 we derive the zero dynamics form for DAE systems with autonomous zero dynamics; in [131] it is not clear that the assumptions (A1), (A3) are equivalent to autonomous zero dynamics (note that (A2) is superfluous for ODEs).

Definition 4.1.6 (Zero dynamics form)**.**
A system $[E, A, B, C] \in \Sigma_{l,n,m,p}$ is said to be in *zero dynamics form* if, and only if,

$$E = \begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix}, \ A = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ 0 & A_6 \end{bmatrix}, \ B = \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, \ C = [0, C_2] \qquad (4.1.6)$$

such that $k = \dim \max(E, A, B; \ker C)$,

$$\max\left(\begin{bmatrix} E_4 \\ E_6 \end{bmatrix}, \begin{bmatrix} A_4 \\ A_6 \end{bmatrix}, \begin{bmatrix} I_m \\ 0 \end{bmatrix}, C_2\right) = \{0\}, \qquad (4.1.7)$$

and $A_1 \in \mathbb{R}^{k\times k}$, $E_2 \in \mathbb{R}^{k\times(n-k)}$, $A_2 \in \mathbb{R}^{k\times(n-k)}$, $A_3 \in \mathbb{R}^{m\times k}$ $E_4 \in \mathbb{R}^{m\times(n-k)}$, $A_4 \in \mathbb{R}^{m\times(n-k)}$, $E_6 \in \mathbb{R}^{(l-k-m)\times(n-k)}$, $A_6 \in \mathbb{R}^{(l-k-m)\times(n-k)}$, $C_2 \in \mathbb{R}^{p\times(n-k)}$.

Theorem 4.1.7 (Zero dynamics form)**.**
Consider $[E, A, B, C] \in \Sigma_{l,n,m,p}$ and suppose that the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are autonomous. Let $V \in \mathbb{R}^{n\times k}$ be such that $\operatorname{im} V = \max(E, A, B; \ker C)$ and $\operatorname{rk} V = k$. Then there exist $W \in \mathbb{R}^{n\times(n-k)}$ and $S \in \mathbf{Gl}_l(\mathbb{R})$ such that $[V, W] \in \mathbf{Gl}_n(\mathbb{R})$ and

$$[E, A, B, C] \overset{S,[V,W]}{\sim} [\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}], \qquad (4.1.8)$$

where $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}]$ is in zero dynamics form.

For uniqueness we have: If $[E, A, B, C]$, $[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \in \Sigma_{l,n,m,p}$ are in zero dynamics form and

$$[E, A, B, C] \overset{S,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}] \quad \text{for some } S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), \qquad (4.1.9)$$

then

$$S = \begin{bmatrix} S_1 & 0 & S_3 \\ 0 & I_m & S_6 \\ 0 & 0 & S_9 \end{bmatrix}, \quad T = \begin{bmatrix} S_1^{-1} & T_2 \\ 0 & T_4 \end{bmatrix},$$

where $S_1 \in \mathbf{Gl}_k(\mathbb{R}), S_9 \in \mathbf{Gl}_{l-k-m}(\mathbb{R}), T_4 \in \mathbf{Gl}_{n-k}(\mathbb{R})$ and S_3, S_6, T_2 are of appropriate sizes. In particular the dimensions of the matrices in (4.1.6) are unique and A_1 is unique up to similarity, hence $\sigma(A_1)$ is unique.

Proof: *Step 1*: We prove (4.1.8) and (4.1.6). By Proposition 4.1.5, autonomous zero dynamics are equivalent to the conditions (A1)–(A3). These conditions imply that $k + m \leq l$. Then we may find $W \in \mathbb{R}^{n\times(n-k)}$ such that $[V, W] \in \mathbf{Gl}_n(\mathbb{R})$. Considering the transformed system $\big(E[V, W], A[V, W], B, C[V, W]\big)$, we find that $CV = 0$, since $\operatorname{im} V \subseteq \ker C$. Further observe that EV has full column rank by (A2) and, since B has full column rank by (A1) and $\operatorname{im} EV \cap \operatorname{im} B = \{0\}$ by (A3), we obtain that $[EV, B]$ has full column rank. Hence, we find $S \in \mathbf{Gl}_l(\mathbb{R})$ such that

$$S\,[EV, B] = \begin{bmatrix} I_k & 0 \\ 0 & I_m \\ 0 & 0 \end{bmatrix}.$$

Therefore,

$$\begin{aligned}[E, A, B, C] &\overset{S,[V,W]}{\sim} \big(SE[V, W], SA[V, W], SB, C[V, W]\big) \\ &= \left[\begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix}, \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ A_5 & A_6 \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, [0, C_2]\right].\end{aligned}$$

By (A, E, B)-invariance of $\operatorname{im} V$, there exist $N \in \mathbb{R}^{k\times k}$, $M \in \mathbb{R}^{m\times k}$ such that $AV = EVN + BM$, thus

$$S^{-1}\begin{bmatrix} A_1 \\ A_3 \\ A_5 \end{bmatrix} = AV = EVN + BM = S^{-1}\begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix} N + S^{-1}\begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} M,$$

which gives $A_5 = 0$.

Step 2: We show (4.1.7). Let $(\overline{E}, \overline{A}, \overline{B}, \overline{C}) := \left(\left[\begin{smallmatrix} E_4 \\ E_6 \end{smallmatrix}\right], \left[\begin{smallmatrix} A_4 \\ A_6 \end{smallmatrix}\right], \left[\begin{smallmatrix} I_m \\ 0 \end{smallmatrix}\right], C_2\right)$ and $q \in \mathbb{N}_0$, $Z \in \mathbb{R}^{(n-k)\times q}$, $X \in \mathbb{R}^{q\times q}$, $Y \in \mathbb{R}^{m\times q}$ be such that

$$\overline{A}Z = \overline{E}ZX + \overline{B}Y \quad \wedge \quad \overline{C}Z = 0.$$

We show that $\mathcal{V} := \operatorname{im}[V, WZ]$ is (A, E, B)-invariant and included in $\ker C$.

Step 2a: We show (A, E, B)-invariance of $\mathcal{V}$. Since $AV = EVA_1 + BA_3$, this follows from

$$\begin{aligned}
&A[V, WZ] \\
&= \left[EVA_1 + BA_3, S^{-1} \begin{bmatrix} A_2 \\ A_4 \\ A_6 \end{bmatrix} Z \right] \\
&= \left[EVA_1 + BA_3, S^{-1} \begin{bmatrix} A_2 Z \\ E_4 ZX + Y \\ E_6 ZX \end{bmatrix} \right] \\
&= \left[EVA_1 + BA_3, S^{-1} \left(\begin{bmatrix} E_2 \\ E_4 \\ E_6 \end{bmatrix} ZX + \begin{bmatrix} A_2 Z - E_2 ZX \\ 0 \\ 0 \end{bmatrix} + \tilde{B} Y \right) \right] \\
&= [EVA_1 + BA_3, EWZX + EV(A_2 Z - E_2 ZX) + BY] \\
&= E[V, WZ] \begin{bmatrix} A_1 & A_2 Z - E_2 ZX \\ 0 & X \end{bmatrix} + B[A_3, Y].
\end{aligned}$$

Step 2b: We show that $\mathcal{V}$ is included in $\ker C$. This is immediate from

$$C[V, WZ] = [0, C_2 Z] = 0.$$

Now, since $\operatorname{im} V$ is the largest (A, E, B)-invariant subspace included in $\ker C$, if follows that $\operatorname{im}[V, WZ] \subseteq \operatorname{im} V$ and hence, since $\operatorname{im} V \cap \operatorname{im} W = \{0\}$ and W has full column rank, $Z = 0$. This implies (4.1.7).

Step 3: We show the uniqueness property. To this end we first show that

$$\max(SET, SAT, SB; \ker CT) = T^{-1} \max(E, A, B; \ker C).$$

Let $V \in \mathbb{R}^{n \times k}$ with full column rank such that $\operatorname{im} V = \max(E, A, B; \ker C)$. By definition of $\max(E, A, B; \ker C)$ there exist $N \in \mathbb{R}^{k \times k}, M \in \mathbb{R}^{m \times k}$ such that $AV = EVN + BM$ and $CV = 0$. Then

$$(SAT)(T^{-1}V) = SEVN + SBM = (SET)(T^{-1}V)N + (SB)M$$

and $(CT)(T^{-1}V) = CV = 0$, which proves the assertion. This shows in particular that

$$\dim \max(SET, SAT, SB; \ker CT) = \dim \max(E, A, B; \ker C)$$

and hence the block structures of $[E, A, B, C]$ and $[\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ coincide. We now show that

$$\max(E, A, B; \ker C) = \max(SET, SAT, SB; \ker CT) = \operatorname{im} \begin{bmatrix} I_k \\ 0 \end{bmatrix}. \tag{4.1.10}$$

First, we consider $\max(E, A, B; \ker C)$. Since

$$A \begin{bmatrix} I_k \\ 0 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_3 \\ 0 \end{bmatrix} = E \begin{bmatrix} I_k \\ 0 \end{bmatrix} A_1 + BA_3 \quad \wedge \quad C \begin{bmatrix} I_k \\ 0 \end{bmatrix} = 0,$$

we find that $\operatorname{im} \left[\begin{smallmatrix} I_k \\ 0 \end{smallmatrix}\right]$ is (A, E, B)-invariant and included in $\ker C$. In order to show maximality, let $V = \left[\begin{smallmatrix} V_1 \\ V_2 \end{smallmatrix}\right] \in \mathbb{R}^{n \times q}$, $N \in \mathbb{R}^{q \times q}$, $M \in \mathbb{R}^{m \times q}$ be such that

$$AV = EVN + BM \quad \wedge \quad CV = 0.$$

In particular, this implies that

$$\begin{bmatrix} A_4 \\ A_6 \end{bmatrix} V_2 = \begin{bmatrix} E_4 \\ E_6 \end{bmatrix} V_2 N + \begin{bmatrix} I_m \\ 0 \end{bmatrix} [M, A_3 V_1] \quad \wedge \quad C_2 V_2 = 0,$$

and hence (4.1.7) implies that $V_2 = 0$, thus $\operatorname{im} V \subseteq \operatorname{im} \left[\begin{smallmatrix} I_k \\ 0 \end{smallmatrix}\right]$. Since $[SET, SAT, SB, CT]$ has the same block structure as $[E, A, B, C]$, we have proved (4.1.10).
From (4.1.10) we obtain

$$\operatorname{im} \begin{bmatrix} I_k \\ 0 \end{bmatrix} = \max(SET, SAT, SB; \ker CT) = T^{-1} \max(E, A, B; \ker C) = \operatorname{im} T^{-1} \begin{bmatrix} I_k \\ 0 \end{bmatrix},$$

by which T takes the form $T = \left[\begin{smallmatrix} T_1 & T_2 \\ 0 & T_4 \end{smallmatrix}\right]$, $T_1 \in \mathbf{Gl}_k(\mathbb{R})$, $T_4 \in \mathbf{Gl}_{n-k}(\mathbb{R})$. Moreover,

$$\begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} = \hat{B} = SB = S \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, \quad \text{and hence} \quad S = \begin{bmatrix} S_1 & 0 & S_3 \\ S_4 & I_m & S_6 \\ S_7 & 0 & S_9 \end{bmatrix}.$$

Now,

$$\begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix} = \hat{E} \begin{bmatrix} I_k \\ 0 \end{bmatrix} = SET \begin{bmatrix} I_k \\ 0 \end{bmatrix} = \begin{bmatrix} S_1 T_1 \\ S_4 T_1 \\ S_7 T_1 \end{bmatrix},$$

by which $T_1 = S_1^{-1}$, $S_4 = 0$ and $S_7 = 0$. This completes the proof of the theorem. □

Remark 4.1.8 (Zero dynamics form).
The name 'zero dynamics form' for the form (4.1.6) may be justified since the zero dynamics are decoupled in this form. If $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$, then, applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ from Theorem 4.1.7, gives $x = Vz_1 + Wz_2$ and from Proposition 4.1.4 we obtain $x(t) \in \operatorname{im} V$ for almost all $t \in \mathbb{R}$. Then $\operatorname{im} V \cap \operatorname{im} W = \{0\}$ gives $z_2 \overset{\text{a.e.}}{=} 0$ and $w := z_1 + E_2 z_2 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^k)$ (which satisfies $w \overset{\text{a.e.}}{=} z_1$) and u solve

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} w &\overset{\text{a.e.}}{=} A_1 w, \\ 0 &\overset{\text{a.e.}}{=} A_3 w + u. \end{aligned}$$

Therefore, w as the solution of an ODE characterizes the 'dynamics' within the zero dynamics (almost everywhere) and z_2 and u are given by algebraic equations depending on w.

Remark 4.1.9 (How close is the ZDF to a canonical form?).
Recall the definition of a canonical form from Definition 2.2.20. In the present setup, the group is $G = \mathbf{Gl}_l(\mathbb{R}) \times \mathbf{Gl}_n(\mathbb{R})$, the considered set is $\mathcal{S} = \Sigma_{l,n,m,p}$ and the group action $\alpha\big((S, T), [E, A, B, C]\big) = [SET, SAT, SB, CT]$ corresponds to $\overset{S,T}{\sim}$. However, Theorem 4.1.7 does not provide a mapping γ. That means the ZDF is not a unique representative within the equivalence class and hence it is not a canonical form. The entries E_2, A_2, E_4, A_4 are not even unique up to matrix equivalence (recall that two matrices $M, N \in \mathbb{R}^{l \times n}$ are *equivalent* if, and only if, there exist $S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R})$ such that $SMT = N$): it is easy to construct an example such that (4.1.9) is satisfied and in the respective forms we have, e.g., $A_2 = 0$ and $\hat{A}_2 \neq 0$. However, the last statement in Theorem 4.1.7 provides that A_1, A_3, E_6, A_6 and C_2 are unique up to similarity or equivalence, resp.

Next we characterize the condition (4.1.7) in Definition 4.1.6 by trivial zero dynamics and by left invertibility of the system pencil; this becomes important for a further refinement of the ZDF (4.1.6) in Theorem 4.2.7.

Proposition 4.1.10 (Invariant subspace, trivial zero dynamics and system pencil)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$*. Then the following statements are equivalent:*

(i) $\operatorname{rk} B = m$ *and* $\max(E, A, B; \ker C) = \{0\}$,

(ii) $\mathcal{ZD}_{[E,A,B,C]} \subseteq \left\{ w \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m) \;\middle|\; w \overset{a.e.}{=} 0 \right\}$,

(iii) $\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}$ *is left invertible over* $\mathbb{R}[s]$.

Proof: (i)⇒(ii): Let $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$ and observe that by $\max(E, A, B; \ker C) = \{0\}$ and Proposition 4.1.4 we have $x \overset{\text{a.e.}}{=} 0$. Then $\operatorname{rk} B = m$ implies $u \overset{\text{a.e.}}{=} 0$ and (ii) is shown.

(ii)⇒(iii): Since the zero dynamics are trivial (almost everywhere), they are autonomous, and by Proposition 4.1.5 the system pencil has full column rank. Hence, invoking Lemma 4.1.3, in a QKF (4.1.4) of the system pencil it holds $\ell(\beta) = 0$. Furthermore, we obtain $n_s = 0$, since otherwise we could find nontrivial solutions of the ODE $\dot z = A_s z$ which would lead to nontrivial trajectories within the zero dynamics. Now,

$$(sN_\alpha - I_{|\alpha|})^{-1} = -I_{|\alpha|} - sN_\alpha - \ldots - s^{\nu-1} N_\alpha^{\nu-1},$$

where ν is the index of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$. Furthermore, by a permutation of the rows of $sK_\gamma^\top - L_\gamma^\top$ we may achieve that there exists $S \in \mathbf{Gl}_{|\gamma|}(\mathbb{R})$ such that

$$S(sK_\gamma^\top - L_\gamma^\top) = \begin{bmatrix} s\tilde N - I_{|\gamma|-\ell(\gamma)} \\ s\tilde K - \tilde L \end{bmatrix},$$

where $\tilde N \in \mathbb{R}^{(|\gamma|-\ell(\gamma))\times(|\gamma|-\ell(\gamma))}$ is nilpotent and $\tilde K, \tilde L$ are matrices of appropriate sizes. Then $sK_\gamma^\top - L_\gamma^\top$ has left inverse $[(s\tilde N - I_{|\gamma|-\ell(\gamma)})^{-1}, 0]S$ over $\mathbb{R}[s]$.

(iii)⇒(i): Clearly, (iii) implies $\operatorname{rk} B = m$. In order to show $\max(E, A, B; \ker C) = \{0\}$ we prove that for all $k \in \mathbb{N}$, $V \in \mathbb{R}^{n\times k}$, $N \in \mathbb{R}^{k\times k}$ and $M \in \mathbb{R}^{m\times k}$ the implication

$$\big(AV = EVN + BM \quad\wedge\quad CV = 0\big) \quad\Longrightarrow\quad V = 0$$

holds. If the left hand side holds true, then we have

$$\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} \begin{bmatrix} V \\ -M \end{bmatrix} = \begin{bmatrix} EV \\ 0 \end{bmatrix} (sI_k - N) \in \mathbb{R}[s]^{(l+p)\times k}.$$

By existence of a left inverse $L(s) \in \mathbb{R}[s]^{(n+m)\times(l+p)}$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ we find that

$$\begin{bmatrix} V \\ -M \end{bmatrix} = L(s) \begin{bmatrix} EV \\ 0 \end{bmatrix} (sI_k - N) = \begin{bmatrix} L_1(s)EV(sI_k - N) \\ L_2(s)EV(sI_k - N) \end{bmatrix}$$

with $L_1(s) = \sum_{i=0}^{q} s^i L_1^i \in \mathbb{R}[s]^{n\times l}$ and $L_2(s) \in \mathbb{R}[s]^{m\times l}$. Comparison of coefficients of the first equation gives

$$V = -L_1^0 EVN,\ L_1^0 EV = L_1^1 EVN,\ L_1^1 EV = L_1^2 EVN, \\ \ldots,\ L_1^{q-1} EV = L_1^q EVN,\ L_1^q EV = 0,$$

and backward solution yields $V = 0$, which concludes the proof of the proposition. □

Remark 4.1.11 (Zero dynamics and system pencil/Kronecker form)**.** We stress the difference in the characterization of autonomous and trivial zero dynamics in terms of the system pencil as they arise from Propositions 4.1.5 and 4.1.10: The zero dynamics are autonomous if, and only if, the system pencil has full column rank over $\mathbb{R}[s]$; they are trivial if, and only if, the system pencil is left invertible over $\mathbb{R}[s]$.

Using the quasi-Kronecker form, it follows that the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are

(i) autonomous if, and only if, in a QKF (4.1.4) of the system pencil no underdetermined blocks are present, i.e., $\ell(\beta) = 0$. The dynamics within the zero dynamics are then characterized by the ODE $\dot{z} = A_s z$.

(ii) trivial if, and only if, in a QKF (4.1.4) of the system pencil no underdetermined blocks and no ODE blocks are present, i.e., $\ell(\beta) = 0$ and $n_s = 0$. The remaining nilpotent and overdetermined blocks then have trivial solutions only.

The ZDF also allows to derive a vector space isomorphism between the largest (A, E, B)-invariant subspace included in $\ker C$ and the zero dynamics provided that the lower right corner in the ZDF has unique solutions.

Corollary 4.1.12 (Vector space isomorphism).
Suppose that $[E,A,B,C] \in \Sigma_{l,n,m,p}$ *satisfies the following:*

(i) *The zero dynamics* $\mathcal{ZD}_{[E,A,B,C]}$ *are autonomous.*

(ii) *Using the notation from Theorem 4.1.7 and the form* (4.1.6) *it holds that*
$$\mathrm{rk}_{\mathbb{R}[s]}\left(s\begin{bmatrix}E_4\\E_6\end{bmatrix}-\begin{bmatrix}A_4\\A_6\end{bmatrix}\right)=n-k.$$

Then the linear mapping, described in terms of Theorem 4.1.7,

$$\begin{aligned}\varphi\colon\ &\max(E,A,B;\ker C)\\ &\to \mathcal{ZD}_{[E,A,B,C]}\cap\left(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n)\times\mathcal{C}(\mathbb{R};\mathbb{R}^m)\times\mathcal{C}(\mathbb{R};\mathbb{R}^p)\right),\\ x^0&\mapsto(x,\ Fx,\ Cx),\\ &\textit{where } F:=[-A_3,0]\,[V,W]^{-1} \textit{ and } x \textit{ solves}\\ &E\dot{x}=(A+BF)\,x,\ x(0)=x^0,\end{aligned}$$

is a vector space isomorphism.

Proof: *Step 1*: We show that φ is well-defined, that means to show that for arbitrary $x^0\in\max(E,A,B;\ker C)$, the (continuously differentiable) solution of

$$E\dot{x}=(A+BF)\,x,\qquad x(0)=x^0 \tag{4.1.11}$$

is unique and global and satisfies

$$(x,u,y):=(x,Fx,Cx)\in\mathcal{ZD}_{[E,A,B,C]}. \tag{4.1.12}$$

Applying the coordinate transformation $(z_1^\top,z_2^\top)^\top=[V,W]^{-1}x$ from Theorem 4.1.7 and invoking

$$BF=S^{-1}\begin{bmatrix}0\\I_m\\0\end{bmatrix}[-A_3,0][V,W]^{-1}=S^{-1}\begin{bmatrix}0&0\\-A_3&0\\0&0\end{bmatrix}[V,W]^{-1},$$

we find

$$\begin{aligned}\dot{z}_1+E_2\dot{z}_2&=A_1z_1+A_2z_2,\\ E_4\dot{z}_2&=A_4z_2,\\ E_6\dot{z}_2&=A_6z_2,\end{aligned}$$

and the initial value satisfies

$$Vz_1(0) + Wz_2(0) = x(0) \in \operatorname{im} V.$$

Therefore, $Wz_2(0) = x(0) - Vz_1(0) \in \operatorname{im} W \cap \operatorname{im} V = \{0\}$, by which $Wz_2(0) = 0$ and hence, invoking the full column rank of W, $z_2(0) = 0$. Now, by (ii), Proposition 3.2.3, Lemma 4.1.3 and a straightforward calculation of the solution of the system in QKF, we find that $\left[\begin{smallmatrix} E_4 \\ E_6 \end{smallmatrix}\right] \dot{y} = \left[\begin{smallmatrix} A_4 \\ A_6 \end{smallmatrix}\right] y$ satisfies uniqueness, i.e., any local solution $y \in \mathcal{C}^1(I; \mathbb{R}^{n-k})$, $I \subseteq \mathbb{R}$ an interval, can be extended to a unique global solution on all of $\mathbb{R}$. This yields $z_2 = 0$. Therefore, $x = Vz_1$ and z_1 satisfies $\dot{z}_1 = A_1 z_1$, $z_1(0) = [I_k, 0][V, W]^{-1} x(0)$, which is a unique and global solution. Finally, $x(t) = Vz_1(t) \in \operatorname{im} V \subseteq \ker C$ for all $t \in \mathbb{R}$ and hence $y = Cx = 0$.

Step 2: We show that φ is injective. Let $x^1, x^2 \in \max(E, A, B; \ker C)(0)$ so that $\varphi(x^1)(\cdot) = \varphi(x^2)(\cdot)$. Then

$$(x^1, *, *) = \varphi(x^1)(\cdot)\big|_{t=0} = \varphi(x^2)(\cdot)\big|_{t=0} = (x^2, *, *).$$

Step 3: We show that φ is surjective. Let

$$(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]} \cap \left(\mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^p)\right).$$

Then Proposition 4.1.4 yields that $x(t) \in \max(E, A, B; \ker C)$ for all $t \in \mathbb{R}$. Hence, applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1} x$ from Theorem 4.1.7 to (4.1.1) gives $Vz_1(t) + Wz_2(t) = x(t) \in \operatorname{im} V$ for all $t \in \mathbb{R}$ and, similar to Step 1, we may conclude $z_2 = 0$. Therefore,

$$\begin{array}{rcl} \dot{z}_1 & = & A_1 z_1, \\ 0 & = & A_3 z_1 + u. \end{array} \tag{4.1.13}$$

The second equation in (4.1.13) now gives

$$u = -A_3 z_1 = -A_3 [I, 0][V, W]^{-1} x = Fx.$$

Finally, a simple calculation shows that $x = Vz_1$ satisfies $E\dot{x} = (A + BF)x$ and, clearly, $x(0) = Vz_1(0) \in \max(E, A, B; \ker C)$. □

In the remainder of this section we derive a representation of the zero dynamics in terms of the ZDF, show that the behavior can be decomposed into a direct sum of the zero dynamics and some summand, and prove that the zero dynamics are a dynamical system.

Remark 4.1.13 (Representation of zero dynamics).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ have autonomous zero dynamics and use the notation from Theorem 4.1.7 and the form (4.1.6). Invoking the considerations in Remark 4.1.8, the zero dynamics may be written as

$$\begin{aligned}\mathcal{ZD}_{[E,A,B,C]} = \big\{(x,u,y) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \mid x = Vz_1 + Wz_2,\\ z_1 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^k),\ z_2 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{n-k}),\\ z_1 + E_2 z_2 = e^{A_1 \cdot} z_1^0,\ z_1^0 \in \mathbb{R}^k,\ z_2 \overset{\text{a.e.}}{=} 0,\\ u \overset{\text{a.e.}}{=} -A_3 z_1,\ y \overset{\text{a.e.}}{=} 0\big\}.\end{aligned}$$

Remark 4.1.14 (Zero dynamics and behavior).
It may be interesting to see that for any $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics the (continuous part of the) behavior $\mathfrak{B}_{[E,A,B,C]}$ can be decomposed, in terms of the transformation matrix $[V, W]$ from Theorem 4.1.7, into a direct sum of the zero dynamics and summand as

$$\begin{aligned}&\mathfrak{B}_{[E,A,B,C]} \cap \big(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}(\mathbb{R};\mathbb{R}^p)\big)\\ &= \mathcal{ZD}_{[E,A,B,C]} \cap \big(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}(\mathbb{R};\mathbb{R}^p)\big)\\ &\oplus \big\{(x,u,y) \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}(\mathbb{R};\mathbb{R}^p) \mid (x,u,y)\\ &\qquad \text{solves (4.1.1) and } [I_k, 0][V, W]^{-1} x(0) = 0\big\}\end{aligned}$$

This decomposition is immediate from Remark 4.1.13.

Remark 4.1.15 (Zero dynamics are a dynamical system).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ have autonomous zero dynamics and use the notation from Theorem 4.1.7 and the form (4.1.6). Define

$$X := \left\{ \begin{pmatrix} x^0 \\ u^0 \end{pmatrix} \in \mathbb{R}^{n+m} \;\middle|\; \begin{array}{l} x^0 \in \max(E, A, B; \ker C) \\ \wedge\ u^0 = A_3 V [I_k, 0][V, W]^{-1} x^0 \end{array} \right\}$$

and

$$\mathcal{D}_\varphi := \mathbb{R} \times \mathbb{R} \times X \times \{0\}.$$

Then, let the *state transition map* be defined as

$$\varphi : \mathcal{D}_\varphi \to X,\ \left(t, t_0, \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}, u(\cdot)\right) \mapsto \begin{pmatrix} V e^{A_1 (t-t_0)} [I_k, 0][V, W]^{-1} x^0 \\ A_3 V e^{A_1 (t-t_0)} [I_k, 0][V, W]^{-1} x^0 \end{pmatrix},$$

and the *output map* as

$$\eta : \mathbb{R} \times X \times \{0\} \to \mathbb{R}^p, \quad \left(t, \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}, u(\cdot)\right) \mapsto 0.$$

Invoking that $x^0 \in \max(E, A, B; \ker C) = \operatorname{im} V$ if, and only if, $x^0 = V z_1^0$ for some $z_1^0 \in \mathbb{R}^k$, it is readily verified that the structure $(\mathbb{R}, \mathbb{R}^m, \{0\}, \mathbb{R}^n, \mathbb{R}^p, \varphi, \eta)$ is an $\mathbb{R}$-linear time-invariant dynamical system as defined in [115, Defs. 2.1.1, 2.1.24, 2.1.26].

By Remark 4.1.13 we have that

$$(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]} \cap \left(\mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^p)\right)$$

if, and only if,

$$\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \varphi\left(t, t_0, \begin{pmatrix} x(t_0) \\ u(t_0) \end{pmatrix}, 0\right) \ \wedge \ y(t) = \eta\left(t, \begin{pmatrix} x(t_0) \\ u(t_0) \end{pmatrix}, 0\right),$$

and hence the map

$$\begin{aligned} \Psi \colon \mathcal{D}_{\varphi,0} \to \ & \mathcal{ZD}_{[E,A,B,C]} \cap \left(\mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^p)\right), \\ & \left(0, 0, \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}, u(\cdot)\right) \\ & \mapsto \left([I_n, 0]\varphi\left(\cdot, 0, \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}, u(\cdot)\right), [0, I_m]\varphi\left(\cdot, 0, \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}, u(\cdot)\right), \right. \\ & \qquad \left. \eta\left(\cdot, \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}, u(\cdot)\right)\right), \end{aligned}$$

where

$$\mathcal{D}_{\varphi,0} := \left\{ \left(0, 0, \begin{pmatrix} x^0 \\ u^0 \end{pmatrix}, u(\cdot)\right) \in \mathcal{D}_\varphi \right\} \subseteq \mathcal{D}_\varphi,$$

is a vector space isomorphism. In this sense, we may say that

$$\mathcal{ZD}_{[E,A,B,C]} \cap \left(\mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}(\mathbb{R}; \mathbb{R}^p)\right)$$

is a dynamical system.

4.2 System inversion

In this section we investigate the properties of left-invertibility, right-invertibility, and invertibility of DAE systems and relate them to the zero dynamics. In order to treat these problems we derive a refinement of the zero dynamics form from Definition 4.1.6.

In the following we give the definition of left- and right-invertibility of a system, which are from [230, Sec. 8.2] - generalized to the DAE case. A detailed survey of left- and right-invertibility of ODE systems can also be found in [205].

Definition 4.2.1 (System invertibility).
$[E, A, B, C] \in \Sigma_{l,n,m,p}$ is called

(i) *left-invertible* if, and only if,

$$\forall\,(x_1, u_1, y_1), (x_2, u_2, y_2) \in \mathfrak{B}_{[E,A,B,C]} : \\ \big[y_1 \overset{\text{a.e.}}{=} y_2 \;\wedge\; Ex_1(0) = Ex_2(0) = 0\big] \implies u_1 \overset{\text{a.e.}}{=} u_2.$$

(ii) *right-invertible* if, and only if,

$$\forall\, y \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^p)\; \exists\,(x, u) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \times \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m) : \\ (x, u, y) \in \mathfrak{B}_{[E,A,B,C]}.$$

(iii) *invertible* if, and only if, $[E, A, B, C]$ is left-invertible and right-invertible.

Remark 4.2.2 (Left-invertibility).
By linearity of the behavior $\mathfrak{B}_{[E,A,B,C]}$, left-invertibility of $[E, A, B, C] \in \Sigma_{l,n,m,p}$ is equivalent to

$$\forall\,(x, u, y) \in \mathfrak{B}_{[E,A,B,C]} : \;\big[y \overset{\text{a.e.}}{=} 0 \;\wedge\; Ex(0) = 0\big] \implies u \overset{\text{a.e.}}{=} 0. \quad (4.2.1)$$

Remark 4.2.3 (Inverse system).
For ODE systems, the problem of finding a realization of the inverse of a system $[I, A, B, C] \in \Sigma_{n,n,m,m}$ is usually described as the problem of finding a realization for the inverse of its transfer function $C(sI-A)^{-1}B$, provided it exists, see e.g. [136, p. 557]. This means that in the corresponding behaviors inputs and outputs are interchanged. In the differential-algebraic setting we may generalize this in the following way: a system $[\hat{E}, \hat{A}, \hat{B}, \hat{C}] \in \Sigma_{\hat{l},\hat{n},p,m}$ is called the *inverse* of

$[E,A,B,C] \in \Sigma_{l,n,m,p}$ if, and only if,

$$\begin{aligned}\forall\,(u,y) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m) \times \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^p) :&\\ &\left[\exists\, x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) :\ (x,u,y) \in \mathfrak{B}_{[E,A,B,C]}\right]\\ \iff\ &\left[\exists\, \hat{x} \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{\hat{n}}) :\ (\hat{x},y,u) \in \mathfrak{B}_{[\hat{E},\hat{A},\hat{B},\hat{C}]}\right]. \quad (4.2.2)\end{aligned}$$

In fact, in the differential-algebraic framework condition (4.2.2) is so weak that it is possible to show that any system $[E,A,B,C] \in \Sigma_{l,n,m,p}$ has an inverse - thus, the existence of an inverse is in no relation to the notion of invertibility of the system.

Let $[E,A,B,C] \in \Sigma_{l,n,m,p}$ with $\mathrm{rk}\, B = q \le m$. Then there exist $S_1 \in \mathbb{R}^{q \times l}, S_2 \in \mathbb{R}^{(l-q)\times l}$ and $T \in \mathbf{Gl}_m(\mathbb{R})$ such that $S_1 B T = [I_q, 0]$ and $S_2 B T = 0$. Let $(x,u,y) \in \mathfrak{B}_{[E,A,B,C]}$ with $\tilde{u} := T^{-1}u = (\tilde{u}_1^\top, \tilde{u}_2^\top)^\top$, where $\tilde{u}_1 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^q)$, $\tilde{u}_2 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{m-q})$. Then

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} S_1 E x &\overset{\text{a.e.}}{=} S_1 A x + \tilde{u}_1\\ \tfrac{\mathrm{d}}{\mathrm{d}t} S_2 E x &\overset{\text{a.e.}}{=} S_2 A x\\ y &\overset{\text{a.e.}}{=} C x.\end{aligned}$$

Now, $\tilde{u}_1$ depends on the derivative of $S_1 E x$, so we introduce the new variable $w \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^q)$ such that $w \overset{\text{a.e.}}{=} \tfrac{\mathrm{d}}{\mathrm{d}t} S_1 E x$; and $\tilde{u}_2$ is the vector of free inputs (which are free outputs in the inverse system), so we introduce the new variable $z := \tilde{u}_2$, which will not be restricted in the inverse system. Clearly, adding these equations to the original system does not change it. Switching the roles of inputs and outputs and using the new augmented state $(x^\top, w^\top, z^\top)^\top \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n+q+(m-q)})$ we may rewrite the system as follows:

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t} S_1 E x &\overset{\text{a.e.}}{=} w\\ \tfrac{\mathrm{d}}{\mathrm{d}t} S_2 E x &\overset{\text{a.e.}}{=} S_2 A x\\ 0 &\overset{\text{a.e.}}{=} -Cx + y\\ \tilde{u}_1 &\overset{\text{a.e.}}{=} -S_1 A x + w\\ \tilde{u}_2 &\overset{\text{a.e.}}{=} z.\end{aligned}$$

Therefore, an inverse of $[E,A,B,C]$ is

$$\left[\begin{bmatrix} S_1E & 0 & 0\\ S_2E & 0 & 0\\ 0 & 0 & 0\end{bmatrix}, \begin{bmatrix} 0 & I_q & 0\\ S_2A & 0 & 0\\ -C & 0 & 0\end{bmatrix}, \begin{bmatrix} 0\\ 0\\ I_p\end{bmatrix}, T\begin{bmatrix} -S_1A & I_q & 0\\ 0 & 0 & I_{m-q}\end{bmatrix}\right] \in \Sigma_{l+p,n+m,p,m}.$$

Note also that for $(x^\top, w^\top, z^\top)^\top \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n+q+(m-q)})$ we have

$$Ex \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \iff \begin{bmatrix} S_1E & 0 & 0 \\ S_2E & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ w \\ z \end{pmatrix} \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n+m}).$$

Next, we will show that a DAE system with autonomous zero dynamics is left-invertible. However, the converse does, in general, not hold true as the following example illustrates.

Example 4.2.4.
Consider the system (4.1.1) with

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},\ B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},\ C = [0, 0, 1].$$

Let $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$ and $x = (x_1, x_2, x_3)^\top$. Then $y \overset{\text{a.e.}}{=} 0$ and hence $x_3 = 0$ and $u \overset{\text{a.e.}}{=} 0$, but $x_1 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R})$ is free and $x_2 \overset{\text{a.e.}}{=} \frac{\mathrm{d}}{\mathrm{d}t}x_1$. Therefore, the zero dynamics are not autonomous. However, $[E, A, B, C]$ is left-invertible since (4.2.1) is satisfied.

Lemma 4.2.5 (Autonomous zero dynamics imply left-invertibility)**.**
If $[E, A, B, C] \in \Sigma_{l,n,m,p}$ has autonomous zero dynamics, then it is left-invertible.

Proof: We show that (4.2.1) is satisfied. To this end let $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]}$ with $y \overset{\text{a.e.}}{=} 0$ and $Ex(0) = 0$. Hence, $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$ and applying the coordinate transformation $(z_1^\top, z_2^\top)^\top = [V, W]^{-1}x$ from Theorem 4.1.7 yields $Vz_1(t) + Wz_2(t) = x(t) \overset{\text{Prop. 4.1.4}}{\in} \operatorname{im} V$ for almost all $t \in \mathbb{R}$. Therefore, $z_2 \overset{\text{a.e.}}{=} 0$ and we have that $w := z_1 + E_2z_2 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$ satisfies $w \overset{\text{a.e.}}{=} z_1$ and $\frac{\mathrm{d}}{\mathrm{d}t}w \overset{\text{a.e.}}{=} A_1w$. Since $Ex(0) = 0$ we obtain from (4.1.6) that $w(0) = 0$, and hence it follows that $w = 0$ and thus $z_1 \overset{\text{a.e.}}{=} 0$ and $u \overset{\text{a.e.}}{=} -A_3z_1 \overset{\text{a.e.}}{=} 0$. □

In the following we investigate right-invertibility for systems with autonomous zero dynamics. In order for $[E, A, B, C] \in \Sigma_{l,n,m,p}$ to be right invertible it is necessary that C has full row rank (i.e., $\operatorname{im} C = \mathbb{R}^p$). This additional assumption leads to the following system inversion form for $[E, A, B, C]$ specializing the zero dynamics form (4.1.6); the derivation of the system inversion form is the main result of this section and proved in Theorem 4.2.7.

Definition 4.2.6 (System inversion form)**.**
A system $[E, A, B, C] \in \Sigma_{l,n,m,p}$ is said to be in *system inversion form* if, and only if,

$$E = \begin{bmatrix} I_k & 0 & 0 \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & N \\ 0 & E_{42} & E_{43} \end{bmatrix}, \; A = \begin{bmatrix} Q & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & A_{42} & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, \; C = [0, I_p, 0], \tag{4.2.3}$$

such that $k = \dim \max(E, A, B; \ker C)$ and $N \in \mathbb{R}^{n_3 \times n_3}$, $n_3 = n-k-p$, is nilpotent with $N^\nu = 0$ and $N^{\nu-1} \neq 0$, $\nu \in \mathbb{N}$, $E_{22}, A_{22} \in \mathbb{R}^{m \times p}$ and all other matrices are of appropriate sizes.

Theorem 4.2.7 (System inversion form)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *have autonomous zero dynamics and* $\operatorname{rk} C = p$*. Then there exist* $S \in \mathbf{Gl}_l(\mathbb{R})$ *and* $T \in \mathbf{Gl}_n(\mathbb{R})$ *such that*

$$[E, A, B, C] \overset{S,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}], \tag{4.2.4}$$

where $[\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ *is in system inversion form.*

Proof: By Theorem 4.1.7 system $[E, A, B, C]$ is equivalent to $[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}]$ in zero dynamics form (4.1.6). Since C and therefore C_2 has full row rank, there exists $\tilde{T} \in \mathbf{Gl}_{n-k}(\mathbb{R})$ such that $C_2\tilde{T} = [I_p, 0]$. Hence,

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}] \overset{I, \left[\begin{smallmatrix} I & 0 \\ 0 & \tilde{T} \end{smallmatrix}\right]}{\sim} \left[\begin{bmatrix} I_k & \tilde{E}_{12} & \tilde{E}_{13} \\ 0 & \tilde{E}_{22} & \tilde{E}_{23} \\ 0 & \tilde{E}_{32} & \tilde{E}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix}, [0, I_p, 0] \right].$$

Now, since

$$\max\left(\begin{bmatrix} \tilde{E}_{22} & \tilde{E}_{23} \\ \tilde{E}_{32} & \tilde{E}_{33} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{22} & \tilde{A}_{23} \\ \tilde{A}_{32} & \tilde{A}_{33} \end{bmatrix}, \begin{bmatrix} I_m \\ 0 \end{bmatrix}; \ker [I_p, 0] \right) = \{0\}$$

by Theorem 4.1.7 and (4.1.7), we may infer from Proposition 4.1.10

that there exists $X(s) \in \mathbb{R}[s]^{(n+m-k)\times(l+p-k)}$ such that

$$\begin{bmatrix} X_{11}(s) & X_{12}(s) & X_{13}(s) \\ X_{21}(s) & X_{22}(s) & X_{23}(s) \\ X_{31}(s) & X_{32}(s) & X_{33}(s) \end{bmatrix} \begin{bmatrix} s\tilde{E}_{22} - \tilde{A}_{22} & s\tilde{E}_{23} - \tilde{A}_{23} & I_m \\ s\tilde{E}_{32} - \tilde{A}_{32} & s\tilde{E}_{33} - \tilde{A}_{33} & 0 \\ I_p & 0 & 0 \end{bmatrix} = \begin{bmatrix} I_p & 0 & 0 \\ 0 & I_{n-k-p} & 0 \\ 0 & 0 & I_m \end{bmatrix}.$$

Obviously, $X_{21}(s) = 0$ and hence $X_{22}(s)(s\tilde{E}_{33} - \tilde{A}_{33}) = I_{n-k-p}$, i.e., $s\tilde{E}_{33} - \tilde{A}_{33}$ is left invertible over $\mathbb{R}[s]$. This implies that in a QKF (3.2.3) of $s\tilde{E}_{33} - \tilde{A}_{33}$ it holds $n_s = 0$ and $\ell(\beta) = 0$. By a permutation of the rows in the block $sK_\gamma^\top - L_\gamma^\top$ we may achieve that there exists $\hat{S} \in \mathbf{Gl}_{l-k-m}(\mathbb{R})$, $\hat{T} \in \mathbf{Gl}_{n-k-p}(\mathbb{R})$ such that $\hat{S}(s\tilde{E}_{33} - \tilde{A}_{33})\hat{T} = \begin{bmatrix} sN - I_{n_3} \\ s\hat{E}_{43} - \hat{A}_{43} \end{bmatrix}$, where N is nilpotent. Hence,

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}] \overset{\left[\begin{smallmatrix} I & 0 \\ 0 & \left[\begin{smallmatrix} I & 0 \\ 0 & \hat{S} \end{smallmatrix}\right] \end{smallmatrix}\right], \left[\begin{smallmatrix} I & 0 \\ 0 & \tilde{T}\cdot\left[\begin{smallmatrix} I & 0 \\ 0 & \hat{T} \end{smallmatrix}\right] \end{smallmatrix}\right]}{\sim} \left[\begin{bmatrix} I_k & \tilde{E}_{12} & \tilde{E}_{13} \\ 0 & \tilde{E}_{22} & \tilde{E}_{23} \\ 0 & \hat{E}_{32} & N \\ 0 & \hat{E}_{42} & \hat{E}_{43} \end{bmatrix}, \begin{bmatrix} \tilde{A}_{11} & \tilde{A}_{12} & \tilde{A}_{13} \\ \tilde{A}_{21} & \tilde{A}_{22} & \tilde{A}_{23} \\ 0 & \hat{A}_{32} & I_{n_3} \\ 0 & \hat{A}_{42} & \hat{A}_{43} \end{bmatrix}, \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, [0, I_p, 0] \right].$$

Applying additional elementary row and column operations we obtain that

$$[E, A, B, C] \overset{\overline{S}, \overline{T}}{\sim} [\overline{E}, \overline{A}, \overline{B}, \overline{C}]$$

for some $\overline{S} \in \mathbf{Gl}_l(\mathbb{R})$ and $\overline{T} \in \mathbf{Gl}_n(\mathbb{R})$, where

$$\overline{E} = \begin{bmatrix} I_k & 0 & E_{13} \\ 0 & E_{22} & E_{23} \\ 0 & E_{32} & N \\ 0 & E_{42} & E_{43} \end{bmatrix}, \overline{A} = \begin{bmatrix} Q & A_{12} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & I_{n_3} \\ 0 & A_{42} & 0 \end{bmatrix}, \overline{B} = \begin{bmatrix} 0 \\ I_m \\ 0 \\ 0 \end{bmatrix}, \overline{C} = [0, I_p, 0].$$

It only remains to show that by an additional transformation we can obtain that $E_{13} = 0$. To this end consider

$$\check{S} := \begin{bmatrix} I & 0 & QL & 0 \\ 0 & I & A_{21}L & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix}, \quad \check{T} := \begin{bmatrix} I & -QLE_{32} & -L \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad L := \sum_{i=0}^{\nu-1} Q^i E_{13} N^i,$$

and observe that $\check{S}\overline{B} = \overline{B} = \hat{B}$, $\overline{C}\check{T} = \overline{C} = \hat{C}$ and $\check{S}\overline{E}\check{T}$, $\check{S}\overline{A}\check{T}$ have the same block structure as $\hat{E}, \hat{A}$ and N is nilpotent. □

Remark 4.2.8 (Uniqueness)**.**
Uniqueness of the entries in the form (4.2.3) may be analyzed similar to the last statement in Theorem 4.1.7. It is easy to see that Q is unique up to similarity, and that there are entries which are not even unique up to matrix equivalence (cf. Remark 4.1.9). In particular, the form (4.2.3) is not a canonical form.

Remark 4.2.9 (DAE of system inversion form and inverse system)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ have autonomous zero dynamics and $\operatorname{rk} C = p$. The behavior of the DAE (4.1.1) may be interpreted, in terms of the system inversion form (4.2.4), (4.2.3) in Theorem 4.2.7, as follows: $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \cap \big(\mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \times \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^m) \times \mathcal{W}^{\nu+1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^p)\big)$ if, and only if, (Tx, u, y) solves

$$\boxed{\begin{array}{rcl} \dot{x}_1 &=& Qx_1 + A_{12}y \\ 0 &=& -E_{22}\dot{y} - \sum_{k=0}^{\nu-1} E_{23}N^kE_{32}y^{(k+2)} + A_{21}x_1 + A_{22}y + u \\ x_3 &=& \sum_{k=0}^{\nu-1} N^kE_{32}y^{(k+1)} \\ 0 &=& -E_{42}\dot{y} - \sum_{k=0}^{\nu-1} E_{43}N^kE_{32}y^{(k+2)} + A_{42}y, \end{array}} \tag{4.2.5}$$

where $Tx = (x_1^\top, y^\top, x_3^\top)^\top \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^{k+p+n_3})$; see also Figure 4.1.

From the form (4.2.3), also the inverse system can be read off immediately. Introducing the new variables $x_2 = y$ and $x_4 = \frac{\mathrm{d}}{\mathrm{d}t}\big(E_{22}x_2 + E_{23}x_3\big)$, an inverse system, with state $(x_1^\top, x_2^\top, x_3^\top, x_4^\top)^\top$, is given by

$$\left[\begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & E_{22} & E_{23} & 0 \\ 0 & E_{32} & N & 0 \\ 0 & E_{42} & E_{43} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} Q & A_{12} & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & I_{n_3} & 0 \\ 0 & A_{42} & 0 & 0 \\ 0 & -I_p & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ I_p \end{bmatrix}, \begin{bmatrix} -A_{21}^\top \\ -A_{22}^\top \\ 0 \\ I_m \end{bmatrix}^\top\right] \in \Sigma_{l+p,n+m,p,m}.$$

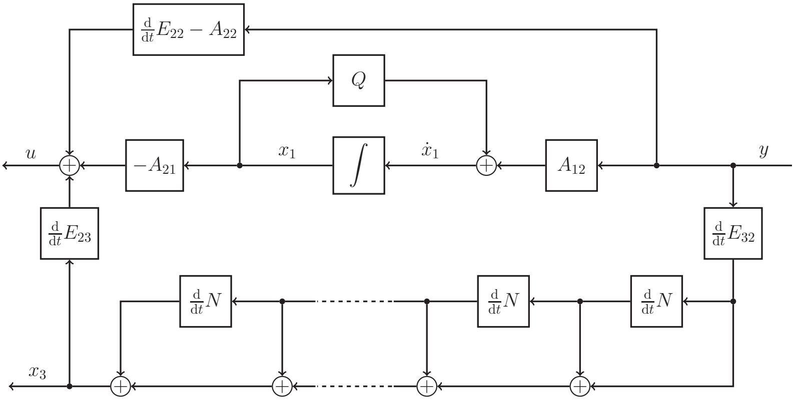


Figure 4.1: *System* $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *in form* (4.2.3)

Remark 4.2.10 (Index of nilpotency)**.**
The index of nilpotency ν of the matrix N in the system inversion form (4.2.3) from Definition 4.2.6 may be larger than the index of the pencil $sE - A$: Consider

$$sE - A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & s \\ s & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad C = [1, 0, 0].$$

It is easy to see, that $[E, A, B, C]$ is in the form (4.2.3) with $k = 0$, $n_3 = 2$, $E_{32} = \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$, $N = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ and all other entries in $\hat{E}$ in (4.2.3) are zero or not present. Hence $\nu = 2$, but the index of $sE - A$ is 1, since there exist $S, T \in \mathbf{Gl}_3(\mathbb{R})$ such that

$$S(sE - A)T = \begin{bmatrix} sK_1^\top - L_1^\top & 0 \\ 0 & sK_3 - L_3 \end{bmatrix},$$

i.e., we have an overdetermined block of size 1×0 and an underdetermined block of size 2×3 (cf. the QKF 3.2.3).

The next corollary follows directly from Theorem 4.2.7 and the form (4.2.3).

Corollary 4.2.11 (Asymptotically stable zero dynamics)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *have autonomous zero dynamics and* $\operatorname{rk} C =$

p. Then, using the notation from Theorem 4.2.7 and the form (4.2.3)*, the zero dynamics* $\mathcal{ZD}_{[E,A,B,C]}$ *are asymptotically stable if, and only if,* $\sigma(Q) \subseteq \mathbb{C}_-$*.*

As discussed in Remark 4.2.9, a realization of the inverse system can be found for $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and $\operatorname{rk} C = p$. However, due to the last equation in (4.2.5), $[E, A, B, C]$ is in general not right-invertible. Necessary and sufficient conditions for the latter are derived next.

Proposition 4.2.12 (System invertibility)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *have autonomous zero dynamics. Then, in terms of the form* (4.2.3) *from Theorem 4.2.7,*

$$[E, A, B, C] \text{ is invertible} \iff \begin{cases} \operatorname{rk} C = p, E_{42} = 0, A_{42} = 0 \ \text{ and} \\ E_{43} N^j E_{32} = 0 \text{ for } j = 0, \ldots, \nu - 1. \end{cases}$$

Proof: By Lemma 4.2.5, $[E, A, B, C]$ is left-invertible, so it remains to show the equivalence for right-invertibility.

$\Rightarrow$: It is is clear that $\operatorname{rk} C = p$, otherwise we might choose any constant $y \equiv y^0$ with $y^0 \notin \operatorname{im} C$, which cannot be attained by the output of the system. Now, by Theorem 4.2.7 we may assume, without loss of generality, that the system is in the form (4.2.3). Assume that $A_{42} \neq 0$. Hence, there exists $y^0 \in \mathbb{R}^p$ such that $A_{42} y^0 \neq 0$. Then, for $y \equiv y^0$ and all $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$, $u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m)$, it holds that $(x, u, y) \notin \mathfrak{B}_{[E,A,B,C]}$ (since the last equation in (4.2.5) is not satisfied), which contradicts right-invertibility. Therefore, we have $A_{42} = 0$. Repeating the argument for E_{42} and $E_{43} N^j E_{32}$ with $y(t) = t y^0$ and $y(t) = t^{j+2} y^0$, resp., yields that $E_{42} = 0$ and $E_{43} N^j E_{32} = 0$, $j = 0, \ldots, \nu - 1$.

$\Leftarrow$: This is immediate from (4.2.5) since $y \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^p)$. □

Remark 4.2.13.
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ have autonomous zero dynamics. If $l = n$, $p = m$ and $\operatorname{rk} C = m$, then $[E, A, B, C]$ is invertible. This can be seen using the form (4.2.3) from Theorem 4.2.7.

4.3 Asymptotically stable zero dynamics

In this section we give a characterization of asymptotically stable zero dynamics and introduce the concepts of transmission zeros and detectability in the behavioral sense. These are exploited to prove the main result of this section, that is that for the class of right-invertible systems with autonomous zero dynamics, the asymptotic stability of the zero dynamics is equivalent to the three conditions: stabilizability in the behavioral sense, detectability in the behavioral sense, and the condition that all transmission zeros of the system are in the open left complex half-plane.

4.3.1 Transmission zeros

In this subsection we introduce the concept of transmission zeros for systems $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics and give a characterization for them.

In order to define transmission zeros, we need the concepts of poles and zeros of rational matrix functions. To this end, we introduce the Smith-McMillan form.

Definition 4.3.1 (Smith-McMillan form [136, Sec. 6.5.2])**.**
Let $G(s) \in \mathbb{R}(s)^{m \times p}$ with $\operatorname{rk}_{\mathbb{R}(s)} G(s) = r$. Then there exist unimodular $U(s) \in \mathbf{Gl}_m(\mathbb{R}[s])$, $V(s) \in \mathbf{Gl}_p(\mathbb{R}[s])$ such that

$$U(s)G(s)V(s) = \operatorname{diag}\left(\frac{\varepsilon_1(s)}{\psi_1(s)}, \ldots, \frac{\varepsilon_r(s)}{\psi_r(s)}, 0_{(m-r)\times(p-r)}\right),$$

where $\varepsilon_i(s), \psi_i(s) \in \mathbb{R}[s]$ are monic, coprime and satisfy $\varepsilon_i(s) | \varepsilon_{i+1}(s)$, $\psi_{i+1}(s) | \psi_i(s)$ for $i = 1, \ldots, r-1$. The number $s_0 \in \mathbb{C}$ is called a *zero* of $G(s)$ if, and only if, $\varepsilon_r(s_0) = 0$ and a *pole* of $G(s)$ if, and only if, $\psi_1(s_0) = 0$.

In the following we give the definition of transmission zeros for the system $[E, A, B, C]$. In fact, there are many different possibilities to define transmission zeros of control systems, even in the ODE case, see [97]; and they are not equivalent. We go along with the definition given by Rosenbrock [210]: For $[I, A, B, C] \in \Sigma_{n,n,m,p}$, the transmission zeros are the zeros of the transfer function $C(sI - A)^{-1}B$. This definition has been generalized to regular DAE systems with transfer function $C(sE - A)^{-1}B$ in [35, Def. 5.3]. In the present framework, we do not require regularity of $sE - A$ and so a transfer function does in general not exist. However, it is possible to give a generalization of the inverse transfer function if the zero dynamics of $[E, A, B, C] \in \Sigma_{l,n,m,p}$

are autonomous: Let $L(s)$ be a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$ (which exists by Proposition 4.1.5) and define

$$H(s) := -[0, I_m]L(s)\begin{bmatrix} 0 \\ I_p \end{bmatrix} \in \mathbb{R}(s)^{m\times p}. \tag{4.3.1}$$

We show that $H(s)$ is independent of the choice of the left inverse $L(s)$ and if $sE - A$ is regular and $m = p$, then $H(s) = \big(C(sE - A)^{-1}B\big)^{-1}$, i.e., $H(s)$ is indeed the inverse of the transfer function in case of regularity.

In the following we parameterize all left inverses of the system pencil for right-invertible systems with autonomous zero dynamics; this is important to read off some properties of the block matrices in the form (4.2.3). Furthermore, it is shown that the lower right block in any left inverse is well-defined and therefore $H(s)$ in (4.3.1) is well-defined. The existence of a left inverse of the system pencil over $\mathbb{R}(s)$ is clear, since by Proposition 4.1.5 autonomous zero dynamics lead to a full column rank of the system pencil over $\mathbb{R}[s]$.

Lemma 4.3.2 (Left inverse of system pencil)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *be right-invertible and have autonomous zero dynamics. Then* $L(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ *is a left inverse of* $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ *if, and only if, using the notation from Theorem 4.2.7 and the form* (4.2.3),

$$L(s) = \begin{bmatrix} T & 0 \\ 0 & I_m \end{bmatrix} \begin{bmatrix} (sI_k - Q)^{-1} & 0 & 0 & X_{14}(s) & X_{15}(s) \\ 0 & 0 & 0 & X_{24}(s) & I_p \\ 0 & 0 & (sN - I_{n_3})^{-1} & X_{34}(s) & X_{35}(s) \\ X_{41}(s) & I_m & X_{43}(s) & X_{44}(s) & X_{45}(s) \end{bmatrix} \begin{bmatrix} S & 0 \\ 0 & I_p \end{bmatrix}, \tag{4.3.2}$$

where $[X_{14}(s)^\top, X_{24}(s)^\top, X_{34}(s)^\top, X_{44}(s)^\top]^\top \in \mathbb{R}(s)^{(n+m)\times(l+p-n-m)}$ *and*

$$\begin{aligned} X_{15}(s) &= (sI - Q)^{-1}A_{12}, \qquad & X_{35}(s) &= -s(sN - I)^{-1}E_{32}, \\ X_{41}(s) &= A_{21}(sI - Q)^{-1}, \qquad & X_{43}(s) &= -sE_{23}(sN - I)^{-1}, \\ X_{45}(s) &= -(sE_{22} - A_{22}) + A_{21}(sI - Q)^{-1}A_{12} + s^2E_{23}(sN - I)^{-1}E_{32}, \end{aligned}$$

and $L(s)$ *is partitioned according to the block structure of* (4.2.3).

If $L_1(s), L_2(s) \in \mathbb{R}(s)^{(n+m)\times(l+p)}$ *are two left inverse matrices of* $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$*, then*

$$[0, I_m]L_1(s)\begin{bmatrix}0\\ I_p\end{bmatrix} = [0, I_m]L_2(s)\begin{bmatrix}0\\ I_p\end{bmatrix}.$$

Proof: By Proposition 4.2.12 we have $\operatorname{rk} C = p$ and hence the assumptions of Theorem 4.2.7 are satisfied. The statements can then be verified by a simple calculation. □

Remark 4.3.3 (Regular systems)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ be such that $sE - A$ is regular. If $L(s)$ is a left inverse of the system pencil, then we have

$$\begin{aligned}\begin{bmatrix}I_n & (sE-A)^{-1}B\\ 0 & I_m\end{bmatrix} &= L(s)\begin{bmatrix}sE-A & -B\\ -C & 0\end{bmatrix}\begin{bmatrix}I_n & (sE-A)^{-1}B\\ 0 & I_m\end{bmatrix}\\ &= L(s)\begin{bmatrix}sE-A & 0\\ -C & -C(sE-A)^{-1}B\end{bmatrix},\end{aligned}$$

and therefore $C(sE - A)^{-1}B$ is invertible over $\mathbb{R}(s)$, and $H(s)$ as in (4.3.1) satisfies $H(s) = \big(C(sE - A)^{-1}B\big)^{-1}$, i.e., $H(s)$ is exactly the inverse transfer function of the system $[E, A, B, C]$. Note that, if $sE - A$ is not regular, then the transfer function $C(sE - A)^{-1}B$ does not exist.

Remark 4.3.3 and the fact that the zeros of $H(s)^{-1}$ are the poles of $H(s)$ and vice versa motivate the following definition.

Definition 4.3.4 (Transmission zeros)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ have autonomous zero dynamics, $L(s)$ be a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$ and let $H(s)$ be given as in (4.3.1). Then $s_0 \in \mathbb{C}$ is called *transmission zero* of $[E, A, B, C]$ if, and only if, s_0 is a pole of $H(s)$.

Concluding this subsection we characterize the transmission zeros in terms of the form (4.2.3).

Corollary 4.3.5 (Transmission zeros in decomposition)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *be right-invertible and have autonomous zero dynamics. Let* $L(s)$ *be a left inverse of* $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ *over* $\mathbb{R}(s)$ *and let*

$H(s)$ be given as in (4.3.1). Then, using the notation from Theorem 4.2.7 and the form (4.2.3),

$$H(s) = sE_{22} - A_{22} - A_{21}(sI_k - Q)^{-1}A_{12} - s^2E_{23}(sN - I_{n_3})^{-1}E_{32}$$

and $s_0 \in \mathbb{C}$ is a transmission zero of $[E, A, B, C]$ if, and only if, s_0 is a pole of

$$A_{21}(sI_k - Q)^{-1}A_{12}.$$

Proof: The representation of $H(s)$ follows from Lemma 4.3.2 and the characterization of transmission zeros is then immediate since $sE_{22} - A_{22} - s^2E_{23}(sN - I)^{-1}E_{32}$ is a polynomial as N is nilpotent and hence

$$(sN - I)^{-1} = -I - sN - \ldots - s^{\nu-1}N^{\nu-1}. \tag{4.3.3}$$

□

4.3.2 Detectability

In this subsection we introduce and characterize the concept of detectability in the behavioral sense. Detectability has been first defined and characterized for regular systems in [20]. For general DAE systems, a definition and characterization can be found in [118]; see also the equivalent definition in [198, Sec. 5.3.2]. The latter definition is given within the behavioral framework, however it is yet too restrictive for our purposes and it is not dual to the respective stabilizability concept from Definition 3.1.5. We use the following concept of behavioral detectability.

Definition 4.3.6 (Detectability)**.**
$[E, A, B, C] \in \Sigma_{l,n,m,p}$ is called *detectable in the behavioral sense* if, and only if,

$$\forall\, (x, 0, 0) \in \mathfrak{B}_{[E,A,B,C]}\ \exists\, (\tilde{x}, 0, 0) \in \mathfrak{B}_{[E,A,B,C]} :\quad x|_{(-\infty,0)} \overset{\text{a.e.}}{=} \tilde{x}|_{(-\infty,0)}\ \wedge\ \lim_{t\to\infty} \text{ess-sup}_{[t,\infty)} \|\tilde{x}\| = 0.$$

In order to characterize behavioral detectability we derive a duality result for detectability and stabilizability in the behavioral sense. To this end, we also use the concept of behavioral stabilizability introduced for DAEs $[E, A]$ in Definition 3.4.1. We call a system $[E, A, B, C] \in \Sigma_{l,n,m,p}$ stabilizable in the behavioral sense if, and only if, $[E, A, B]$ is stabilizable in the behavioral sense as in Definition 3.1.5.

Lemma 4.3.7 (Duality).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$*. Then the following statements are equivalent:*

(i) $[E, A, B, C]$ *is stabilizable in the behavioral sense.*

(ii) $[[E, 0], [A, B]]$ *is stabilizable in the behavioral sense.*

(iii) $\left[\left[\begin{smallmatrix} E^\top \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A^\top \\ B^\top \end{smallmatrix}\right]\right]$ *is stabilizable in the behavioral sense.*

(iv) $[E^\top, A^\top, C^\top, B^\top]$ *is detectable in the behavioral sense.*

Proof: It follows from the definition that (i)⇔(ii) and (iii)⇔(iv). By Corollary 3.4.4, (ii) is equivalent to

$$\forall\, \lambda \in \overline{\mathbb{C}}_+ :\ \mathrm{rk}_\mathbb{C}[\lambda E - A, -B] = \mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B].$$

Since ranks are invariant under matrix transpose, we find that (ii) is equivalent to

$$\forall\, \lambda \in \overline{\mathbb{C}}_+ :\ \mathrm{rk}_\mathbb{C} \begin{bmatrix} \lambda E^\top - A^\top \\ -B^\top \end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE^\top - A^\top \\ -B^\top \end{bmatrix},$$

which, again by Corollary 3.4.4, is equivalent to (iv). This completes the proof. □

In view of Lemma 4.3.7 and Corollary 3.4.4 we may infer the following.

Corollary 4.3.8 (Characterization of stabilizability and detectability).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$*. Then* $[E, A, B, C]$ *is detectable in the behavioral sense if, and only if,*

$$\forall\, \lambda \in \overline{\mathbb{C}}_+ :\ \mathrm{rk}_\mathbb{C} \begin{bmatrix} \lambda E - A \\ -C \end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ -C \end{bmatrix}.$$

4.3.3 Characterization of stable zero dynamics

In this subsection we derive some characterizations of asymptotically stable zero dynamics and, in particular, the main result of this section, that is Theorem 4.3.12.

In terms of the system pencil we get the following characterization of asymptotically stable zero dynamics.

Lemma 4.3.9 (Characterization of asymptotically stable zero dynamics).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$. *Then*

$$\mathcal{ZD}_{[E,A,B,C]} \text{ are asymptotically stable} \iff \forall\, \lambda \in \overline{\mathbb{C}}_+ : \ \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} = n + m.$$

Proof: $\Rightarrow$: Suppose there exist $\lambda \in \overline{\mathbb{C}}_+$ and $x^0 \in \mathbb{R}^n$, $u^0 \in \mathbb{R}^m$ such that

$$\begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} \begin{pmatrix} x^0 \\ u^0 \end{pmatrix} = 0.$$

Let $x : \mathbb{R} \to \mathbb{R}^n,\ t \mapsto e^{\lambda t} x^0$ and $u : \mathbb{R} \to \mathbb{R}^m,\ t \mapsto e^{\lambda t} u^0$. Then

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = e^{\lambda t}(\lambda E x^0) = e^{\lambda t}(A x^0 + B u^0) = Ax(t) + Bu(t), \quad Cu(t) = 0,$$

hence $(x, u, 0) \in \mathcal{ZD}_{[E,A,B,C]}$, which contradicts asymptotic stability of $\mathcal{ZD}_{[E,A,B,C]}$.

$\Leftarrow$: The rank condition implies that the system pencil must have full column rank over $\mathbb{R}[s]$. Therefore, by Lemma 4.1.3, in a QKF (4.1.4) of the system pencil it holds that $\ell(\beta) = 0$. It is also immediate that $\sigma(A_s) \subseteq \mathbb{C}_-$. The asymptotic stability of $\mathcal{ZD}_{[E,A,B,C]}$ then follows from a consideration of the solutions to each block in the OKF (4.1.4). $\square$

Remark 4.3.10 (Asymptotically stable zero dynamics are autonomous).
If follows from Lemma 4.3.9 that for any $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with asymptotically stable zero dynamics, the system pencil must have full column rank for some and hence almost all $s \in \mathbb{C}$. This implies full column rank over $\mathbb{R}[s]$. Therefore, by Proposition 4.1.5, the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are autonomous.

For the proof of the main result of this section, that is Theorem 4.3.12, we first show the following technical lemma.

Lemma 4.3.11 (Eigenvalues and poles).
Consider $[I_n, A, B, C] \in \Sigma_{n,n,m,p}$ *and assume that* $\mu \in \sigma(A)$ *is not a pole of* $C(sI - A)^{-1}B \in \mathbb{R}(s)^{p \times m}$. *Then*

$$\mathrm{rk}_{\mathbb{C}}\, [\mu I - A,\ B] < n \quad \vee \quad \mathrm{rk}_{\mathbb{C}}\, [\mu I - A^\top,\ C^\top] < n\,.$$

Proof: Without loss of generality, we assume that A is in Jordan canonical form and A, B, C are partitioned as follows

$$A = \begin{bmatrix} \lambda_1 I_{n_1} + N_1 & & \\ & \ddots & \\ & & \lambda_k I_{n_k} + N_k \end{bmatrix}, \quad B = \begin{bmatrix} B_1 \\ \vdots \\ B_k \end{bmatrix}, \quad C = \begin{bmatrix} C_1, & \ldots, & C_k \end{bmatrix},$$

where $\sigma(A) = \{\lambda_1, \ldots, \lambda_k\}$, $\lambda_1, \ldots, \lambda_k$ pairwise distinct, $\mu = \lambda_1$ and $N_1, \ldots, N_k$ are nilpotent with indices of nilpotency $\nu_1, \ldots, \nu_k$ and appropriate formats. Then

$$C(sI - A)^{-1}B \;=\; \sum_{i=1}^{k} \sum_{j=0}^{\nu_i - 1} \frac{C_i N_i^j B_i}{(s - \lambda_i)^{j+1}}$$

and the set of poles of $C(sI - A)^{-1}B$ is given by

$$\left\{ \lambda_i \in \sigma(A) \;\middle|\; i \in \{1 \ldots, k\} \;\wedge\; \exists j \in \{0, \ldots, \nu_i - 1\} : \; C_i N_i^j B_i \neq 0 \right\}.$$

Suppose μ is not a pole of $C(sI - A)^{-1}B$ and $\mathrm{rk}_{\mathbb{C}}\,[\mu I - A^\top, C^\top] = n$; then $C_1 N_1^{\nu_1 - 1} B_1 = 0$ and $\mathrm{rk}\,[N_1^\top, C_1^\top] = n_1$. Since

$$\begin{bmatrix} N_1 \\ C_1 \end{bmatrix} (N_1^{\nu_1 - 1} B_1) = 0,$$

we conclude $N_1^{\nu_1 - 1} B_1 = 0$ and so

$$N_1^{\nu_1 - 1} \cdot [\, N_1 \,,\, B_1 \,] = 0.$$

Since $N_1^{\nu_1 - 1} \neq 0$, it follows that $\mathrm{rk}\,[N_1, B_1] < n_1$ and thus $\mathrm{rk}_{\mathbb{C}}\,[\mu I - A, B] < n$.
Analogously, one may show that 'μ is not a pole of $C(sI - A)^{-1}B$ and $\mathrm{rk}_{\mathbb{C}}\,[\mu I - A, B] = n$' yields $\mathrm{rk}_{\mathbb{C}}\,[\mu I - A^\top, C^\top] < n$; this is omitted. This completes the proof of the lemma. □

Theorem 4.3.12 (Stable zero dynamics vs. transmission zeros, detectability & stabilizability)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible and have autonomous zero dynamics. Then the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are asymptotically stable if, and only if, the following three conditions hold:

(i) $[E, A, B, C]$ *is stabilizable in the behavioral sense,*

(ii) $[E, A, B, C]$ *is detectable in the behavioral sense,*

(iii) $[E, A, B, C]$ *has no transmission zeros in* $\overline{\mathbb{C}}_+$.

Proof: Since right-invertibility of $[E, A, B, C]$ implies, by Proposition 4.2.12, that $\operatorname{rk} C = p$, the assumptions of Theorem 4.2.7 are satisfied and we may assume that, without loss of generality, $[E, A, B, C]$ is in the form (4.2.3).

$\Rightarrow$: *Step 1*: We show (i). Let

$$T_1(s) := \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ -A_{21} & sE_{22} - A_{22} & sE_{23} & -I_m \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}[s])$$

and observe that, since $E_{42} = A_{42} = 0$ by Proposition 4.2.12,

$$[sE - A, -B]T_1(s) = \begin{bmatrix} sI_k - Q & -A_{12} & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & sE_{32} & sN - I_{n_3} & 0 \\ 0 & 0 & sE_{43} & 0 \end{bmatrix}.$$

Then, with

$$T_2(s) := \begin{bmatrix} I_k & (sI_k - Q)^{-1}A_{12} & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & 0 & I_{n_3} & 0 \\ 0 & 0 & 0 & -I_m \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}(s)),$$

and

$$T_3(s) := \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & -s(sN - I_{n_3})^{-1}E_{32} & I_{n_3} & 0 \\ 0 & 0 & 0 & -I_m \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}[s]),$$

where we note that it follows from (4.3.3) that $T_3(s)$ is a polynomial, we obtain

$$[sE - A, -B]T_1(s)T_2(s)T_3(s) = \begin{bmatrix} sI_k - Q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & sN - I_{n_3} & 0 \\ 0 & X(s) & sE_{43} & 0 \end{bmatrix},$$

where $X(s) = -s^2 E_{43}(sN - I_{n_3})^{-1} E_{32} = 0$ by Proposition 4.2.12 and (4.3.3). Finally,

$$S_1(s) := \begin{bmatrix} I_k & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ 0 & & I_{n_3} & 0 \\ 0 & 0 & -sE_{43}(sN - I_{n_3})^{-1} & -I_m \end{bmatrix} \in \mathbf{Gl}_{n+m}(\mathbb{R}[s])$$

yields

$$S_1(s)[sE - A, -B]T_1(s)T_2(s)T_3(s) = \begin{bmatrix} sI_k - Q & 0 & 0 & 0 \\ 0 & 0 & 0 & I_m \\ 0 & 0 & sN - I_{n_3} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and hence $\mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B] = k + n_3 + m = n + m - p$, since $n_3 = n - k - p$ by Theorem 4.2.7. Now let $\lambda \in \overline{\mathbb{C}}_+$ and observe that, by Corollary 4.2.11, $\lambda I_k - Q$ is invertible. Hence, the matrices $T_1(\lambda), T_2(\lambda), T_3(\lambda)$ and $S_1(\lambda)$ exist and are invertible. Thus, using the same transformations as above for fixed $\lambda \in \overline{\mathbb{C}}_+$ now, we find that $\mathrm{rk}_{\mathbb{C}}[\lambda E - A, -B] = n + m - p$. This proves (i).

Step 2: We show (ii). Similar to Step 1 it can be shown that

$$\forall\, \lambda \in \overline{\mathbb{C}}_+ :\ \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ -C \end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ -C \end{bmatrix} = n.$$

Step 3: We show (iii). By Corollary 4.3.5, the transmission zeros of $[E, A, B, C]$ are the poles of

$$F(s) := A_{21}(sI_k - Q)^{-1} A_{12}.$$

Every pole of $F(s)$ is also an eigenvalue of Q. In view of Corollary 4.2.11, we have that $\sigma(Q) \subseteq \mathbb{C}_-$ and so (iii) follows.

$\Leftarrow$: By Corollary 4.2.11, we have to show that if $\lambda \in \sigma(Q)$, then $\lambda \in \mathbb{C}_-$. Let $\lambda \in \sigma(Q)$. We distinguish two cases:

Case 1: λ is a pole of $F(s)$. Then, by Corollary 4.3.5, λ is a transmission zero of $[E, A, B, C]$ and by (iii) we obtain $\lambda \in \mathbb{C}_-$.

Case 2: λ is not a pole of $F(s)$. Then Lemma 4.3.11 applied to $[I_k, Q, A_{12}, A_{21}]$ and λ yields that

$$\text{(a)}\ \mathrm{rk}_{\mathbb{C}}[\lambda I_k - Q, A_{12}] < k \quad \text{or} \quad \text{(b)}\ \mathrm{rk}_{\mathbb{C}}[\lambda I_k - Q^\top, A_{21}^\top] < k.$$

If (a) holds, then there exists $v_1 \in \mathbb{C}^k \setminus \{0\}$ such that

$$v_1^\top [\lambda I_k - Q, A_{12}] = 0.$$

Let $v_4 \in \mathbb{C}^{(l-n)+(p-m)}$ be arbitrary and define

$$v_3^\top := -\lambda v_4^\top E_{43}(\lambda N - I_{n_3})^{-1}.$$

Now observe that

$$(v_1^\top, 0, v_3^\top, v_4^\top) \begin{bmatrix} \lambda I_k - Q & -A_{12} & 0 & 0 \\ -A_{21} & \lambda E_{22} - A_{22} & \lambda E_{23} & I_m \\ 0 & \lambda E_{32} & \lambda N - I_{n_3} & 0 \\ 0 & 0 & \lambda E_{43} & 0 \end{bmatrix} = (0, w^\top, 0, 0),$$

where

$$w^\top = -v_1^\top A_{12} + \lambda v_3^\top E_{32} = -\lambda^2 v_4^\top E_{43}(\lambda N - I_{n_3})^{-1} E_{32} = 0$$

by Proposition 4.2.12 and (4.3.3). This implies that $\mathcal{K} := \ker [\lambda E - A, -B]^\top \subseteq \mathbb{C}^l$ has dimension $\dim \mathcal{K} \geq (l-n) + (p-m) + 1$. Therefore,

$$\begin{aligned} \mathrm{rk}_{\mathbb{C}}[\lambda E - A, -B] &\leq l - \dim \mathcal{K} \leq n + m - p - 1 \\ &= \mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B] - 1 < \mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B], \end{aligned} \tag{4.3.4}$$

where $\mathrm{rk}_{\mathbb{R}(s)}[sE - A, -B] = n + m - p$ has been proved in Step 1 of "$\Rightarrow$". Hence, (4.3.4) together with (i) implies that $\lambda \in \mathbb{C}_-$.

If (b) holds, then there exists $v_1 \in \mathbb{C}^k \setminus \{0\}$ such that $v_1^\top [\lambda I_k - Q^\top, A_{21}^\top] = 0$. Therefore,

$$\begin{bmatrix} \lambda I_k - Q & -A_{12} & 0 \\ -A_{21} & \lambda E_{22} - A_{22} & \lambda E_{23} \\ 0 & \lambda E_{32} & \lambda N - I_{n_3} \\ 0 & 0 & \lambda E_{43} \\ 0 & I_p & 0 \end{bmatrix} \begin{pmatrix} v_1 \\ 0 \\ 0 \end{pmatrix} = 0$$

and thus

$$\mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ -C \end{bmatrix} < n = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A \\ -C \end{bmatrix}, \tag{4.3.5}$$

where $\mathrm{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE - A \\ -C \end{smallmatrix} \right] = n$ has been proved in Step 2 of "$\Rightarrow$". Hence, (4.3.5) together with (ii) implies that $\lambda \in \mathbb{C}_-$. This completes the proof of the theorem. □

For regular systems with invertible transfer function we may characterize asymptotic stability of the zero dynamics by Hautus criteria for stabilizability and detectability and the absence of zeros of the transfer function in the closed right complex half-plane (recall Definition 4.3.1 for the definition of a zero of a rational matrix function).

Corollary 4.3.13 (Regular systems)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *be such that* $sE - A$ *is regular and* $G(s) := C(sE - A)^{-1}B$ *is invertible over* $\mathbb{R}(s)$*. Then the zero dynamics* $\mathcal{ZD}_{[E,A,B,C]}$ *are asymptotically stable if, and only if, the following three conditions hold:*

(i) $\forall\, \lambda \in \overline{\mathbb{C}}_+ :\ \operatorname{rk}_{\mathbb{C}}[\lambda E - A, -B] = n,$

(ii) $\forall\, \lambda \in \overline{\mathbb{C}}_+ :\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A \\ -C \end{bmatrix} = n,$

(iii) $G(s)$ *has no zeros in* $\overline{\mathbb{C}}_+$*.*

Proof: Since $G(s) \in \mathbf{Gl}_m(\mathbb{R}(s))$ it is a simple calculation that $\operatorname{rk}_{\mathbb{R}[s]} \left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right] = n + m$ and hence it follows from Proposition 4.1.5 that $\mathcal{ZD}_{[E,A,B,C]}$ are autonomous. Furthermore, $\operatorname{rk} C = m$ and hence we may infer from Remark 4.2.13 that $[E, A, B, C]$ is right-invertible. Now, we may apply Theorem 4.3.12 to deduce that $\mathcal{ZD}_{[E,A,B,C]}$ are asymptotically stable if, and only if,

(a) $[E, A, B, C]$ is stabilizable in the behavioral sense,

(b) $[E, A, B, C]$ is detectable in the behavioral sense,

(c) $[E, A, B, C]$ has no transmission zeros in $\overline{\mathbb{C}}_+$.

Since regularity of $sE-A$ gives that $\operatorname{rk}_{\mathbb{R}(s)}[sE-A, -B] = \operatorname{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A \\ -C \end{smallmatrix}\right] = n$, we find that (i)$\Leftrightarrow$(a) and (ii)$\Leftrightarrow$(b). (iii)$\Leftrightarrow$(c) follows from the fact that by Remark 4.3.3 we have $H(s) = G(s)^{-1}$ for $H(s)$ as in (4.3.1) and that transmission zeros of $[E, A, B, C]$ are, by definition, exactly the poles of $H(s)$. □

4.4 Stabilization

In this section, we introduce the Lyapunov exponent for DAEs and show that for any right-invertible system with autonomous zero dynamics there exists a compatible control such that the Lyapunov exponent of the interconnected system is equal to the Lyapunov exponent of the zero dynamics of the nominal system.

In order to get a most general definition of the Lyapunov exponent, we use a definition similar to the Bohl exponent for DAEs in [26, Def. 3.4], not requiring a fundamental solution matrix as in [165].

Definition 4.4.1 (Lyapunov exponent)**.**
Let $E, A \in \mathbb{R}^{l\times n}$. The *Lyapunov exponent* of $[E, A]$ is defined as

$$k_L(E,A) := \inf \left\{ \mu \in \mathbb{R} \;\middle|\; \begin{array}{r} \exists\, M_\mu > 0\ \forall\, x \in \mathfrak{B}_{[E,A]} \text{ for a.a. } t \geq s : \\ \|x(t)\| \leq M_\mu e^{\mu(t-s)} \|x(s)\| \end{array} \right\}.$$

Note that we use the convention $\inf \emptyset = +\infty$.

The (infimal) exponential decay rate of the (asymptotically stable) zero dynamics of a system can be determined by the Lyapunov exponent of the DAE $[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right]]$.

Lemma 4.4.2 (Lyapunov exponent and stable zero dynamics)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *have autonomous zero dynamics and* $\operatorname{rk} C = p$*. Then, using the notation from Theorem 4.2.7, the form* (4.2.3) *and* $k = \dim \max(E, A, B; \ker C)$*, we have*

$$\begin{aligned} & k_L(\mathcal{ZD}_{[E,A,B,C]}) \\ & := \inf \left\{ \mu \in \mathbb{R} \;\middle|\; \begin{array}{r} \exists\, M_\mu > 0\ \forall\, w \in \mathcal{ZD}_{[E,A,B,C]} \text{ for a.a. } t \geq s : \\ \|w(t)\| \leq M_\mu e^{\mu(t-s)} \|w(s)\| \end{array} \right\} \\ & = k_L\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ C & 0 \end{bmatrix} \right) \\ & = \begin{cases} \max \{\, \operatorname{Re} \lambda \mid \lambda \in \sigma(Q) \,\}, & \text{if } k > 0 \\ -\infty, & \text{if } k = 0. \end{cases} \end{aligned}$$

Proof: The first equality follows from the fact that the trajectories in $\mathcal{ZD}_{[E,A,B,C]}$ can be identified with those in the behavior $\mathfrak{B}_{[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & B \\ C & 0 \end{smallmatrix}\right]]}$.

The second equality can be seen by using the decomposition (4.2.3): Since the Lyapunov exponent is invariant under transformation of the system (see e.g. [26, Prop. 3.17]) we may assume that, without loss of generality, $[E, A, B, C]$ is in the form (4.2.3). Now observe that $(x, u, y) \in \mathcal{ZD}_{[E,A,B,C]}$, where $x = (x_1, y, x_3)$, if, and only if, $y \overset{\text{a.e.}}{=} 0$, $x_3 \overset{\text{a.e.}}{=} 0$ and $x_1 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^k)$, $u \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^m)$ satisfy

$$\tfrac{\mathrm{d}}{\mathrm{d}t} x_1 \overset{\text{a.e.}}{=} Q x_1, \quad u \overset{\text{a.e.}}{=} -A_{21} x_1.$$

This equivalence of solution trajectories yields the assertion. □

Note that it follows from Corollary 4.2.11 and Lemma 4.4.2 that asymptotic stability of the zero dynamics implies exponential stability of the zero dynamics, i.e., any trajectory tends to zero exponentially.

We are now in a position to prove the main result of this subsection, which states that for right-invertible systems with autonomous zero dynamics there exists a compatible control such that the Lyapunov exponent of the interconnected system is equal to the Lyapunov exponent of the zero dynamics of the nominal system; in particular, this shows that asymptotic stability of the zero dynamics implies that the system can be asymptotically stabilized in the sense that every solution of the interconnected system tends to zero. Recall the concepts of compatible control and interconnected system (behavior) from Subsection 3.4.3; we call a control $K \in \mathbb{R}^{q \times (n+m)}$ compatible for $[E, A, B, C] \in \Sigma_{l,n,m,p}$ if, and only if, K is compatible for $[E, A, B]$ as in Definition 3.4.8.

Theorem 4.4.3 (Compatible and stabilizing control).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible with autonomous zero dynamics. If $\dim \max(E, A, B; \ker C) > 0$, then there exists a compatible control matrix $K = [K_x, K_u] \in \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ for $[E, A, B, C]$ such that

$$k_L\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix} \right) = k_L(\mathcal{ZD}_{[E,A,B,C]}). \tag{4.4.1}$$

If $\dim \max(E, A, B; \ker C) = 0$, then for all $\mu \in \mathbb{R}$ there exists a compatible control matrix $K = [K_x, K_u] \in \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times m}$ for $[E, A, B, C]$ such that

$$k_L\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix} \right) \le \mu. \tag{4.4.2}$$

Proof: Since the Lyapunov exponent is invariant under transformation of the system (see e.g. [26, Prop. 3.17]) we may, similar to the proof of Theorem 4.3.12, assume that, without loss of generality, $[E, A, B, C]$ is in the form (4.2.3). Then, with similar transformations as in Step 1 of the proof of Theorem 4.3.12, it can be shown that

$$\forall\,\lambda \in \mathbb{C}: \ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E_{22} - A_{22} & \lambda E_{23} & I_m \\ \lambda E_{32} & \lambda N - I_{n_3} & 0 \\ 0 & \lambda E_{43} & 0 \end{bmatrix} = \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE_{22} - A_{22} & sE_{23} & I_m \\ sE_{32} & sN - I_{n_3} & 0 \\ 0 & sE_{43} & 0 \end{bmatrix},$$

and hence, by Proposition 3.3.3, the system

$$\left[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}\right] := \left[\begin{bmatrix} E_{22} & E_{23} \\ E_{32} & N \\ 0 & E_{43} \end{bmatrix}, \begin{bmatrix} A_{22} & 0 \\ 0 & I_{n_3} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_m \\ 0 \\ 0 \end{bmatrix}, [I_p, 0]\right]$$

is controllable in the behavioral sense.

We will now mimic the proof of Theorem 3.4.10 without repeating all of its arguments: It follows from the above controllability in the behavioral sense and Corollary 3.2.8 that in the feedback form (3.2.9) of $[\tilde{E}, \tilde{A}, \tilde{B}]$ we have $n_{\overline{c}} = 0$. Therefore, for any given $\mu \in \mathbb{R}$ and $\varepsilon > 0$, it is possible to choose F_{11} and K_x in the proof of Theorem 3.4.10 such that the resulting control matrix $\tilde{K} = [K_1, K_2] \in \mathbb{R}^{q\times(n-k)} \times \mathbb{R}^{q\times m}$ is compatible for $\left[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}\right]$ and satisfies

$$k_L\left(\begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A} & \tilde{B} \\ K_1 & K_2 \end{bmatrix}\right) \le \mu - \varepsilon. \tag{4.4.3}$$

We show that

$$K = [K_x \,|\, K_u] := [K_2 A_{21}, K_1 \,|\, K_2] \in \mathbb{R}^{q\times k} \times \mathbb{R}^{q\times(n-k)} \times \mathbb{R}^{q\times m},$$

is compatible for $[E, A, B, C]$ and satisfies (4.4.1) or (4.4.2), resp.

Step 1: We show compatibility. Let

$$x^0 \in \left\{ x^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists\,(x, u, y) \in \mathfrak{B}_{(4.1.1)}: \ x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n) \\ \wedge\ Ex(0) = Ex^0 \end{array} \right\}$$

and partition $x^0 = \left((x_1^0)^\top, (x_2^0)^\top\right)^\top$ with $x_1^0 \in \mathbb{R}^k$, $x_2^0 \in \mathbb{R}^{n-k}$. Then there exist $x_1 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$, $x_2 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n-k})$ and $u \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^m)$ such that

$$\left.\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}x_1 & \overset{\text{a.e.}}{=} & Qx_1 + [A_{12}, 0]x_2, \\ \frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}x_2 & \overset{\text{a.e.}}{=} & \begin{bmatrix} A_{21} \\ 0 \\ 0 \end{bmatrix} x_1 + \tilde{A}x_2 + \tilde{B}u, \\ x_1(0) & = & x_1^0, \\ \tilde{E}x_2(0) & = & \tilde{E}x_2^0. \end{array}\right\} \tag{4.4.4}$$

Therefore,

$$x_2^0 \in \left\{ x_2^0 \in \mathbb{R}^n \;\middle|\; \begin{array}{r} \exists\, (x_2, u, \tilde{C}x_2) \in \mathfrak{B}_{[\tilde{E},\tilde{A},\tilde{B},\tilde{C}]} : \; x_2 \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n-k}) \\ \wedge\; \tilde{E}x_2(0) = \tilde{E}x_2^0 \end{array} \right\},$$

and by compatibility of $[K_1, K_2]$ for $\left[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}\right]$ there exists $(x_2, v) \in \mathfrak{B}^{[K_1,K_2]}_{[\tilde{E},\tilde{A},\tilde{B}]}$ such that

$$\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}\tilde{E}x_2 & \overset{\text{a.e.}}{=} & \tilde{A}x_2 + \tilde{B}v, \\ 0 & \overset{\text{a.e.}}{=} & K_1x_2 + K_2v, \end{array} \tag{4.4.5}$$

and $\tilde{E}x_2(0) = \tilde{E}x_2^0$. Define

$$x_1(t) := e^{Qt}x_1^0 + \int_0^t e^{Q(t-s)}[A_{12}, 0]x_2(s)\; \mathrm{d}s\,, \quad t \in \mathbb{R},$$

which is well-defined since $x_2 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n-k})$, and let $u := v - A_{21}x_1$. Then (x_1, x_2, u) solves (4.4.4) and satisfies

$$K_2A_{21}x_1 + K_1x_2 + K_2u \overset{\text{a.e.}}{=} K_2A_{21}x_1 + K_1x_2 + K_2v - K_2A_{21}x_1 \overset{\text{a.e.}}{=} 0,$$

which proves that $[K_2A_{21}, K_1, K_2]$ is compatible for $[E, A, B, C]$.

Step 2: We show that (4.4.2) is satisfied in case that $k = \dim \max(E, A, B; \ker C) = 0$. This follows from (4.4.3) since

$$k_L\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix}\right) = k_L\left(\begin{bmatrix} \tilde{E} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \tilde{A} & \tilde{B} \\ K_1 & K_2 \end{bmatrix}\right) \le \mu - \varepsilon$$

with arbitrary $\mu \in \mathbb{R}$ and $\varepsilon > 0$.

Step 3: We show that (4.4.1) is satisfied in case that $k > 0$. Denote

$$\mu := k_L(\mathcal{ZD}_{[E,A,B,C]}) \overset{\text{Lem. 4.4.2}}{=} \max\{\ \operatorname{Re}\lambda \mid \lambda \in \sigma(Q)\ \}$$

and let $\rho > 0$ be arbitrary. Then there exists $M_\rho > 0$ such that, for all $t \geq 0$, $\|e^{Qt}\| \leq M_\rho e^{(\mu+\rho)t}$.
Step 3a: We show "$\geq$" in (4.4.1). Since, for any solution $x_1 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^k)$ of $\frac{\mathrm{d}}{\mathrm{d}t}x_1 = Qx_1$ we have

$$\big((x_1^\top, 0)^\top, -A_{21}x_1, 0\big) \in \mathfrak{B}^K_{[E,A,B]},$$

it follows that

$$k_L\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix}\right) \geq \mu.$$

Step 3b: We show "$\leq$" in (4.4.1). Let $(x,u) \in \mathfrak{B}^K_{[E,A,B]}$ and write $x = \big(x_1^\top, x_2^\top\big)^\top$ with $x_1 \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R};\mathbb{R}^k)$ and $x_2 \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^{n-k})$. Then we have

$$\begin{aligned}
\tfrac{\mathrm{d}}{\mathrm{d}t}x_1 &\overset{\text{a.e.}}{=} Qx_1 + [A_{12}, 0]x_2, \\
\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{E}x_2 &\overset{\text{a.e.}}{=} \begin{bmatrix} A_{21} \\ 0 \\ 0 \end{bmatrix} x_1 + \tilde{A}x_2 + \tilde{B}u, \\
0 &\overset{\text{a.e.}}{=} K_2A_{21}x_1 + K_1x_2 + K_2u.
\end{aligned}$$

Observe that $(x_2, w := u + A_{21}x_1)$ solves (4.4.5) and hence, by (4.4.3) for μ and some $\varepsilon > 0$, there exists $M_1 > 0$ such that

$$\text{for a.a. } t \geq s: \left\|\begin{pmatrix} x_2(t) \\ w(t) \end{pmatrix}\right\| \leq M_1 e^{(\mu-\varepsilon)(t-s)} \left\|\begin{pmatrix} x_2(s) \\ w(s) \end{pmatrix}\right\|.$$

Therefore,

$$\begin{aligned}
\|x_1(t)\| &\leq \|e^{Q(t-s)}\| \cdot \|x_1(s)\| + \int_s^t \|e^{Q(t-\tau)}\| \cdot \|[A_{12}, 0]\| \cdot \left\|\begin{pmatrix} x_2(\tau) \\ w(\tau) \end{pmatrix}\right\| \,\mathrm{d}\tau \\
&\leq M_\rho e^{(\mu+\rho)(t-s)}\|x_1(s)\| \\
&\quad + M_1M_\rho e^{(\mu+\rho)(t-s)} \cdot \|[A_{12}, 0]\| \cdot \left\|\begin{pmatrix} x_2(s) \\ w(s) \end{pmatrix}\right\| \underbrace{\int_s^t e^{-(\varepsilon+\rho)(t-\tau)} \,\mathrm{d}\tau}_{\leq 1/\varepsilon}
\end{aligned}$$

for almost all $t, s \in \mathbb{R}$ with $t \geq s$. This implies that

$$k_L\left(\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix}\right) \leq \mu + \rho$$

and since $\rho > 0$ is arbitrary the claim is shown. □

Remark 4.4.4 (Construction of the control)**.**
The construction of the control K in the proof of Theorem 4.4.3 relies on the construction used in Theorem 3.4.10. Here we make it precise. We have split up the procedure into several steps.

(i) The first step is to transform the given system $[E, A, B, C] \in \Sigma_{l,n,m,p}$ into the form (4.2.3). The first transformation which has to be applied in order to achieve this is the ZDF stated in Theorem 4.1.7 and uses the maximal (A, E, B)-invariant subspace included in $\ker C$. This subspace can be obtained easily via a subspace iteration as described in Lemma 4.1.2. The second transformation which has to be applied is stated in Theorem 4.2.7. Denote the resulting system by

$$\begin{bmatrix} s\overline{E} - \overline{A} & \overline{B} \\ \overline{C} & 0 \end{bmatrix} = \begin{bmatrix} P & 0 \\ 0 & I_p \end{bmatrix} \begin{bmatrix} sE - A & B \\ C & 0 \end{bmatrix} \begin{bmatrix} Q & 0 \\ 0 & I_m \end{bmatrix}.$$

(ii) Next we have to consider the subsystem

$$\left[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{C}\right] := \left[\begin{bmatrix} E_{22} & E_{23} \\ E_{32} & N \\ 0 & E_{43} \end{bmatrix}, \begin{bmatrix} A_{22} & 0 \\ 0 & I_{n_3} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} I_m \\ 0 \\ 0 \end{bmatrix}, [I_p, 0]\right]$$

and transform it into a feedback form (3.2.9). Since $[\tilde{E}, \tilde{A}, \tilde{B}]$ is controllable in the behavioral sense and $\operatorname{rk} \tilde{B} = m$ this form can be simplified, i.e., there exist $S \in \mathbf{Gl}_{l-k}(\mathbb{R})$, $T \in \mathbf{Gl}_{n-k}(\mathbb{R})$, $V \in \mathbf{Gl}_m(\mathbb{R})$, $F \in \mathbb{R}^{m \times (n-k)}$ such that

$$[s\hat{E} - \hat{A}, \hat{B}] = S[s\tilde{E} - \tilde{A}, \tilde{B}] \begin{bmatrix} T & 0 \\ -F & V \end{bmatrix},$$

where

$$[\hat{E}, \hat{A}, \hat{B}] = \left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 \\ 0 & K_\beta & 0 & 0 & 0 \\ 0 & 0 & L_\gamma^\top & 0 & 0 \\ 0 & 0 & 0 & K_\delta^\top & 0 \\ 0 & 0 & 0 & 0 & N_\kappa \end{bmatrix}, \begin{bmatrix} N_\alpha^\top & 0 & 0 & 0 & 0 \\ 0 & L_\beta & 0 & 0 & 0 \\ 0 & 0 & K_\gamma^\top & 0 & 0 \\ 0 & 0 & 0 & L_\delta^\top & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} \end{bmatrix}, \begin{bmatrix} E_\alpha & 0 \\ 0 & 0 \\ 0 & E_\gamma \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \right],$$

for some multi-indices $\alpha, \beta, \gamma, \delta, \kappa$.

(iii) Let $\mu \in \mathbb{R}$ be arbitrary. We construct a compatible control in the behavioral sense for $[\hat{E}, \hat{A}, \hat{B}]$ such that the interconnected system has Lyapunov exponent smaller or equal to μ. Let $F_{11} \in \mathbb{R}^{\ell(\alpha)\times|\alpha|}$ be such that

$$\max\{ \text{ Re } \lambda \mid \lambda \in \sigma(N_\alpha + E_\alpha F_{11}) \} \leq \mu.$$

This can be achieved as follows: For $j = 1, \ldots, \ell(\alpha)$, consider vectors

$$a_j = -[a_{j\alpha_j-1}, \ldots, a_{j0}] \in \mathbb{R}^{1\times\alpha_j}.$$

Then, for

$$F_{11} = \text{diag}\,(a_1, \ldots, a_{\ell(\alpha)}) \in \mathbb{R}^{\ell(\alpha)\times|\alpha|},$$

the matrix $N_\alpha + E_\alpha F_{11}$ is diagonally composed of companion matrices, whence, for

$$p_j(s) = s^{\alpha_j} + a_{j\alpha_j-1}s^{\alpha_j-1} + \ldots + a_{j0} \in \mathbb{R}[s]$$

the characteristic polynomial of $N_\alpha + E_\alpha F_{11}$ is given by

$$\det(sI_{|\alpha|} - (N_\alpha + E_\alpha F_{11})) = \prod_{j=1}^{\ell(\alpha)} p_j(s).$$

Hence, choosing the coefficients a_{ji}, $j = 1, \ldots, \ell(\alpha)$, $i = 0, \ldots, \alpha_j$ such that the roots of the polynomials $p_1(s), \ldots, p_{\ell(\alpha)}(s) \in \mathbb{R}[s]$ are all smaller or equal to μ yields the assertion.

Now we find that

$$k_L\left(\begin{bmatrix} I_{|\alpha|} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} N_\alpha & E_\alpha \\ F_{11} & -I_{\ell(\alpha)} \end{bmatrix} \right) \leq \mu.$$

Furthermore, by the same reasoning as above, for

$$a_j = [a_{j\beta_j-2}, \ldots, a_{j0}, 1] \in \mathbb{R}^{1\times\beta_j}$$

with the property that the roots of the polynomials

$$p_j(s) = s^{\beta_j} + a_{j\beta_j-1}s^{\beta_j-1} + \ldots + a_{j0} \in \mathbb{R}[s]$$

are all smaller or equal to μ for $j = 1, \ldots, \ell(\alpha)$, the choice

$$K_x = \operatorname{diag}(a_1, \ldots, a_{\ell(\beta)}) \in \mathbb{R}^{\ell(\beta)\times|\beta|}$$

leads to

$$k_L\left(\begin{bmatrix} K_\beta \\ 0 \end{bmatrix}, \begin{bmatrix} L_\beta \\ K_x \end{bmatrix}\right) \le \mu.$$

Therefore, the control matrix

$$\hat{K} = [\hat{K}_1, \hat{K}_2] = \begin{bmatrix} F_{11} & 0 & 0 & 0 & 0 & -I_{\ell(\alpha)} & 0 \\ 0 & K_x & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{q\times(n-k)} \times \mathbb{R}^{q\times m},$$

where $q = \ell(\alpha) + \ell(\beta)$, establishes that

$$k_L\left(\begin{bmatrix} \hat{E} & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{K}_1 & \hat{K}_2 \end{bmatrix}\right) \le \mu.$$

Since the differential variables can be arbitrarily initialized in any of the previously discussed subsystems, the constructed control $\hat{K}$ is also compatible for $[\hat{E}, \hat{A}, \hat{B}]$.

(iv) We show that $\hat{K}$ leads to a compatible control $\tilde{K}$ for $[\tilde{E}, \tilde{A}, \tilde{B}]$ such that the interconnected system has Lyapunov exponent-smaller or equal to μ. Observe that

$$\begin{bmatrix} S^{-1} & 0 \\ 0 & I_q \end{bmatrix} \begin{bmatrix} s\hat{E} - \hat{A} & \hat{B} \\ \hat{K}_1 & \hat{K}_2 \end{bmatrix} \begin{bmatrix} T^{-1} & 0 \\ V^{-1}FT^{-1} & V^{-1} \end{bmatrix} = \begin{bmatrix} s\tilde{E} - \tilde{A} & \tilde{B} \\ \hat{K}_1 + \hat{K}_2V^{-1}FT^1 & \hat{K}_2V^{-1} \end{bmatrix}$$

and hence, by invariance of the Lyapunov exponent under transformation of the system (see e.g. [26, Prop. 3.17]), we find that for

$$[K_1, K_2] := [\hat{K}_1 + \hat{K}_2V^{-1}FT^1, \hat{K}_2V^{-1}] \in \mathbb{R}^{q\times(n-k)} \times \mathbb{R}^{q\times m},$$

we have

$$k_L\left(\begin{bmatrix}\tilde{E} & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}\tilde{A} & \tilde{B}\\ K_1 & K_2\end{bmatrix}\right) \le \mu.$$

(v) If $k = \dim \max(E, A, B; \ker C) = 0$, then we can choose $\mu \in \mathbb{R}$ as we like and obtain

$$k_L\left(\begin{bmatrix}\overline{E} & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}\overline{A} & \overline{B}\\ \overline{K}_x := K_1 & \overline{K}_u := K_2\end{bmatrix}\right) = k_L\left(\begin{bmatrix}\tilde{E} & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}\tilde{A} & \tilde{B}\\ K_1 & K_2\end{bmatrix}\right) \le \mu.$$

If $k > 0$, then we can choose $\mu < k_L(\mathcal{ZD}_{[E,A,B,C]})$ and obtain, with

$$[\overline{K}_x \mid \overline{K}_u] := [K_2 A_{21}, K_1 \mid K_2] \in \mathbb{R}^{q\times k} \times \mathbb{R}^{q\times(n-k)} \times \mathbb{R}^{q\times m},$$

that

$$k_L\left(\begin{bmatrix}\overline{E} & 0\\ 0 & 0\end{bmatrix}, \begin{bmatrix}\overline{A} & \overline{B}\\ \overline{K}_x & \overline{K}_u\end{bmatrix}\right) = k_L(\mathcal{ZD}_{[E,A,B,C]}).$$

This is shown in the proof of Theorem 4.4.3.

(vi) The desired compatible control K for $[E, A, B, C]$ is now given by

$$K = [\overline{K}_x Q^{-1}, \overline{K}_u].$$

Note that the practical computation of the decompositions in (i) and (ii) is in general not numerically stable. This can be achieved by using orthogonal transformations and condensed forms as in [49]. It seems that with some effort the form (4.2.3) in (i) can also be obtained with orthogonal transformations, but this needs to be investigated in detail. Instead of the feedback form (3.2.9) in (ii) a condensed form from [49] could be used. However, in the present thesis we do not focus on the numerical aspect.

Remark 4.4.5 (Implementation of the control)**.** As explained in Subsection 3.4.3, the control law

$$K_x x(t) + K_u u(t) = 0$$

cannot necessarily be solved for $u(t)$. This rises the question for the implementation of the controller. There are basically two perspectives in this regard:

(i) In order to implement the controller it is necessary that all free variables of the open-loop system $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t)+Bu(t)$ can be manipulated. The free variables of the system can be identified via the quasi-Kronecker form (see Proposition 3.2.3) of the pencil $s[E, 0] - [A, B]$; each underdetermined block $sK_{\beta_i} - L_{\beta_i}$ in the quasi-Kronecker form yields one free variable, i.e., there are $\ell(\beta)$ free variables in the system. The set of free variables may consist of input variables as well as state variables and not necessarily all input variables are free variables. For the implementation of the control, the free variables are treated as controls and the control law can be solved for the free variables. A similar approach has been discussed in [65].

(ii) For an alternative approach, where we do not wish to reinterpret variables, we use the fact that (cf. also Remark 4.4.4 (iii)) the control law can be rewritten in the form

$$\begin{bmatrix} K_1 \\ K_2 \end{bmatrix} x(t) + \begin{bmatrix} I_r & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = 0,$$

where a suitable input space transformation has been performed. Then we may solve the first row for $u_1(t)$ and implement this control. It only remains to implement the algebraic condition $K_2x(t) = 0$. In practice, this relation can be implemented by integrating appropriate components (such as dampers or resistors) into the given plant. In particular, it is not necessary to (actively) manipulate specific state variables, only the implementation of an algebraic relation between some of the state variables is necessary.

Theorem 4.4.3 shows that right-invertible systems $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with asymptotically stable zero dynamics can be stabilized by a compatible control so that any solution of the interconnected system tends to zero. It is well known [131, Rem. 6.1.3] that any linear ODE system with asymptotically stable zero dynamics (and $p = m$) is stabilizable by *state feedback*, i.e., the compatible control is of the form $u = Fx$, cf. also Subsection 3.4.2. For time-invariant (nonlinear) ODE systems, stabilization by state feedback is well investigated [55,57,131]. For regular DAE systems this has not been investigated yet.

Proposition 4.4.6 (Regular systems and state feedback).
If $sE - A \in \mathbb{R}[s]^{n\times n}$ in Theorem 4.4.3 is regular, then the compatible control K can be chosen as a state feedback in each case, i.e., $K = [K_x, -I_m] \in \mathbb{R}^{m\times n} \times \mathbb{R}^{m\times m}$.

Proof: We use the procedure presented in Remark 4.4.4 and modify it at some instances.

(i) Consider the system $[\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ from Remark 4.4.4 (ii) and observe that, using the same argument as in Remark 3.4.7 (i), we obtain $\ell(\delta) = 0$ and $\ell(\beta) = \ell(\gamma)$.

(ii) For any multi-index $\eta \in \mathbb{N}^l$ let

$$F_\eta := \operatorname{diag}(e_1^{[\eta_1]}, \ldots, e_1^{[\eta_l]}) \in \mathbb{R}^{|\eta|\times l}.$$

A straightforward calculation yields that there exists a permutation matrix $P \in \mathbb{R}^{\xi\times\xi}$, $\xi := |\beta| + |\gamma| - \ell(\gamma)$, such that

$$P\left(s\begin{bmatrix} K_\beta & 0 \\ 0 & L_\gamma^\top \end{bmatrix} - \begin{bmatrix} L_\beta & 0 \\ E_\gamma F_\beta^\top & K_\gamma^\top \end{bmatrix}\right) = sN - I_\xi,$$

where $N \in \mathbb{R}^{\xi\times\xi}$ is nilpotent.

(iii) Changing the control matrix $\hat{K}$ in Remark 4.4.4 (iii) to

$$\begin{aligned} \hat{K} &= [\hat{K}_1, \hat{K}_2] \\ &= \begin{bmatrix} F_{11} & 0 & 0 & 0 & 0 & -I_{\ell(\alpha)} & 0 \\ 0 & F_\beta^\top & 0 & 0 & 0 & 0 & -I_{\ell(\gamma)} \end{bmatrix} \in \mathbb{R}^{m\times(n-k)} \times \mathbb{R}^{m\times m}, \end{aligned}$$

where it is worth noting that $\ell(\alpha) + \ell(\gamma) = m$, and invoking the observation in (ii), we obtain the same result for the Lyapunov exponent, and the control can be equivalently expressed as a state feedback

$$u_1 = F_{11}x_1, \quad u_2 = F_\beta^\top x_2.$$

Since $\hat{K}_2 = -I_m$ we can write $[K_1, K_2]$ in Remark 4.4.4 (iv) as

$$[K_1, K_2] = [V\hat{K}_1 - FT^{-1}, -I_m]$$

and, furthermore, we have $\overline{K}_u = -I_m$ in Remark 4.4.4 (vi). Therefore, the compatible control K is a state feedback. □

4.5 Notes and References

(i) Autonomous zero dynamics seem to have their first appearance in [257], where it is mentioned that in the Byrnes-Isidori form (BIF) of a nonlinear ODE system the equation for the internal dynamics describes autonomous zero dynamics in case of zero output. However, it seems that this concept was not perceived very well within the control theory community and the only real utilization (except for the previously published results of the present thesis) seems to be [195], whose authors use it for the analysis of inverted pendulum systems. However, as mentioned in Section 4.1, it were Ilchmann and Wirth who recognized that the assumptions imposed by Isidori in [131, Rem. 6.1.3], in order to derive conditions for the stabilization of linear systems, are equivalent to autonomous zero dynamics after all.

(ii) The name 'zero dynamics form' is also used in the literature as a synonym for the Byrnes-Isidori form for linear ODE systems, see e.g. [156,179]. The BIF was originally developed in [131, Sec. 5.1] for nonlinear systems; for time-invariant multi-input multi-output systems see [127], for time-varying systems see [122]. In fact, this can be justified because in the BIF the zero dynamics of the system are decoupled, similarly to the ZDF (4.1.6), cf. Remark 4.1.8.

(iii) System inversion for linear ODE systems was first discussed in [256] and already well investigated in the sixties [47, 88, 220]. For DAEs, only few results are available: for regular systems, system inversion was first studied by Lewis [157] and investigated further in [160, 226]; nonregular systems have been investigated in [91], although several additional assumptions are made on the DAE system. Left- and right-invertibility, as introduced in Definition 4.2.1, have been studied in [160] for regular DAE systems.

(iv) Zero dynamics have been the essential tool for the theory of stable system inversion developed in [69, 258]. Compared to right-invertibility as in Definition 4.2.1, according to [69] we may call a system (4.1.1) stably invertible if, and only if, for any sufficiently smooth reference trajectory y with compact support (i.e., $y(t) = 0$ for all $t \in \mathbb{R} \setminus K$, $K \subseteq \mathbb{R}$ compact) there exists an input

u and a state x, both continuous, such that $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]}$ and $\lim_{t\to\infty} \|u(t)\| = \lim_{t\to\infty} \|x(t)\| = 0$. It is clear that stable invertibility implies right-invertibility. Furthermore, as it can be deduced from Theorem 4.2.7 and Corollary 4.2.11, a system $[E, A, B, C] \in \Sigma_{l,n,m,p}$ with autonomous zero dynamics is stably invertible if, and only if, its zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ are asymptotically stable.

5 High-gain and funnel control

In this chapter we study high-gain and funnel control for linear differential-algebraic control systems

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\,, \end{aligned}$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{m\times n}$, i.e., we assume that the number of inputs and outputs coincide. As in Chapter 4 we use the notation $\Sigma_{l,n,m,m}$ for the set of these systems.

Throughout this chapter we restrict ourselves to right-invertible systems
$[E, A, B, C] \in \Sigma_{l,n,m,m}$ with autonomous zero dynamics. An important property which will be exploited is asymptotically stable zero dynamics; this property has been investigated in detail in Section 4.3 for the considered class of systems. We will show in Section 5.1 that constant high-gain output feedback (1.4.1) (cf. also Section 1.4) achieves stabilization of the system for sufficiently large $k > 0$ (we say that the system has the *high-gain property*), provided that the zero dynamics are asymptotically stable and the matrix Γ in (5.1.2) exists and satisfies $\Gamma = \Gamma^\top \geq 0$. This specializes the stabilization results of Section 4.4 in a certain sense: under the given assumptions, the compatible control $K_x x + K_u u = 0$ in Theorem 4.4.3 can be chosen as $u = -kCx = -ky$ for some $k > 0$, i.e., $K_x = kC$ and $K_u = I_m$.

A drawback of high-gain control (1.4.1) is that it is not known a priori how large the constant $k > 0$ must be chosen. This problem can be resolved by the funnel controller (1.4.3) introduced in [125] (cf. Section 1.4). For feasibility of funnel control it is not necessary to know specific parts of the system or to use an identification method; only structural assumptions have to be made. We will show in Section 5.2 that funnel control is feasible for right-invertible systems $[E, A, B, C] \in$

$\Sigma_{l,n,m,m}$ with asymptotically stable zero dynamics for which Γ in (5.1.2) exists and satisfies $\Gamma = \Gamma^\top \geq 0$. The proof of this result exploits that systems within this class have the high-gain property.

In the context of vector relative degree, as discussed in Section 5.3 for regular systems, the above assumption of existence of Γ as in (5.1.2) turns out to be satisfied, provided that the vector relative degree is componentwise smaller or equal to one. High-gain and funnel control for regular system with positive strict relative degree and positive definite high-frequency gain matrix is studied as well in Section 5.3.

The funnel control results of this chapter are illustrated by a simulation of position and velocity control of a 'real world' mechanical system in Subsections 5.2.2 and 5.3.3. To point out the peculiarities of the singular case, a simulation of an academic example is also included in Subsection 5.2.2.

Parts of Sections 5.1 and 5.2 and Subsection 5.3.1 are contained in the manuscript [28] and the short note [27], which are submitted for publication. The results of Subsection 5.3.2 were obtained in a joint work with Achim Ilchmann and Timo Reis [34], which also contains the simulations of the mechanical system from Subsections 5.2.2 and 5.3.3.

5.1 High-gain control

In this section we consider constant high-gain proportional output feedback given by

$$u(t) = -k\, y(t) \tag{5.1.1}$$

applied to DAE systems $[E, A, B, C] \in \Sigma_{l,n,m,m}$. It is our aim to derive conditions which guarantee high-gain stabilizability.

Definition 5.1.1 (High-gain stabilizability)**.**
A system $[E, A, B, C] \in \Sigma_{l,n,m,m}$ is called *high-gain stabilizable* if, and only if,

$$\exists\, k^* \geq 0\ \forall\, k \geq k^*\ \forall\, x \in \mathfrak{B}_{[E,A-kBC]} \cap \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) :\ \lim_{t\to\infty} x(t) = 0.$$

Roughly speaking, in view of Definition 5.1.1, a system $[E, A, B, C] \in \Sigma_{l,n,m,m}$ is high-gain stabilizable if, and only if, the closed-loop system (4.1.1), (5.1.1), that is $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = (A - kBC)x(t)$, is asymptotically stable.

In order to prove high-gain stabilizability for a certain class of systems, we derive a simplification of the form (4.2.3) under the condition that for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ the matrix

$$\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)\begin{bmatrix} 0 \\ I_p \end{bmatrix} \in \mathbb{R}^{m\times p} \tag{5.1.2}$$

exists. This simplified form then provides an integral-differential-algebraic equation as shown in Lemma 5.1.3. Note that by Lemma 4.3.2 we have that if Γ in (5.1.2) exists, then it does not depend on the choice of $L(s)$.

For single-input, single-output systems with transfer function $g(s) = \frac{p(s)}{q(s)} \in \mathbb{R}(s)\setminus\{0\}$, the matrix Γ in (5.1.2) exists if, and only if, $\deg q(s) - \deg p(s) \le 1$, i.e., $g(s)$ has strict relative degree smaller or equal to one (cf. Section 5.3). For a connection of the existence of Γ in (5.1.2) to the (vector) relative degree of the system $[E, A, B, C]$ see Subsection 5.3.1.

Now we investigate the consequences of the assumption of existence of Γ in (5.1.2).

Lemma 5.1.2 (Consequences of existence of Γ)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ be right-invertible and have autonomous zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (5.1.2) exists. Then, using the notation from Theorem 4.2.7 and Definition 4.2.6, we have

$$\forall\, k = 0, \ldots, \nu-1 : \; E_{23}N^k E_{32} = 0. \tag{5.1.3}$$

and, furthermore, $\Gamma = E_{22}$.

Proof: The left inverse $L(s)$ is given in (4.3.2) and Γ is independent of the choice of $L(s)$ by Lemma 4.3.2. By existence of Γ the matrix $s^{-1}[0, I_m]L(s)[0, I_p]^\top$ is proper, which implies that

$$s^{-1}X_{45}(s) = -(E_{22} - s^{-1}A_{22}) + s^{-1}A_{21}(sI-Q)^{-1}A_{12} + sE_{23}(sN-I)^{-1}E_{32}$$

is proper. Hence, $sE_{23}(sN-I)^{-1}E_{32} = \sum_{k=0}^{\nu-1} E_{23}N^kE_{32}s^{k+1}$ has to be proper. This yields (5.1.3) and the last statement is then an immediate consequence of $\Gamma = -\lim_{s\to\infty} s^{-1}X_{45}(s) = E_{22}$. □

The following simplification of the form (4.2.3) relies on partially solving the equations (4.2.5) using the condition (5.1.3) derived in Lemma 5.1.2. This form is used in the proofs of Theorems 5.1.4 and 5.2.3.

Lemma 5.1.3 (Behavior and underlying equations).
Let $[E, A, B, C] \in \Sigma_{l,n,m,p}$ *be right-invertible and have autonomous zero dynamics. Suppose that, for a left inverse* $L(s)$ *of* $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ *over* $\mathbb{R}(s)$*, the matrix* Γ *in* (5.1.2) *exists. Then, using the notation from Theorem 4.2.7 and Definition 4.2.6, for any* $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \cap \big(\mathcal{C}^1(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}^0(\mathbb{R};\mathbb{R}^m) \times \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^p)\big)$ *and* $Tx = (x_1^\top, y^\top, x_3^\top)^\top \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{k+p+n_3})$*,* (Tx, u, y) *solves*

$$\boxed{\begin{array}{rcl} \dot{x}_1(t) &=& Q\,x_1(t) + A_{12}\,y(t) \\ \Gamma\,\dot{y}(t) &=& A_{22}\,y(t) + \Psi(x_1(0), y)(t) + u(t) \\ x_3(t) &=& \sum_{k=0}^{\nu-1} N^k E_{32}\,y^{(k+1)}(t), \\ 0 &=& 0, \end{array}} \tag{5.1.4}$$

where

$$\begin{aligned} \Psi : \mathbb{R}^k \times \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^m) &\to \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m), \\ (x_1^0, y) &\mapsto \left(t \mapsto A_{21}e^{Qt}x_1^0 + \int_0^t A_{21}e^{Q(t-\tau)}A_{12}\,y(\tau)\,\mathrm{d}\tau\right). \end{aligned}$$

Ψ *is linear in each argument and, if* $\sigma(Q) \subseteq \mathbb{C}_-$*, then* Ψ *has the property*

$$\Psi\big(\mathbb{R}^k \times (\mathcal{L}^\infty(\mathbb{R};\mathbb{R}^p) \cap \mathcal{C}^\nu(\mathbb{R};\mathbb{R}^p))\big) \subseteq \mathcal{L}^\infty(\mathbb{R};\mathbb{R}^m) \cap \mathcal{C}^{\nu+1}(\mathbb{R};\mathbb{R}^m). \tag{5.1.5}$$

Proof: The assumptions of Theorem 4.2.7 are satisfied and, invoking Lemma 5.1.2, it is clear that the respective first and third equations in (4.2.5) and (5.1.4) coincide. By Proposition 4.2.12 and right-invertibility of $[E, A, B, C]$, the fourth equation in (4.2.5) reads $0 = 0$. Therefore, it remains to show that under the additional assumption of existence of Γ, the second equation in (5.1.4) follows from (4.2.5). To this end, observe that by Lemma 5.1.2, namely (5.1.3), the second equation in (4.2.5) reads

$$E_{22}\dot{y}(t) = A_{22}y(t) + A_{21}x_1(t) + u(t). \tag{5.1.6}$$

Insertion of the solution of the first equation in (4.2.5) into (5.1.6) then yields the assertion.

Statement (5.1.5) about Ψ is obvious from the representation of Ψ and the fact that if $\sigma(Q) \subseteq \mathbb{C}_-$, then there exist $\mu, M > 0$ such that

$$\forall\, t \geq 0: \; \|e^{Qt}\| \leq M e^{-\mu t}. \qquad \square$$

We are now in the position to prove the main result of this section.

Theorem 5.1.4 (High-gain control).
Let $[E,A,B,C] \in \Sigma_{l,n,m,m}$ be right-invertible and have autonomous zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (5.1.2) exists and satisfies $\Gamma = \Gamma^\top \geq 0$. Then

$$\mathcal{ZD}_{[E,A,B,C]} \text{ is asymptotically stable} \quad \Longrightarrow \quad [E,A,B,C] \text{ is high-gain stabilizable.}$$

The converse implication is in general false even for ODE systems.

Proof: We prove "⇒". By virtue of Lemma 5.1.3 we may without loss of generality consider $[E,A,B,C]$ in the form (5.1.4) with $A_{21}x_1 = \Psi(x_1(0), y)$ and $x = T^{-1}(x_1^\top, y^\top, x_3^\top)^\top \in \mathcal{C}^1(\mathbb{R};\mathbb{R}^{k+p+n_3})$. Application of the high-gain controller (5.1.1) leads to

$$\begin{aligned} \dot{x}_1(t) &= Qx_1(t) + A_{12}y(t) \\ \Gamma\dot{y}(t) &= A_{21}x_1(t) + (A_{22} - kI_m)y(t) \\ x_3(t) &= \textstyle\sum_{k=0}^{\nu-1} N^k E_{32} y^{(k+1)}(t). \end{aligned} \tag{5.1.7}$$

Since $\Gamma = \Gamma^\top \geq 0$, there exists an orthogonal matrix $V \in \mathbf{Gl}_m(\mathbb{R})$ and a diagonal matrix $D \in \mathbb{R}^{m_1 \times m_1}$ with only positive entries for some $0 \leq m_1 \leq m$, such that

$$\Gamma = V^\top \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V.$$

In order to decouple the second equation in (5.1.7) we introduce the new variables $y_1(\cdot) = [I_{m_1}, 0]Vy(\cdot)$ and $y_2(\cdot) = [0, I_{m-m_1}]Vy(\cdot)$. Rewriting (5.1.7) and using $A_{12}V = [\tilde{A}_1, \tilde{A}_2] \in \mathbb{R}^{k \times m_1} \times \mathbb{R}^{k \times (m-m_1)}$ and

$$V^\top A_{22} V = \begin{bmatrix} \hat{A}_{11} & \hat{A}_{12} \\ \hat{A}_{21} & \hat{A}_{22} \end{bmatrix}, \qquad \hat{A}_{11} \in \mathbb{R}^{m_1 \times m_1},\, \hat{A}_{22} \in \mathbb{R}^{(m-m_1)\times(m-m_1)},$$

this leads to the system, without the third equation in (5.1.7),

$$\begin{aligned}
\dot{x}_1(t) &= Qx_1(t) + \tilde{A}_1 y_1(t) + \tilde{A}_2 y_2(t) \\
\dot{y}_1(t) &= [D^{-1}, 0]V^\top A_{21} x_1(t) + D^{-1}(\hat{A}_{11} - kI_{m_1})y_1(t) + D^{-1}\hat{A}_{12} y_2(t) \\
0 &= [0, I_{m-m_1}]V^\top A_{21} x_1(t) + \hat{A}_{21} y_1(t) + (\hat{A}_{22} - kI_{m-m_1})y_2(t).
\end{aligned}$$

Assuming that $k \geq \|\hat{A}_{22}\|$ we may solve the third equation for y_2, i.e.,

$$y_2(t) = -(\hat{A}_{22} - kI_{m-m_1})^{-1}\Big([0, I_{m-m_1}]V^\top A_{21} x_1(t) + \hat{A}_{21} y_1(t)\Big), \tag{5.1.8}$$

and insert this into the first and second equation, thus obtaining

$$\begin{aligned}
\dot{x}_1(t) &= \big(Q - \tilde{A}_2(\hat{A}_{22} - kI_{m-m_1})^{-1}[0, I_{m-m_1}]V^\top A_{21}\big)x_1(t) \\
&\quad + \big(\tilde{A}_1 - \tilde{A}_2(\hat{A}_{22} - kI_{m-m_1})^{-1}\hat{A}_{21}\big)y_1(t) \\
\dot{y}_1(t) &= \big([D^{-1}, 0]V^\top A_{21} - D^{-1}\hat{A}_{12}(\hat{A}_{22} - kI_{m-m_1})^{-1} \\
&\quad \times [0, I_{m-m_1}]V^\top A_{21}\big)x_1(t) + \big(D^{-1}(\hat{A}_{11} - kI_{m_1}) \\
&\quad - D^{-1}\hat{A}_{12}(\hat{A}_{22} - kI_{m-m_1})^{-1}\hat{A}_{21}\big)y_1(t).
\end{aligned} \tag{5.1.9}$$

In order to show asymptotic stability of (5.1.9) we use a Lyapunov function approach. By $\sigma(Q) \subseteq \mathbb{C}_-$ and [212, Thm. 7.10] there exists a solution $P \in \mathbf{Gl}_k(\mathbb{R})$ with $P = P^\top$ and

$$\beta_1 I_k \leq P \leq \beta_2 I_k, \qquad \beta_1, \beta_2 > 0, \tag{5.1.10}$$

of the Lyapunov equation

$$Q^\top P + PQ = -I_k.$$

Define the Lyapunov function

$$V : \mathbb{R}^k \times \mathbb{R}^{m_1} \to \mathbb{R},\ (x_1, y_1) \mapsto x_1^\top P x_1 + y_1^\top y_1.$$

Then, for any solution $(x_1, y_1) \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^k \times \mathbb{R}^{m_1})$ of (5.1.9) and all

$t \in \mathbb{R}$, we have

$$\begin{aligned}
&\tfrac{\mathrm{d}}{\mathrm{d}t} V(x_1(t), y_1(t)) \\
&\quad = 2x_1(t)^\top P \Big(\big(Q - \tilde{A}_2(\hat{A}_{22} - kI_{m-m_1})^{-1}[0, I_{m-m_1}]V^\top A_{21}\big) x_1(t) \\
&\qquad + \big(\tilde{A}_1 - \tilde{A}_2(\hat{A}_{22} - kI_{m-m_1})^{-1}\hat{A}_{21}\big) y_1(t) \Big) \\
&\qquad + 2y_1(t)^\top \Big(D^{-1}\big([I_{m_1}, 0]V^\top A_{21} - \hat{A}_{12}(\hat{A}_{22} - kI_{m-m_1})^{-1} \\
&\qquad \times [0, I_{m-m_1}]V^\top A_{21}\big) x_1(t) + D^{-1}\big((\hat{A}_{11} - kI_{m_1}) \\
&\qquad - \hat{A}_{12}(\hat{A}_{22} - kI_{m-m_1})^{-1}\hat{A}_{21}\big) y_1(t) \Big).
\end{aligned}$$

Observe that there exist positive constants $M_1, \ldots, M_5$ such that the following inequalities hold:

$$\begin{aligned}
&-2x_1(t)^\top P\tilde{A}_2(\hat{A}_{22} - kI_{m-m_1})^{-1}[0, I_{m-m_1}]V^\top A_{21}x_1(t) \\
&\qquad \le M_1 \|(\hat{A}_{22} - kI_{m-m_1})^{-1}\| \|x_1(t)\|^2 \le \frac{M_1}{k - \|\hat{A}_{22}\|}\|x_1(t)\|^2,
\end{aligned}$$

$$\begin{aligned}
&-2x_1(t)^\top P\tilde{A}_2(\hat{A}_{22} - kI_{m-m_1})^{-1}\hat{A}_{21}y_1(t) \\
&\quad \le \frac{M_2}{k - \|\hat{A}_{22}\|}\|x_1(t)\| \cdot \|y_1(t)\| \le \frac{M_2}{k - \|\hat{A}_{22}\|}(\|x_1(t)\|^2 + \|y_1(t)\|^2),
\end{aligned}$$

$$2x_1(t)^\top P\tilde{A}_1 y_1(t) + 2y_1(t)^\top [D^{-1}, 0]V^\top A_{21}x_1(t) \le \frac{1}{2}\|x_1(t)\|^2 + M_3\|y_1(t)\|^2,$$

$$\begin{aligned}
&-2y_1(t)^\top D^{-1}\hat{A}_{12}(\hat{A}_{22} - kI_{m-m_1})^{-1}[0, I_{m-m_1}]V^\top A_{21}x_1(t) \\
&\qquad \le \frac{M_4}{k - \|\hat{A}_{22}\|}(\|x_1(t)\|^2 + \|y_1(t)\|^2),
\end{aligned}$$

$$-2y_1(t)^\top D^{-1}\hat{A}_{12}(\hat{A}_{22} - kI_{m-m_1})^{-1}\hat{A}_{21}y_1(t) \le \frac{M_5}{k - \|\hat{A}_{22}\|}\|y_1(t)\|^2.$$

Furthermore, since D^{-1} is positive definite, there exist $\alpha > 0$ and $M_6 > 0$ such that

$$2y_1(t)^\top D^{-1}(\hat{A}_{11} - kI_{m_1})y_1(t) \le M_6\|y_1(t)\|^2 - \alpha k\|y_1(t)\|^2.$$

Incorporating these inequalities we find that

$$\tfrac{\mathrm{d}}{\mathrm{d}t}V(x_1(t),y_1(t)) \le \Bigg(\underbrace{\frac{K_1}{k-\|\hat{A}_{22}\|}-\frac{1}{2}}_{=:\gamma_1(k)}\Bigg)\|x_1(t)\|^2 + \Bigg(\underbrace{K_2+\frac{K_1}{k-\|\hat{A}_{22}\|}-\alpha k}_{=:\gamma_2(k)}\Bigg)\|y_1(t)\|^2.$$

Now let $k^* > \|\hat{A}_{22}\|$ be such that $\gamma_1 := -\gamma_1(k^*) > 0$ and $\gamma_2 := -\gamma_2(k^*) > 0$, and define $\gamma := \min\left\{\frac{\gamma_1}{\beta_2}, \gamma_2\right\} > 0$. Then, for all $k \ge k^*$,

$$\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}V(x_1(t),y_1(t)) &\overset{(5.1.10)}{\le} -\gamma\left(x_1(t)^\top P x_1(t) + y_1(t)^\top y_1(t)\right)\\ &= -\gamma V(x_1(t),y_1(t)).\end{aligned}$$

By integrating both sides of the inequality and using (5.1.10) we may conclude that

$$\lim_{t\to\infty} x_1(t) = 0 \quad\wedge\quad \lim_{t\to\infty} y_1(t) = 0.$$

Invoking (5.1.8), this implies $\lim_{t\to\infty} y_2(t) = 0$ and, since y_1 and y_2 decay exponentially, $\lim_{t\to\infty} x_3(t) = 0$. Therefore,

$$\lim_{t\to\infty} x(t) = \lim_{t\to\infty} T^{-1}(x_1^\top, y^\top, x_3^\top)^\top = 0.$$

We have shown that $[E,A,B,C]$ is high-gain stabilizable.

To see that "$\Leftarrow$" does, in general, not hold true, consider system (4.1.1) for

$$E = I, \quad A = \begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}, \quad B = \begin{bmatrix}0\\ 1\end{bmatrix}, \quad C = [0,1], \tag{5.1.11}$$

which is in form (4.2.3) and in particular $[E,A,B,C]$ is right-invertible with autonomous zero dynamics. We may observe that $Q = 0$ and therefore the zero dynamics of (5.1.11) are not asymptotically stable. However, the closed-loop system (5.1.11), (5.1.1) takes the form

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x(t) = A_k x(t) = \begin{bmatrix}0 & -1\\ 1 & -k\end{bmatrix} x(t),$$

which is, since $\sigma(A_k) = \left\{-k/2 \pm \sqrt{k^2/4\ - 1}\right\}$, asymptotically stable for all $k > 0$. This shows that $[E, A, B, C]$ is high-gain stabilizable. □

5.2 Funnel control

In this section we show that funnel control is feasible for the class of right-invertible systems $[E, A, B, C] \in \Sigma_{l,n,m,m}$ with asymptotically stable zero dynamics for which Γ in (5.1.2) exists and satisfies $\Gamma = \Gamma^\top \geq 0$. It was shown in Section 5.1 that this class has the high-gain property - this is exploited here. We conclude the section with some illustrative simulations.

5.2.1 Main result

We consider funnel control for systems $[E, A, B, C] \in \Sigma_{l,n,m,m}$. For a motivation of funnel control we consider some classical control strategies: One possibility is constant high-gain control (5.1.1) as considered in Section 5.1. High-gain stabilization can be achieved under the assumptions of Theorem 5.1.4. However, it is not known a priori how large the high gain constant must be and the transient behavior is not taken into account.

An alternative strategy, that is adaptive high-gain control (1.4.2) as discussed in Section 1.4, has the advantage that it resolves the above mentioned problem by adaptively increasing the high gain. However, the gain function is monotonically increasing and potentially so large that the input is very sensitive to output perturbations.

To overcome these drawbacks, the concept of 'funnel control' is introduced (see [123] and the references therein): For any function φ belonging to

$$\Phi^\mu := \left\{ \varphi \in \mathcal{C}^\mu(\mathbb{R}_{\geq 0};\mathbb{R}) \cap \mathcal{B}^1(\mathbb{R}_{\geq 0};\mathbb{R}) \;\middle|\; \begin{array}{l} \varphi(0) = 0,\ \ \varphi(s) > 0 \\ \text{for all } s > 0 \text{ and} \\ \liminf_{s\to\infty} \varphi(s) > 0 \end{array} \right\}$$

for $\mu \in \mathbb{N}$, we associate the *performance funnel*

$$\mathcal{F}_\varphi := \left\{ (t, e) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \;\middle|\; \varphi(t)\|e\| < 1 \right\}, \tag{5.2.1}$$

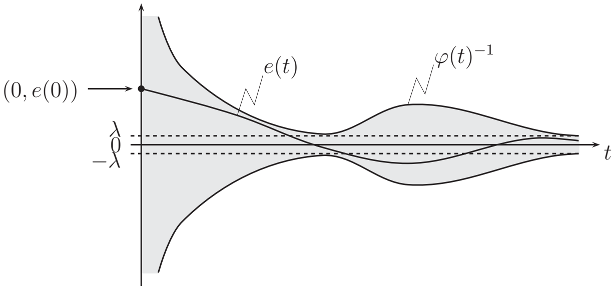


Figure 5.1: *Error evolution in a funnel $\mathcal{F}_\varphi$ with boundary $\varphi(t)^{-1}$ which has a pole at $t = 0$.*

see Figure 5.1. The control objective is feedback control so that the tracking error $e = y - y_{\text{ref}}$, where y_{ref} is the reference signal, evolves within $\mathcal{F}_\varphi$ and all variables are bounded. More specific, the transient behavior is supposed to satisfy

$$\forall\, t > 0: \ \|e(t)\| < \varphi(t)^{-1},$$

and, moreover, if φ is chosen so that $\varphi(t) \geq 1/\lambda$ for all t sufficiently large, then the tracking error remains smaller than λ.

By choosing $\varphi(0) = 0$ we ensure that the width of the funnel is infinity at $t = 0$, see Figure 5.1. In the following we only treat 'infinite' funnels for technical reasons, since if the funnel is finite, that is $\varphi(0) > 0$, then we need to assume that the initial error is within the funnel boundaries at $t = 0$, i.e., $\varphi(0)\|Cx^0 - y_{\text{ref}}(0)\| < 1$, and this assumption suffices.

As indicated in Figure 5.1, we do not assume that the funnel boundary decreases monotonically. Certainly, in most situations it is convenient to choose a monotone funnel, however there are situations where widening the funnel at some later time might be beneficial, e.g., when it is known that the reference signal varies strongly.

To ensure error evolution within the funnel, we introduce, for $\hat{k} > 0$, the *funnel controller*:

$$\boxed{\begin{aligned} u(t) &= -k(t)\, e(t), \qquad \text{where} \qquad e(t) = y(t) - y_{\text{ref}}(t) \\ k(t) &= \frac{\hat{k}}{1 - \varphi(t)^2\|e(t)\|^2}\,. \end{aligned}} \tag{5.2.2}$$

If we assume asymptotically stable zero dynamics, we see intuitively that, in order to maintain the error evolution within the funnel, high gain values may only be required if the norm $\|e(t)\|$ of the error is close to the funnel boundary $\varphi(t)^{-1}$: k increases if necessary to exploit the high-gain property of the system and decreases if a high gain is not necessary. This intuition underpins the choice of the gain $k(t)$ in (5.2.2), where the constant $\hat{k} > 0$ is only of technical importance, see Remark 5.2.1. The control design (5.2.2) has two advantages: k is non-monotone and (5.2.2) is a static time-varying proportional output feedback of striking simplicity.

We will show that funnel control for systems (4.1.1) is feasible under the following structural assumptions:

- $[E, A, B, C]$ has asymptotically stable zero dynamics,
- $[E, A, B, C]$ is right-invertible,
- the matrix

$$\Gamma = -\lim_{s\to\infty} s^{-1}[0, I_m]L(s)\begin{bmatrix} 0 \\ I_m \end{bmatrix} \in \mathbb{R}^{m\times m} \tag{5.2.3}$$

  exists and satisfies $\Gamma = \Gamma^\top \geq 0$, where $L(s)$ denotes a left inverse of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$,
- $\hat{k}$ in (5.2.2) satisfies

$$\hat{k} > \left\| \lim_{s\to\infty} \left([0, I_m]L(s)\begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma \right) \right\|. \tag{5.2.4}$$

Note that by Lemma 4.3.2, Γ is independent of the choice of $L(s)$.

The condition (5.2.4) on $\hat{k}$ seems undesirable, since at first glance it is not known how large $\hat{k}$ must be chosen; this is just the drawback of high-gain control that we seek to avoid by the introduction of funnel control. However, condition (5.2.4) turns out to be structural, since $-[0, I_m]L(s)[0, I_m]^\top$ is a generalization of the inverse transfer function (cf. Remark 4.3.3) and we only need a bound for the norm of the constant coefficient in its Laurent series. In several important cases it is indeed possible to calculate the bound explicitly, see Remark 5.2.1.

Remark 5.2.1 (Initial gain)**.**
The condition (5.2.4) is specific for DAEs and already appears in [34, 35], but not in the ODE case, see [125]. A careful inspection of the proof of Theorem 5.2.3 shows that we have to ensure that the matrix $\hat{A}_{22}-k(t)I_m$ is invertible for all $t \geq 0$, and in order to avoid singularities we choose, as a simple condition, the 'minimal value' $\hat{k}$ of k to be larger than $\|A_{22}\| \geq \|\hat{A}_{22}\|$. We perform the calculation of the lower bound for $\hat{k}$ in (5.2.4) for some classes of ODEs: Consider the system

$$\begin{aligned} \dot{x}(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t)\,, \end{aligned} \tag{5.2.5}$$

where $A \in \mathbb{R}^{n\times n}, B, C^\top \in \mathbb{R}^{n\times m}, D \in \mathbb{R}^{m\times m}$. System (5.2.5) can be rewritten in the form (4.1.1) as

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} &= \begin{bmatrix} A & 0 \\ 0 & -I_m \end{bmatrix}\begin{pmatrix} x(t) \\ w(t) \end{pmatrix} + \begin{bmatrix} B \\ D \end{bmatrix} u(t) \\ y(t) &= [C, I]\begin{pmatrix} x(t) \\ w(t) \end{pmatrix}. \end{aligned}$$

Observe that $s\left[\begin{smallmatrix} I_n & 0 \\ 0 & 0 \end{smallmatrix}\right] - \left[\begin{smallmatrix} A & 0 \\ 0 & -I_m \end{smallmatrix}\right]$ is regular, and hence applying Remark 4.3.3 gives

$$\begin{aligned} \Gamma = \lim_{s\to\infty} s^{-1}\left([C, I]\begin{bmatrix} sI - A & 0 \\ 0 & I \end{bmatrix}^{-1}\begin{bmatrix} B \\ D \end{bmatrix}\right)^{-1} \\ = \lim_{s\to\infty} s^{-1}\left(C(sI-A)^{-1}B + D\right)^{-1}. \end{aligned}$$

Assume now that $D \in \mathbf{Gl}_m(\mathbb{R})$, i.e., the system has strict relative degree 0 (cf. Section 5.3). Then

$$\Gamma = \lim_{s\to\infty} s^{-1}D^{-1}\sum_{k=0}^{\infty}\left(-D^{-1}C(sI-A)^{-1}B\right)^k = 0,$$

and

$$\begin{aligned} \lim_{s\to\infty}\left([0, I_m]L(s)\begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma\right) \\ = -\lim_{s\to\infty} D^{-1}\sum_{k=0}^{\infty}\left(-D^{-1}C(sI-A)^{-1}B\right)^k = -D^{-1}. \end{aligned}$$

Therefore, (5.2.4) reads $\hat{k} > \|D^{-1}\|$. If $D = 0$ and $CB \in \mathbf{Gl}_m(\mathbb{R})$, i.e., the system has strict relative degree 1 (cf. Section 5.3), then similar calculations lead to $\Gamma = \big(CB\big)^{-1}$ and (5.2.4) simply reads $\hat{k} > 0$; the latter is a general condition compared to the choice $\hat{k} = 1$ in [125].

For single-input, single-output systems the above conditions can also be motivated by just looking at the output equation $y = cx + du$, $c^\top \in \mathbb{R}^n, d \in \mathbb{R}$. If a feedback $u = -ky$, $k > 0$ is applied, then $(1 + dk)y = cx$ and in order to solve this equation for y it is sufficient that either $k > 0$ (no further condition) if $d = 0$, or $k > |d^{-1}|$ if $d \neq 0$.

Before we state our main result, we need to define consistency of the initial value of the closed-loop system and solutions of the latter. Compared to Chapters 2-4, here we require more smoothness of the trajectories.

Definition 5.2.2 (Consistent initial value)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,m}$, $\varphi \in \Phi^1$ and $y_{\text{ref}} \in \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. An initial value $x^0 \in \mathbb{R}^n$ is called *consistent* for the closed-loop system (4.1.1), (5.2.2) if, and only if, there exists a solution of the initial value problem (4.1.1), (5.2.2), $x(0) = x^0$, i.e., a function $x \in \mathcal{C}^1([0, \omega); \mathbb{R}^n)$ for some $\omega \in (0, \infty]$, such that $x(0) = x^0$ and x satisfies (4.1.1), (5.2.2) for all $t \in [0, \omega)$.

Note that, in practice, consistency of the initial state of the 'unknown' system should be satisfied as far as the DAE $[E, A, B, C]$ is the correct model.

We are now in a position to state the main result of this section.

Theorem 5.2.3 (Funnel control)**.**
Let $[E, A, B, C] \in \Sigma_{l,n,m,m}$ be right-invertible and have asymptotically stable zero dynamics. Suppose that, for a left inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$, the matrix Γ in (5.2.3) *exists and satisfies $\Gamma = \Gamma^\top \geq 0$. Let $\hat{k} > 0$ be such that* (5.2.4) *is satisfied. Using the notation from Theorem 4.2.7 and Definition 4.2.6, let $\varphi \in \Phi^{\nu+1}$ define a performance funnel $\mathcal{F}_\varphi$.*

Then, for any reference signal $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ and any consistent initial value $x^0 \in \mathbb{R}^n$, the application of the funnel controller (5.2.2) *to* (4.1.1) *yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution x,*

(i) *x is bounded and the corresponding tracking error $e = Cx - y_{\text{ref}}$ evolves uniformly within the performance funnel $\mathcal{F}_\varphi$; more precisely,*

$$\exists\, \varepsilon > 0\ \forall\, t > 0 :\ \|e(t)\| \le \varphi(t)^{-1} - \varepsilon\,. \tag{5.2.6}$$

(ii) *the corresponding gain function k given by* (5.2.2) *is bounded:*

$$\forall\, t > 0 :\ k(t) \le \frac{\hat{k}}{1 - (1 - \varphi(t)\varepsilon)^2}.$$

Proof: Note that Γ is well-defined by Lemma 4.3.2. We proceed in several steps.

Step 1: By Lemma 5.1.3, the closed-loop system (4.1.1), (5.2.2) is, without loss of generality, in the form

$$\begin{array}{rcl} \dot{x}_1(t) & = & Q\, x_1(t) + A_{12}\, e(t) + A_{12}\, y_{\text{ref}}(t) \\ \Gamma\, \dot{e}(t) & = & (A_{22} - k(t) I_m)\, e(t) + A_{22}\, y_{\text{ref}}(t) - \Gamma\, \dot{y}_{\text{ref}}(t) + \tilde{\Psi}(x_1^0, e)(t) \\ x_3(t) & = & \sum_{k=0}^{\nu-1} N^k E_{32}\, e^{(k+1)}(t) + \sum_{k=0}^{\nu-1} N^k E_{32}\, y_{\text{ref}}^{(k+1)}(t) \\ k(t) & = & \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2}, \end{array} \tag{5.2.7}$$

where $x_1^0 = [I_k, 0, 0] T^{-1} x^0$ and

$$\tilde{\Psi}(x_1^0, e)(t) = \Psi(x_1^0, e)(t) + \Psi(x_1^0, y_{\text{ref}})(t) - A_{21} e^{Qt} x_1^0, \quad t \ge 0.$$

Note that, as $[0, I_m] L(s) [0, I_m]^\top = X_{45}(s)$ for the representation in (4.3.2),

$$\hat{k} > \left\| \lim_{s \to \infty} \left([0, I_m] L(s) [0, I_m]^\top + s\Gamma \right) \right\| = \|A_{22}\|.$$

By consistency of the initial value x^0 there exists a local solution $(x_1, e, x_3, k) \in \mathcal{C}^1([0, \rho); \mathbb{R}^{n+1})$ of (5.2.7) for some $\rho > 0$ and initial data

$$(x_1, e, x_3, k)(0) = \begin{pmatrix} T^{-1} x^0 - \begin{pmatrix} 0 \\ y_{\text{ref}}(0) \\ 0 \end{pmatrix} \\ \hat{k} \end{pmatrix},$$

where the differentiability follows since $y_{\rm ref} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0};\mathbb{R}^m)$ and $\varphi \in \mathcal{C}^{\nu+1}(\mathbb{R}_{\geq 0};\mathbb{R})$. It is clear that $(t,e(t))$ belongs to the set $\mathcal{F}_\varphi$ for all $t \in [0,\rho)$. Even more so, we have that

$$\begin{aligned}\forall\, t \in [0,\rho):\ & (t, x_1(t), e(t), x_3(t), k(t)) \\ & \in \tilde{\mathcal{D}} := \left\{ (t,x_1,e,x_3,k) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^{n+1} \;\middle|\; \varphi(t)\|e\| < 1 \right\}.\end{aligned}$$

We will now, for the time being, ignore the first and third equation in (5.2.7) and construct an integral-differential equation from the second and fourth equation, which is solved by (e,k). To this end, observe that by $\Gamma = \Gamma^\top \geq 0$, there exists an orthogonal matrix $V \in \mathbf{Gl}_m(\mathbb{R})$ and a diagonal matrix $D \in \mathbb{R}^{m_1 \times m_1}$ with only positive entries for some $0 \leq m_1 \leq m$, such that

$$\Gamma = V^\top \begin{bmatrix} D & 0 \\ 0 & 0 \end{bmatrix} V.$$

In order to decouple the second equation in (5.2.7) into an ODE and an algebraic equation, we introduce the new variables $e_1(\cdot) = [I_{m_1}, 0]Ve(\cdot)$ and $e_2(\cdot) = [0, I_{m-m_1}]Ve(\cdot)$. Rewriting (5.2.7) and invoking $\|e(t)\|^2 = \|Ve(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2$, this leads to the system

$$\begin{aligned} \dot{e}_1(t) &= [D^{-1}, 0](VA_{22}V^\top - k(t)I_m) \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} \\ &\quad + [D^{-1}, 0]V\,\Theta_1(e_1,e_2)(t) \\ 0 &= [0, I_{m-m_1}]VA_{22}V^\top \begin{pmatrix} e_1(t) \\ e_2(t) \end{pmatrix} - k(t)e_2(t) \\ &\quad + [0, I_{m-m_1}]V\,\Theta_1(e_1,e_2)(t) \\ k(t) &= \tfrac{\hat{k}}{1-\varphi(t)^2(\|e_1(t)\|^2 + \|e_2(t)\|^2)}, \end{aligned} \tag{5.2.8}$$

on $\mathbb{R}_{\geq 0}$ where

$$\begin{aligned} \Theta_1 : \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^{m_1}) \times \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^{m-m_1}) &\to \mathcal{C}^{\nu+1}(\mathbb{R}_{\geq 0};\mathbb{R}^m), \\ (e_1,e_2) &\mapsto \Big(t \mapsto A_{22}\,y_{\rm ref}(t) - \Gamma\,\dot{y}_{\rm ref}(t) + \tilde{\Psi}(x_1^0, V^\top(e_1^\top, e_2^\top)^\top)(t)\Big). \end{aligned}$$

Introduce the set

$$\mathcal{D} := \left\{ \begin{array}{l} (t,k,e_1,e_2) \in \\ \mathbb{R}_{\geq 0} \times [\hat{k},\infty) \times \mathbb{R}^{m_1} \times \mathbb{R}^{m-m_1} \end{array} \;\middle|\; \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2) < 1 \right\}$$

and define

$$f_1 : \mathcal{D} \times \mathbb{R}^m \to \mathbb{R}^{m_1},\ (t,k,e_1,e_2,\xi) \mapsto [D^{-1},0](VA_{22}V^\top - kI_m)\begin{pmatrix} e_1 \\ e_2 \end{pmatrix} + [D^{-1},0]V\,\xi.$$

By differentiation of the second equation in (5.2.8), and using

$$\hat{A}_{22} = [0, I_{m-m_1}]VA_{22}V^\top \begin{bmatrix} 0 \\ I_{m-m_1} \end{bmatrix}, \quad \hat{A}_{21} = [0, I_{m-m_1}]VA_{22}V^\top \begin{bmatrix} I_{m_1} \\ 0 \end{bmatrix},$$

we get

$$0 = \hat{A}_{21}\dot{e}_1(t) + \hat{A}_{22}\dot{e}_2(t) - \dot{k}(t)e_2(t) - k(t)\dot{e}_2(t) + [0, I_{m-m_1}]V\,\tfrac{\mathrm{d}}{\mathrm{d}t}\Theta_1(e_1,e_2)(t). \tag{5.2.9}$$

Observe that the derivative of k is given by

$$\begin{aligned} \dot{k}(t) &= 2\hat{k}\left(1 - \varphi(t)^2(\|e_1(t)\|^2 + \|e_2(t)\|^2)\right)^{-2} \\ &\times \left(\varphi(t)\dot{\varphi}(t)(\|e_1(t)\|^2 + \|e_2(t)\|^2) + \varphi(t)^2(e_1(t)^\top\dot{e}_1(t) + e_2(t)^\top\dot{e}_2(t))\right). \end{aligned} \tag{5.2.10}$$

Now let

$$\begin{aligned} M : \mathcal{D} \to \mathbf{Gl}_{m-m_1}(\mathbb{R}),\ (t,k,e_1,e_2) \mapsto& \\ \left(\hat{A}_{22} - k\left(I_{m-m_1} + 2\varphi(t)^2\left(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\right)^{-1} e_2e_2^\top\right)\right)&, \end{aligned}$$

$$\begin{aligned} \Theta_2 : \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^{m_1}) \times \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^{m-m_1}) &\to \mathcal{C}^\nu(\mathbb{R}_{\geq 0};\mathbb{R}^m), \\ (e_1,e_2) \mapsto \Bigg(t \mapsto A_{22}\,\dot{y}_{\mathrm{ref}}(t) - \Gamma\,\ddot{y}_{\mathrm{ref}}(t) &+ \tfrac{\mathrm{d}}{\mathrm{d}t}\Psi(x_1^0, y_{\mathrm{ref}})(t) \\ &+ \int_0^t A_{21}Qe^{Q(t-\tau)}A_{12}\,V^\top \begin{pmatrix} e_1(\tau) \\ e_2(\tau) \end{pmatrix}\mathrm{d}\tau\Bigg) \end{aligned}$$

and

$$\begin{aligned} f_2 : \mathcal{D} \times \mathbb{R}^{m_1} \times \mathbb{R}^m \to \mathbb{R}^{(m-m_1)},\ (t,k,e_1,e_2,\tilde{e}_1,\xi) \mapsto& \\ 2\hat{k}\left(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\right)^{-2}\left(\varphi(t)\dot{\varphi}(t)(\|e_1\|^2 + \|e_2\|^2)\right.& \\ \left. + \varphi(t)^2(e_1^\top\tilde{e}_1)\right)e_2 - \hat{A}_{21}\tilde{e}_1 - [0, I_{m-m_1}]V\,\left(A_{21}A_{12}V^\top(e_1^\top, e_2^\top)^\top + \xi\right)&. \end{aligned}$$

We show that M is well-defined. To this end let

$$G : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)},$$
$$(t, k, e_1, e_2) \mapsto 2\varphi(t)^2 \left(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\right)^{-2} e_2 e_2^\top$$

and observe that G is symmetric and positive semi-definite everywhere, hence there exist $\hat{V} : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)}$, $\hat{V}$ orthogonal everywhere, and $\hat{D} : \mathcal{D} \to \mathbb{R}^{(m-m_1)\times(m-m_1)}$, $\hat{D}$ a diagonal matrix with nonnegative entries everywhere, such that $G = \hat{V}^{-1}\hat{D}\hat{V}$. Therefore, $(I + G)^{-1} = \hat{V}^{-1}(I + \hat{D})^{-1}\hat{V}$ and $(I + \hat{D})^{-1}$ is diagonal with entries in $(0, 1]$ everywhere, which implies that $\|(I + G)^{-1}\| \le 1$. Then, for all $(t, k, e_1, e_2) \in \mathcal{D}$, we obtain

$$\|k^{-1}(I + G(t, k, e_1, e_2))^{-1}\hat{A}_{22}\| \le \hat{k}^{-1}\|\hat{A}_{22}\| \le \hat{k}^{-1}\|A_{22}\| < 1.$$

and hence $k^{-1}(I + G(t, e_1, e_2))^{-1}\hat{A}_{22} - I$ is invertible, which gives invertibility of

$$M(t, k, e_1, e_2) = \hat{A}_{22} - k(I + G(t, k, e_1, e_2)).$$

Now, inserting $\dot{k}$ from (5.2.10) into (5.2.9) and rearranging according to $\dot{e}_2$ gives

$$M\big(t, k(t), e_1(t), e_2(t)\big)\, \dot{e}_2(t) = f_2\big(t, k(t), e_1(t), e_2(t), \dot{e}_1(t), \Theta_2(e_1, e_2)(t)\big).$$

With

$$\tilde{f}_2 : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^{(m-m_1)},\ (t, k, e_1, e_2, \xi_1, \xi_2) \mapsto$$
$$M(t, k, e_1, e_2)^{-1} f_2(t, k, e_1, e_2, f_1(t, k, e_1, e_2, \xi_1), \xi_2),$$

and

$$f_3 : \mathcal{D} \times \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R},\ (t, k, e_1, e_2, \xi_1, \xi_2) \mapsto$$
$$2\hat{k}\left(1 - \varphi(t)^2(\|e_1\|^2 + \|e_2\|^2)\right)^{-2} \big(\varphi(t)\dot{\varphi}(t)(\|e_1\|^2 + \|e_2\|^2)$$
$$+ \varphi(t)^2(e_1^\top f_1(t, k, e_1, e_2, \xi_1) + e_2^\top \tilde{f}_2(t, k, e_1, e_2, \xi_1, \xi_2))\big)$$

we get the system

$$\begin{aligned}
\dot{e}_1(t) &= f_1(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t)) \\
\dot{e}_2(t) &= \tilde{f}_2(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)) \\
\dot{k}(t) &= f_3(t, k(t), e_1(t), e_2(t), \Theta_1(e_1, e_2)(t), \Theta_2(e_1, e_2)(t)).
\end{aligned} \tag{5.2.11}$$

$(k, e_1, e_2) \in \mathcal{C}^1([0, \rho); \mathbb{R}^{m+1})$ obtained from (e, k) is a local solution of (5.2.11) with

$$(k, e_1, e_2)(0) = \Big(\hat{k}, V([0, I_m, 0]T^{-1}x^0 - y_{\text{ref}}(0))\Big) =: \eta \qquad (5.2.12)$$

and

$$\forall\, t \in [0, \rho): \ (t, k(t), e_1(t), e_2(t)) \in \mathcal{D}.$$

Step 2: We show that the local solution (x_1, e, x_3, k) can be extended to a maximal solution, the graph of which leaves every compact subset of $\tilde{\mathcal{D}}$.
With $z = (k, e_1^\top, e_2^\top)^\top$ and appropriate $F : \mathcal{D} \times \mathbb{R}^{2m} \to \mathbb{R}^{m+1}$, we may write (5.2.11), (5.2.12) in the form

$$\dot{z}(t) = F(t, z(t), (Tz)(t)), \qquad z(0) = \eta, \qquad (5.2.13)$$

where $Tz = (\Theta_1(e_1, e_2)^\top, \Theta_2(e_1, e_2)^\top)^\top$ and $T : \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{m+1}) \to \mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^{2m})$ is an operator with the properties as in [124, Def. 2.1] (note that in [124] only operators with domain $\mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R})$ are considered, but the generalization to domain $\mathcal{C}(\mathbb{R}_{\geq 0}; \mathbb{R}^q)$ is straightforward). It is immediate that T satisfies properties (i)–(iii) in [124, Def. 2.1]; (iv) follows from the fact that $\sigma(Q) \subseteq \mathbb{C}_-$ by the asymptotically stable zero dynamics (cf. also (5.1.5)) and $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

Furthermore, for $\mu := \max\{1, \nu\}$ and the functions defined in Step 1, we find that f_1 and f_2 are μ-times continuously differentiable (since $\varphi \in \mathcal{C}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R})$). Furthermore, M is μ-times continuously differentiable and invertible on $\mathcal{D}$, hence M^{-1} is μ-times continuously differentiable as well. Finally, this gives that $\tilde{f}_2$ and f_3 are μ-times continuously differentiable and hence we have $F \in \mathcal{C}^\mu(\mathcal{D} \times \mathbb{R}^{2m}; \mathbb{R}^{m+1})$.

Let $\tilde{z} = (k, e_1^\top, e_2^\top)^\top \in \mathcal{C}^1([0, \rho); \mathbb{R}^{m+1})$ be the local solution of (5.2.11) obtained at the end of Step 1. Then $\tilde{z}$ solves (5.2.13). Observe that, since F is μ-times continuously differentiable and T is essentially an integral-operator, i.e., it increments the degree of differentiability, we have $\tilde{z} \in \mathcal{C}^{\mu+1}([0, \rho); \mathbb{R}^{m+1})$. Then [124, Thm. B.1][1] is applicable to the system (5.2.13) and we may conclude that

(a) there exists a solution of (5.2.13), i.e., a function $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ for some $\rho \in (0, \infty]$ such that z is locally absolutely continuous,

[1] In [124] a domain $\mathcal{D} \subseteq \mathbb{R}_{\geq 0} \times \mathbb{R}$ is considered, but the generalization to the higher dimensional case is only a technicality.

$z(0) = \eta$, $(t, z(t)) \in \mathcal{D}$ for all $t \in [0, \rho)$ and (5.2.13) holds for almost all $t \in [0, \rho)$,

(b) every solution can be extended to a maximal solution $z \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$, i.e., z has no proper right extension that is also a solution,

(c) if $z \in \mathcal{C}([0, \rho); \mathbb{R}^{m+1})$ is a maximal solution, then the closure of $\operatorname{graph} z$ is not a compact subset of $\mathcal{D}$.

Property (c) follows since F is locally essentially bounded, as it is at least continuously differentiable. Clearly $\tilde{z}$ is a solution (in the context of (a)) of (5.2.13), hence by (b) it can be extended to a maximal solution $\hat{z} \in \mathcal{C}([0, \omega); \mathbb{R}^{m+1})$. Similar to $\tilde{z}$, $\hat{z}$ is $(\mu + 1)$-times continuously differentiable.

We show that the extended solution $\hat{z}$ leads to an extended solution of (5.2.7). Clearly, $\hat{z}$ is a solution of (5.2.11). Integrating the equations for k and e_2 in (5.2.11) and invoking consistency of the initial values gives that (k, e_1, e_2) also solve the problem (5.2.8) and this leads to a maximal solution $(x_1, e, x_3, k) \in \mathcal{C}^1([0, \omega); \mathbb{R}^{n+1})$, $\omega \in (0, \infty]$, of (5.2.7) (extension of the original local solution (x_1, e, x_3, k) - for brevity we use the same notation) with $\operatorname{graph}(x_1, e, x_3, k) \subseteq \tilde{\mathcal{D}}$. Furthermore, by (c) we have

$$\text{the closure of } \operatorname{graph}(x_1, e, x_3, k) \text{ is not a compact subset of } \tilde{\mathcal{D}}. \tag{5.2.14}$$

Step 3: We show that k is bounded. Seeking a contradiction, assume that $k(t) \to \infty$ for $t \to \omega$. Using $e_1(\cdot) = [I_{m_1}, 0]Ve(\cdot)$ and $e_2(\cdot) = [0, I_{m-m_1}]Ve(\cdot)$, we obtain from (5.2.8) that

$$\begin{aligned} &\|e_2(t)\| \leq \\ &\quad \|\big(\hat{A}_{22} - k(t) I_{m-m_1}\big)^{-1}\| \Big(\|\hat{A}_{21} e_1(t)\| + \|[0, I_{m-m_1}] V \Theta_1(e_1, e_2)(t)\| \Big). \end{aligned}$$

Observing that, since $\|\hat{A}_{22}\| \leq \|A_{22}\| < \hat{k}$,

$$\begin{aligned} \|\big(\hat{A}_{22} - k(t) I_{m-m_1}\big)^{-1}\| &= k(t)^{-1} \|\big(I_{m-m_1} - k(t)^{-1} \hat{A}_{22}\big)^{-1}\| \\ &\leq k(t)^{-1} \frac{1}{1 - k(t)^{-1} \|\hat{A}_{22}\|} \leq k(t)^{-1} \frac{\hat{k}}{\hat{k} - \|\hat{A}_{22}\|}, \end{aligned}$$

and invoking boundedness of e_1 (since e evolves within the funnel) and boundedness of $\Theta_1(e_1, e_2)$ (since $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ and (5.1.5) holds) we obtain

$$\|e_2(t)\| \leq \\ k(t)^{-1} \frac{\hat{k}}{\hat{k} - \|\hat{A}_{22}\|} \left(\|\hat{A}_{21} e_1\|_\infty + \|[0, I_{m-m_1}] V \Theta_1(e_1, e_2)\|_\infty \right) \underset{t \to \omega}{\longrightarrow} 0. \tag{5.2.15}$$

Now, if $m_1 = 0$ then $e = e_2$ and we have $\lim_{t\to\omega} \|e(t)\| = 0$, which implies, by boundedness of φ, $\lim_{t\to\omega} \varphi(t)^2 \|e(t)\|^2 = 0$, hence $\lim_{t\to\omega} k(t) = \hat{k}$, a contradiction. Hence, in the following we assume that $m_1 > 0$.

Let $\delta \in (0, \omega)$ be arbitrary but fix and $\lambda := \inf_{t \in (0,\omega)} \varphi(t)^{-1} > 0$. Since $\dot{\varphi}$ is bounded and $\liminf_{t\to\infty} \varphi(t) > 0$ we find that $\frac{\mathrm{d}}{\mathrm{d}t} \varphi|_{[\delta,\infty)} (\cdot)^{-1}$ is bounded and hence there exists a Lipschitz bound $L > 0$ of $\varphi|_{[\delta,\infty)} (\cdot)^{-1}$. Furthermore, let $\hat{A}_{11} := [I_{m_1}, 0] V A_{22} V^\top [I_{m_1}, 0]^\top$, $\hat{A}_{12} := [I_{m_1}, 0] V A_{22} V^\top [0, I_{m-m_1}]^\top$ and

$$\begin{aligned} \alpha &:= \|D^{-1} \hat{A}_{11}\| \|e_1\|_\infty + \|[D^{-1}, 0] V \Theta_1(e_1, e_2)\|_\infty, \\ \beta &:= \tfrac{2}{\lambda \hat{k}} \|D^{-1} \hat{A}_{12}\|, \\ \gamma &:= \tfrac{\hat{k}}{\hat{k} - \|\hat{A}_{22}\|} \left(\|\hat{A}_{21} e_1\|_\infty + \|[0, I_{m-m_1}] V \Theta_1(e_1, e_2)\|_\infty \right), \\ \kappa &:= \tfrac{\lambda^2 \hat{k}}{4\, \sigma_{\max}(\Gamma)} > 0, \end{aligned}$$

where $\sigma_{\max}(\Gamma)$ denotes the largest eigenvalue of the positive semi-definite matrix Γ and $\sigma_{\max}(\Gamma) > 0$ since $m_1 > 0$.

Choose $\varepsilon > 0$ small enough so that

$$\varepsilon \leq \min \left\{ \frac{\lambda}{2}, \min_{t \in [0,\delta]} (\varphi(t)^{-1} - \|e_1(t)\|) \right\}$$

and

$$L \leq -\alpha - \beta\gamma\varepsilon + \frac{\kappa}{\varepsilon}. \tag{5.2.16}$$

We show that

$$\forall\, t \in (0, \omega): \ \varphi(t)^{-1} - \|e_1(t)\| \geq \varepsilon. \tag{5.2.17}$$

By definition of ε this holds on $(0,\delta]$. Seeking a contradiction suppose that

$$\exists\, t_1 \in [\delta,\omega): \ \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon.$$

Then for

$$t_0 := \max\left\{ \ t \in [\delta, t_1) \ \middle|\ \varphi(t)^{-1} - \|e_1(t)\| = \varepsilon \ \right\}$$

we have for all $t \in [t_0, t_1]$ that

$$\varphi(t)^{-1} - \|e_1(t)\| \le \varepsilon \quad \text{and} \quad \|e_1(t)\| \ge \varphi(t)^{-1} - \varepsilon \ge \lambda - \varepsilon \ge \frac{\lambda}{2}$$

and

$$\begin{aligned} k(t) = \frac{\hat{k}}{1-\varphi(t)^2\|e(t)\|^2} &\ge \frac{\hat{k}}{1-\varphi(t)^2\|e_1(t)\|^2} \\ &= \frac{\hat{k}}{(1-\varphi(t)\|e_1(t)\|)(1+\varphi(t)\|e_1(t)\|} \ge \frac{\hat{k}}{2\varepsilon\varphi(t)} \ge \frac{\lambda\hat{k}}{2\varepsilon}. \end{aligned}$$

Now we have, for all $t \in [t_0, t_1]$,

$$\begin{aligned} \frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 &= e_1(t)^\top \dot{e}_1(t) \\ &= e_1(t)^\top \Big(D^{-1}(\hat{A}_{11} - k(t) I_{m_1}) e_1(t) + D^{-1}\hat{A}_{12} e_2(t) \\ &\quad + [D^{-1}, 0] V\, \Theta_1(e_1, e_2)(t) \Big) \\ &\le \alpha\|e_1(t)\| + \|D^{-1}\hat{A}_{12}\|\|e_2(t)\|\|e_1(t)\| - \frac{\lambda\hat{k}}{2\varepsilon} e_1(t)^\top D^{-1} e_1(t)^\top \\ &\le \alpha\|e_1(t)\| + \|D^{-1}\hat{A}_{12}\|\|e_2(t)\|\|e_1(t)\| - \frac{\lambda\hat{k}}{2\varepsilon\, \sigma_{\max}(\Gamma)}\|e_1(t)\|^2. \end{aligned}$$

Moreover, from the inequality in (5.2.15) we obtain that, for all $t \in [t_0, t_1]$,

$$\|e_2(t)\| \le k(t)^{-1}\gamma \le \frac{2}{\lambda\hat{k}}\gamma\varepsilon.$$

This yields that

$$\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \le \left(\alpha + \beta\gamma\varepsilon - \frac{\kappa}{\varepsilon}\right)\|e_1(t)\| \overset{(5.2.16)}{\le} -L\|e_1(t)\|.$$

Therefore, using

$$\frac{1}{2}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 = \|e_1(t)\|\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|,$$

we find that

$$\begin{aligned}\|e_1(t_1)\| - \|e_1(t_0)\| &= \int_{t_0}^{t_1} \frac{1}{2}\|e_1(t)\|^{-1}\tfrac{\mathrm{d}}{\mathrm{d}t}\|e_1(t)\|^2 \,\mathrm{d}t \\ &\le -L(t_1-t_0) \le -|\varphi(t_1)^{-1} - \varphi(t_0)^{-1}| \le \varphi(t_1)^{-1} - \varphi(t_0)^{-1},\end{aligned}$$

and hence

$$\varepsilon = \varphi(t_0)^{-1} - \|e_1(t_0)\| \le \varphi(t_1)^{-1} - \|e_1(t_1)\| < \varepsilon,$$

a contradiction.

Therefore, (5.2.17) holds and by (5.2.15) there exists $\tilde{t} \in [0,\omega)$ such that $\|e_2(t)\| \le \varepsilon$ for all $t \in [\tilde{t},\omega)$. Then, invoking $\varepsilon \le \frac{\lambda}{2}$, we obtain for all $t \in [\tilde{t},\omega)$

$$\begin{aligned}\|e(t)\|^2 = \|e_1(t)\|^2 + \|e_2(t)\|^2 &\le (\varphi(t)^{-1} - \varepsilon)^2 + \varepsilon^2 \\ &\le \varphi(t)^{-2} - 2\varepsilon\lambda + 2\varepsilon^2 \le \varphi(t)^{-2} - 2\varepsilon^2.\end{aligned}$$

This implies boundedness of k, a contradiction.

Step 4: We show that x_1 and x_3 are bounded. To this end, observe that $z = (k, e_1^\top, e_2^\top)^\top$ solves (5.2.13) and, by Step 3, z is bounded. Using (5.1.5) and $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\ge 0};\mathbb{R}^m)$ we find that Tz is bounded as well. This implies, since F is continuously differentiable, that $\dot{z}$ is bounded. Then again, we obtain that $\frac{\mathrm{d}}{\mathrm{d}t}(Tz)$ is bounded and differentiating (5.2.13) gives boundedness of $\ddot{z}$. Iteratively, we have that

$$\begin{aligned}&\forall\, j = 0,\ldots,\nu+1: \\ &\quad \Big(\exists\, c_0,\ldots,c_j > 0\ \forall\, t \in [0,\omega):\ \|z(t)\| \le c_0,\ldots,\|z^{(j)}(t)\| \le c_j\Big) \\ &\qquad\qquad \Longrightarrow \Big(\exists\, C > 0\ \forall\, t \in [0,\omega):\ \|(Tz)^{(j)}(t)\| \le C\Big)\end{aligned}$$

and successive differentiation of (5.2.13) finally yields that z, $\dot{z}$, ..., $z^{(\nu+1)}$ are bounded. This gives boundedness of e, $\dot{e}$, ..., $e^{(\nu+1)}$. Then, from the first and third equation in (5.2.7) and the fact that $\sigma(Q) \subseteq \mathbb{C}_-$ and $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\ge 0};\mathbb{R}^m)$, it is immediate that x_1 and x_3 are bounded.

Step 5: We show that $\omega = \infty$. First note that by Step 3 and Step 4 we have that $(x_1, e, x_3, k) : [0, \omega) \to \mathbb{R}^{n+1}$ is bounded. Further noting that boundedness of k is equivalent to (5.3.21) (for $t \in [0, \omega)$), the assumption $\omega < \infty$ implies existence of a compact subset $\mathcal{K} \subseteq \widetilde{\mathcal{D}}$ such that $\operatorname{graph}(x_1, e, x_3, k) \subseteq \mathcal{K}$. This contradicts (5.2.14).

Step 6: It remains to show (ii). This follows from

$$\forall\, t > 0: \; k(t) = \hat{k} + k(t)\varphi(t)^2\|e(t)\|^2 \overset{(5.3.21)}{\leq} \hat{k} + k(t)\varphi(t)^2(\varphi(t)^{-1} - \varepsilon)^2 = \hat{k} + k(t)(1 - \varphi(t)\varepsilon)^2. \quad \square$$

Remark 5.2.4.

(i) The problem of finding a solution of (5.2.13) with the properties (a)–(c) as in the proof of Theorem 5.2.3 is not solved just by the consistency of the initial value, i.e., existence of a local solution, since it is not clear that this solution can be extended to a maximal solution which leaves every compact subset of $\mathcal{D}$. Solvability for any other initial value (for (5.2.13)) is required for this.

(ii) Note that ν in Theorem 5.2.3 is in general not known explicitly. However, we have, by Theorem 4.2.7, the estimate $\nu \leq n_3 = n - k - m$, where $k = \dim \max(E, A, B; \ker C)$. Hence, choosing φ and y_{ref} to be $(n - m + 2)$-times continuously differentiable will always suffice.

(iii) The differentiability assumption $y_{\text{ref}} \in \mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ in Theorem 5.2.3 can in general not be relaxed in order to avoid Dirac impulses in the state variables. For an example consider the system

$$x_1 + u = 0, \quad \dot{x}_1 = x_2, \quad \dot{x}_2 = x_3, \quad y = x_1.$$

Application of the funnel controller (5.2.2) leads to the equation

$$y(t) = \frac{k(t)}{k(t) - 1} y_{\text{ref}}(t),$$

where $k(t) \geq \hat{k} > 1$. If y_{ref} is continuous, but not continuously differentiable ($\nu = 2$ here), then $x_2 = \dot{y}$ has jumps and in $x_3 = \dot{x}_2$ we will have Dirac impulses. By adding equations of the form $x_{i+1} = \dot{x}_i$ we may construct systems with Dirac impulses in its

solutions for any $y_{\text{ref}} \in \mathcal{C}^k(\mathbb{R}_{\geq 0}; \mathbb{R}^m) \setminus \mathcal{C}^{k+1}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.
It is not in the scope of Theorem 5.2.3 to treat these impulses, but combining the theory of the present thesis with a distributional solution theory for DAEs as in [227] might result in an appropriate treatment of less smooth reference trajectories.

5.2.2 Simulations

Position control of a mechanical system

We consider a mechanical system, see Figure 5.2, with springs, masses and dampers with single-input spatial distance between the two masses and single-output position of one mass. I am indebted to Professor P.C. Müller (BU Wuppertal) for suggesting this example.

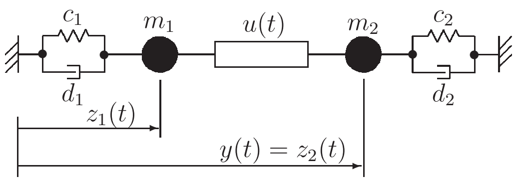


Figure 5.2: *Mass-spring-damper system*

The masses m_1, m_2, damping constants d_1, d_2 and spring constants c_1, c_2 are all assumed to be positive. As output $y(t) = z_2(t)$ we take the position of the mass m_2. The input is chosen as $u(t) = z_2(t) - z_1(t)$, i.e., the spatial distance between the masses m_1 and m_2. The mechanical system in Figure 5.2 may then be modeled by the second-order differential-algebraic equation

$$\begin{aligned} m_1\ddot{z}_1(t) + d_1\dot{z}_1(t) + c_1 z_1(t) - \lambda(t) &= 0 \\ m_2\ddot{z}_2(t) + d_2\dot{z}_2(t) + c_2 z_2(t) + \lambda(t) &= 0 \\ z_2(t) - z_1(t) &= u(t) \\ y(t) &= z_2(t) \end{aligned} \tag{5.2.18}$$

where $\lambda(\cdot)$ is a constraint force viewed as a variable. Defining the state

$$x(t) = (z_1(t), \dot{z}_1(t), z_2(t), \dot{z}_2(t), \lambda(t))^\top, \tag{5.2.19}$$

model (5.2.18) may be rewritten as the linear differential-algebraic input-output system (4.1.1) for

$$sE - A = \begin{bmatrix} s & -1 & 0 & 0 & 0 \\ c_1 & sm_1 + d_1 & 0 & 0 & -1 \\ 0 & 0 & s & -1 & 0 \\ 0 & 0 & c_2 & sm_2 + d_2 & 1 \\ -1 & 0 & 1 & 0 & 0 \end{bmatrix}, \; B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \; C = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}^\top . \tag{5.2.20}$$

We may immediately see that the pencil $sE - A$ is regular and has index $\nu = 3$. The zero dynamics of (5.2.20) are asymptotically stable: setting $y = 0$ in (5.2.18) yields $z_2 = 0$, $\lambda = 0$, $z_1 = -u$ and $m_1 z_1(t) + d_1 \dot{z}_1(t) + c_1 z_1(t) = 0$ for all $t \geq 0$; positivity of m_1, d_1 and c_1 then gives $\lim_{t\to\infty} \dot{z}_1(t) = \lim_{t\to\infty} z_1(t) = 0$. Right-invertibility of (5.2.20) can be easily read off (5.2.18). The transfer function

$$G(s) = C(sE - A)^{-1}B = \frac{m_1 s^2 + d_1 s + c_1}{(m_1 + m_2)s^2 + (d_1 + d_2)s + (c_1 + c_2)}$$

has proper inverse: $\lim_{s\to\infty} G^{-1}(s) = (m_1 + m_2)/m_1$. Therefore, invoking Remark 4.3.3,

$$\Gamma = \lim_{s\to\infty} s^{-1} G(s)^{-1} = 0.$$

Summarizing, system (5.2.20) satisfies the assumptions of Theorem 5.2.3.

As reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}$, we take the first component of the solution of the following initial-value problem for the Lorenz system

$$\begin{aligned} \dot{\xi}_1(t) &= 10\,(\xi_2(t) - \xi_1(t)), & \xi_1(0) &= 5 \\ \dot{\xi}_2(t) &= 28\,\xi_1(t) - \xi_1(t)\,\xi_3(t) - \xi_2(t), & \xi_2(0) &= 5 \\ \dot{\xi}_3(t) &= \xi_1(t)\,\xi_2(t) - \tfrac{8}{3}\,t\,\xi_3(t), & \xi_3(0) &= 5\,. \end{aligned} \tag{5.2.21}$$

This may be viewed as a rather academic choice, however it is well known (see for example [224, App. C]) that the Lorenz system is chaotic (and thus the reference signal is rather 'wild'), the unique global solution of (5.2.21) is bounded with bounded derivative on the positive real axis (and thus our assumptions on the class of reference signals are satisfied). The solution of (5.2.21) is depicted in Figure 5.3.

The funnel $\mathcal{F}_\varphi$ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5\; te^{-t} + 2\;\arctan t\,. \tag{5.2.22}$$

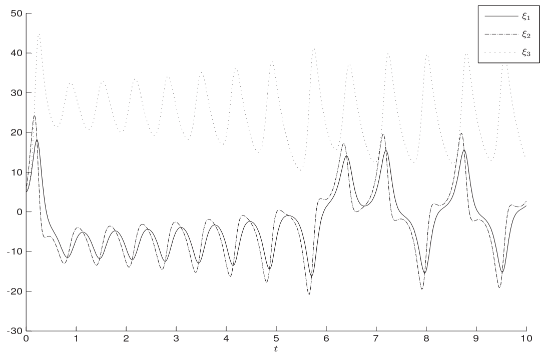


Figure 5.3: *Components* ξ_i *of the Lorenz system* (5.2.21)

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see e.g. Figure 5.4d.

Spring and damping constants, masses and their initial positions are chosen, for the simulations, as

$$\begin{aligned} &m_1 = 1, \quad m_2 = 3, \quad c_1 = 2, \quad c_2 = 1, \quad d_1 = 3, \quad d_2 = 5, \\ &z_1(0) = 101, \quad z_2(0) = 21 \quad \text{and} \qquad \hat{k} = 5. \end{aligned} \tag{5.2.23}$$

Straightforward calculations show that the closed-loop system (5.2.2), (5.2.18) has uniquely determined initial velocities $\dot{z}_1(0)$, $\dot{z}_2(0)$ as well as initial constraint force $\lambda(0)$ and that the initialization is consistent. Since

$$\hat{k} = 5 > 4 = \lim_{s\to\infty} G^{-1}(s) = \left\| \lim_{s\to\infty} \left(G(s)^{-1} + s\Gamma \right) \right\|,$$

all assumptions of Theorem 5.2.3 are satisfied and we may apply the funnel controller (5.2.2) with funnel boundary specified in (5.2.22) and reference signal $y_{\text{ref}} = \xi_1$ given in (5.2.21).

All numerical simulations are performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-5}). The simulations over the time interval $[0, 10]$ are depicted in Figure 5.4: Figure 5.4a shows the output y tracking the rather 'vivid' reference signal y_{ref} within the funnel shown in Figure 5.4d. Note that the input u in Figure 5.4c as well as the gain function k in Figure 5.4b have spikes at those times t when the norm of the error $\|e(t)\|$ is 'close' to the funnel

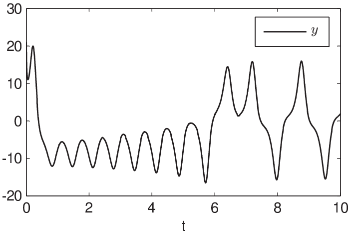


Figure a: Solution y

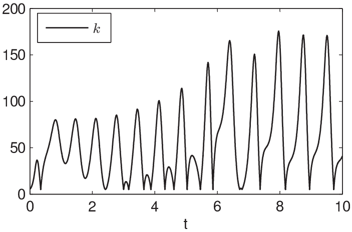


Figure b: Gain k

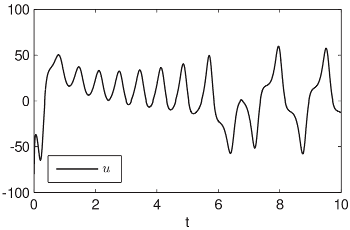


Figure c: Input u

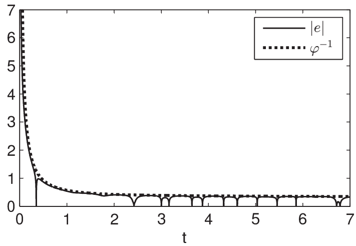


Figure d: Norm of error $|e|$ and funnel boundary φ^{-1}

Figure 5.4: *Simulation of the funnel controller* (5.2.2) *with funnel boundary specified in* (5.2.22) *and reference signal* $y_{\text{ref}} = \xi_1$ *given in* (5.2.21) *applied to the mechanical model* (5.2.18) *with data* (5.2.23).

boundary $\varphi(t)^{-1}$; this is due to rapid change of the reference signal. We stress that the gain function k is nonmonotone.

Singular academic example

For purposes of illustration we consider an example of a singular differential-algebraic system (4.1.1), where Γ is neither zero nor invertible. Consider system (4.1.1) with

$$[E, A, B, C] := \left[\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 1 & -2 \\ 3 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right]. \tag{5.2.24}$$

It is immediate that $[E, A, B, C]$ is right-invertible and in the form (4.2.3), has asymptotically stable zero dynamics, and the matrix

$$\Gamma = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

satisfies (5.2.3) and $\Gamma = \Gamma^\top \geq 0$. We set

$$\hat{k} := 2 > \sqrt{2} = \left\| \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \right\| = \|A_{22}\| = \left\| \lim_{s \to \infty} \left([0, I_m] L(s) [0, I_m]^\top + s\Gamma \right) \right\|, \tag{5.2.25}$$

where $L(s)$ is an inverse of the system pencil, see also Step 1 in the proof of Theorem 5.2.3 for the latter equalities. The (consistent) initial value for the closed-loop system (5.2.24), (5.2.2) is chosen as

$$x^0 = (-4, 3, -2)^\top. \tag{5.2.26}$$

As reference signal $y_{\text{ref}} : \mathbb{R}_{\geq 0} \to \mathbb{R}^2$, we take the first and second component of the solution of the Lorenz system (5.2.21). The funnel $\mathcal{F}_\varphi$ is determined by (5.2.22).

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-5}). In Figure 5.5 the simulation, over the time interval $[0, 10]$, of the funnel controller (5.2.2) with funnel boundary specified in (5.2.22) and reference signal $y_{\text{ref}} = (\xi_1, \xi_2)^\top$ given in (5.2.21), applied to system (5.2.24) with initial data (5.2.25), (5.2.26) is depicted. Figure 5.5a shows the output components y_1 and y_2 tracking the reference signal y_{ref} within the funnel shown in Figure 5.5d. Note that an action of the input components u_1 and u_2 in Figure 5.5c and the gain function k in Figure 5.5b is required only if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$. It can be seen that initially the error is very close to the funnel boundary and hence the gain rises sharply. Then, at approximately $t = 0.2$, the distance between error and funnel boundary gets larger and the gain drops accordingly. After $t = 2$, the error gets close to the funnel boundary again which causes the gain to rise again. This in particular shows that the gain function k is nonmonotone.

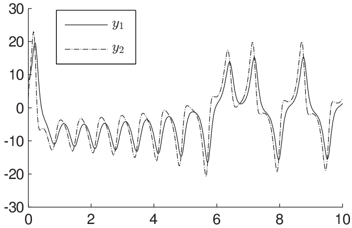


Figure a: Solution components y_1 and y_2

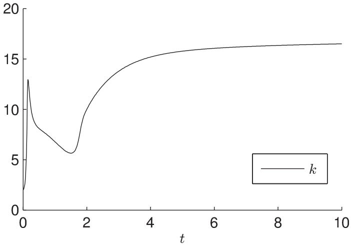


Figure b: Gain k

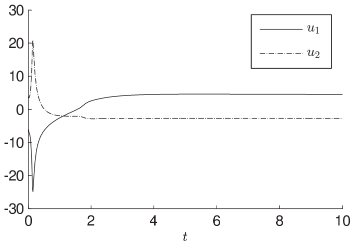


Figure c: Input components u_1 and u_2

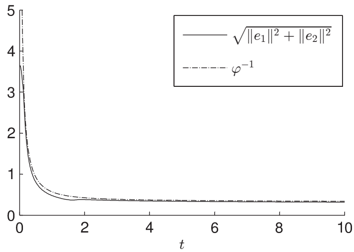


Figure d: Norm of error $\|e\|$ and funnel boundary φ^{-1}

Figure 5.5: *Simulation of the funnel controller* (5.2.2) *with funnel boundary specified in* (5.2.22) *and reference signal* $y_{\text{ref}} = (\xi_1, \xi_2)^\top$ *given in* (5.2.21) *applied to system* (5.2.24) *with initial data* (5.2.25), (5.2.26).

5.3 Regular systems and relative degree

In this section we consider systems $[E, A, B, C] \in \Sigma_{n,n,m,p}$ for which $sE - A$ is regular and hence the *transfer function*

$$G(s) = C(sE - A)^{-1}B \in \mathbb{R}(s)^{p\times m} \tag{5.3.1}$$

is well defined. We will introduce the notions of vector and strict relative degree and show that systems with a vector relative degree which is componentwise smaller or equal to one can be treated within the frameworks of Theorems 5.1.4 and 5.2.3. Furthermore, we show that for systems with positive strict relative degree, high-gain and funnel control is feasible provided the controllers are modified in order to account for output derivatives.

As a preliminary result we derive some characterizations of autonomous zero dynamics and right-invertibility for regular systems.

Proposition 5.3.1 (Autonomous zero dynamics and right-invertibility).
Let $[E, A, B, C] \in \Sigma_{n,n,m,p}$ be such that $sE - A$ is regular and let $G(s)$ be as in (5.3.1)*. Then the following statements hold true:*

(i) $\mathcal{ZD}_{[E,A,B,C]}$ *is autonomous* $\iff$ $\operatorname{rk}_{\mathbb{R}[s]} G(s) = m$.

(ii) $[E, A, B, C]$ *is right-invertible* $\iff$ $\operatorname{rk}_{\mathbb{R}[s]} G(s) = p$.

(iii) *If $p = m$, then the following conditions are equivalent:*
   (a) $\mathcal{ZD}_{[E,A,B,C]}$ *is autonomous,*
   (b) $[E, A, B, C]$ *is right-invertible,*
   (c) $G(s) \in \mathbf{Gl}_m(\mathbb{R}(s))$.

Proof: (i): We show "⇒": If the zero dynamics are autonomous, then the system pencil has a left inverse over $\mathbb{R}(s)$ and it can be concluded, using the same calculation as in Remark 4.3.3, that $G(s)$ has a left inverse over $\mathbb{R}(s)$. Equivalently, $\operatorname{rk}_{\mathbb{R}[s]} G(s) = m$.
We show "⇐": By assumption there exists $G_L(s) \in \mathbb{R}(s)^{m\times p}$ such that $G_L(s)G(s) = I_m$. Calculating

$$\begin{aligned}\begin{bmatrix} (sE-A)^{-1} & -(sE-A)^{-1}BG_L(s) \\ 0 & -G_L(s) \end{bmatrix} \begin{bmatrix} I_n & 0 \\ C(sE-A)^{-1} & I_m \end{bmatrix} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} \\ = \begin{bmatrix} (sE-A)^{-1} & -(sE-A)^{-1}BG_L(s) \\ 0 & -G_L(s) \end{bmatrix} \begin{bmatrix} sE-A & -B \\ 0 & -G(s) \end{bmatrix} = I_{n+m},\end{aligned}$$

yields that the system pencil $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ is left invertible over $\mathbb{R}(s)$. Hence, by Proposition 4.1.5, the zero dynamics are autonomous.

(ii): We show "⇒": Seeking a contradiction assume that $\operatorname{rk}_{\mathbb{R}[s]} G(s) < p$. Hence, there exists $q(s) \in \mathbb{R}(s)^p \setminus \{0\}$ such that $q(s)^\top G(s) = 0$. Let $p(s) \in \mathbb{R}[s] \setminus \{0\}$ be such that $\tilde{q}(s) := p(s)q(s)$ satisfies

$$(\tilde{q}(s)^\top C(sE-A)^{-1}, \tilde{q}(s)^\top)^\top \in \mathbb{R}[s]^{n+p}.$$

Now there exists $y \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^p)$ such that $\tilde{q}(\frac{\mathrm{d}}{\mathrm{d}t})^\top y \neq 0$. By right-invertibility of $[E, A, B, C]$ we find $(x, u) \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m)$ such that

$(x, u, y) \in \mathfrak{B}_{[E,A,B,C]}$. Therefore, heeding that all poles in the rational functions are canceled out,

$$\begin{aligned}\tilde{q}(\tfrac{\mathrm{d}}{\mathrm{d}t})^\top y &= (\tilde{q}(\tfrac{\mathrm{d}}{\mathrm{d}t})^\top C(E\tfrac{\mathrm{d}}{\mathrm{d}t} - A)^{-1}, \tilde{q}(\tfrac{\mathrm{d}}{\mathrm{d}t})^\top) \begin{pmatrix} 0 \\ y \end{pmatrix} \\ &\overset{\text{a.e.}}{=} (\tilde{q}(\tfrac{\mathrm{d}}{\mathrm{d}t})^\top C(E\tfrac{\mathrm{d}}{\mathrm{d}t} - A)^{-1}, \tilde{q}(\tfrac{\mathrm{d}}{\mathrm{d}t})^\top) \begin{bmatrix} E\frac{\mathrm{d}}{\mathrm{d}t} - A & -B \\ -C & 0 \end{bmatrix} \begin{pmatrix} x \\ u \end{pmatrix} \overset{\text{a.e.}}{=} 0,\end{aligned}$$

a contradiction.
We show "$\Leftarrow$": By assumption there exists $G_R(s) \in \mathbb{R}(s)^{m\times p}$ such that $G(s)G_R(s) = I_p$. Similar to (i) we may calculate that that the system pencil $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ is right invertible over $\mathbb{R}(s)$. Then applying Lemma 4.1.3 to its transpose yields that $\ell(\gamma) = 0$ in a QKF (4.1.4) of the system pencil. Let $y \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^p)$ and $S(0, y^\top)^\top =: (y_1^\top, y_2^\top, y_3^\top)^\top$ according to the block structure of (4.1.4). Then, by Theorem 2.4.13, there exist solutions $z_1 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n_s})$, $z_2 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{|\alpha|})$, $z_3 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{|\beta|})$ of $\frac{\mathrm{d}}{\mathrm{d}t} z_1 \overset{\text{a.e.}}{=} A_s z_1 + y_1$, $\frac{\mathrm{d}}{\mathrm{d}t} N_\alpha z_2 \overset{\text{a.e.}}{=} z_2 + y_2$ and $\frac{\mathrm{d}}{\mathrm{d}t} K_\beta z_3 \overset{\text{a.e.}}{=} L_\beta z_3 + y_3$. Therefore,

$$\begin{pmatrix} 0 \\ y \end{pmatrix} \overset{\text{a.e.}}{=} S^{-1} \begin{bmatrix} \frac{\mathrm{d}}{\mathrm{d}t} I_{n_s} - A_s & 0 & 0 \\ 0 & \frac{\mathrm{d}}{\mathrm{d}t} N_\alpha - I_{|\alpha|} & 0 \\ 0 & 0 & \frac{\mathrm{d}}{\mathrm{d}t} K_\beta - L_\beta \end{bmatrix} \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix}$$

and hence $(x^\top, u^\top)^\top := T(z_1^\top, z_2^\top, z_3^\top)^\top$ satisfies $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]}$.
(iii): The assertion is immediate from (i) and (ii). □

A consequence of Proposition 5.3.1 is that the class of regular systems with proper inverse transfer function can be treated within the frameworks of Theorems 5.1.4 and 5.2.3.

Definition 5.3.2 (Proper rational matrix function)**.**
A rational matrix $G(s) \in \mathbb{R}(s)^{p\times m}$ is called *proper* if, and only if, $\lim_{s\to\infty} G(s) = D$ for some $D \in \mathbb{R}^{p\times m}$. It is called *strictly proper* if, and only if, $\lim_{s\to\infty} G(s) = 0$.

Corollary 5.3.3 (Proper inverse transfer function)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *be such that* $sE - A$ *is regular and that* $G(s)$ *as in* (5.3.1) *is invertible over* $\mathbb{R}(s)$ *and* $G(s)^{-1}$ *is proper. Then* $[E, A, B, C]$ *is right-invertible, has autonomous zero dynamics and* Γ *as in* (5.2.3) *exists and satisfies* $\Gamma = 0$.

Proof: Right-invertibility and autonomy of the zero dynamics follow from Proposition 5.3.1. The last assertion is a consequence of Remark 4.3.3 by which

$$\Gamma = \lim_{s\to\infty} s^{-1}G(s)^{-1} = 0. \qquad \square$$

5.3.1 Vector relative degree

In this subsection we give the definition of vector relative degree for transfer functions of regular systems $[E, A, B, C] \in \Sigma_{n,n,m,p}$ and relate this property to the frameworks of Theorems 5.1.4 and 5.2.3.

Definition 5.3.4 (Vector relative degree)**.**
We say that $G(s) \in \mathbb{R}(s)^{p\times m}$ has *vector relative degree* $(\rho_1, \ldots, \rho_p) \in \mathbb{Z}^{1\times p}$ if, and only if, the limit

$$D := \lim_{s\to\infty} \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right) G(s) \in \mathbb{R}^{p\times m}$$

exists and satisfies $\operatorname{rk} D = p$.

Remark 5.3.5 (Vector relative degree)**.**

(i) It is an easy calculation that if $G(s) \in \mathbb{R}(s)^{p\times m}$ has a vector relative degree, then the vector relative degree is unique. However, a vector relative degree does not necessarily exist, even if $G(s)$ is (strictly) proper; see Example 5.3.7.

(ii) Isidori [131, Sec. 5.1] introduced a local version of vector relative degree for nonlinear systems. Definition 5.3.4 coincides with Isidori's definition if strictly proper transfer functions are considered. In this sense, Definition 5.3.4 is a generalization to arbitrary rational transfer functions. For linear ODE systems a global version of the vector relative degree has been stated in [180]. It is straightforward to show that $[I_n, A, B, C] \in \Sigma_{n,n,m,m}$ has vector relative degree $(\rho_1, \ldots, \rho_p)$ in the sense of [180, Def. 2.1] if, and only if, $C(sI - A)^{-1}B$ has vector relative degree $(\rho_1, \ldots, \rho_p)$.

In the following we show that a regular system with transfer function which has componentwise vector relative degree smaller or equal to one, is right-invertible, has autonomous zero dynamics and Γ in (5.2.3) exists.

Proposition 5.3.6 (Vector relative degree ≤ 1 implies existence of Γ)**.**

Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *be such that* $sE - A$ *is regular and* $G(s)$ *in* (5.3.1) *has vector relative degree* $(\rho_1, \ldots, \rho_m)$ *with* $\rho_i \leq 1$ *for all* $i = 1, \ldots, m$. *Then*

(i) $\mathcal{Z}D_{(4.1.1)}$ *are autonomous,*

(ii) $[E, A, B, C]$ *is right-invertible,*

(iii) $\begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}$ *has inverse* $L(s)$ *over* $\mathbb{R}(s)$ *and the matrix* Γ *in* (5.2.3) *exists and satisfies*

$$\forall\, j = 1, \ldots, m: \\ \Gamma e_j = \begin{cases} \left(\lim_{s\to\infty} \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right) G(s)\right)^{-1} e_j, & \text{if } \rho_j = 1, \\ 0, & \text{if } \rho_j < 1. \end{cases} \tag{5.3.2}$$

Proof: *Step 1*: We show that $G(s)$ is invertible over $\mathbb{R}(s)$. To this end, let $F(s) := \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right) G(s)$. Since

$$D := \lim_{s\to\infty} F(s) = \lim_{s\to\infty} \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right) G(s) \in \mathbf{Gl}_m(\mathbb{R})$$

exists, $G_{\mathrm{sp}}(s) := F(s) - D \in \mathbb{R}(s)^{m\times m}$ is strictly proper, i.e., $\lim_{s\to\infty} G_{\mathrm{sp}}(s) = 0$. Since D is invertible, $F(s)$ is invertible as well, as by the Sherman-Morrison-Woodbury formula (see [105, p. 50])

$$F(s)^{-1} = D^{-1} - D^{-1}G_{\mathrm{sp}}(s)(I + D^{-1}G_{\mathrm{sp}}(s))^{-1}D^{-1} \in \mathbb{R}(s)^{m\times m}. \tag{5.3.3}$$

It is then immediate that $G(s)$ has inverse $G(s)^{-1} = F(s)^{-1} \operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right)$ over $\mathbb{R}(s)$.

Step 2: In view of Step 1, (i) and (ii) follow from Proposition 5.3.1.

Step 3: We show (iii). As in Remark 4.3.3 we may conclude that $[0, I_m]L(s)[0, I_m]^\top = -G(s)^{-1}$ and

$$\begin{aligned} s^{-1}G(s)^{-1} &= \left(\operatorname{diag}\left(s^{\rho_1}, \ldots, s^{\rho_p}\right)G(s)\right)^{-1}\operatorname{diag}\left(s^{\rho_1-1}, \ldots, s^{\rho_p-1}\right) \\ &\overset{(5.3.3)}{=} \left(D^{-1} - D^{-1}G_{\mathrm{sp}}(s)(I + D^{-1}G_{\mathrm{sp}}(s))^{-1}D^{-1}\right)\operatorname{diag}\left(s^{\rho_1-1}, \ldots, s^{\rho_p-1}\right). \end{aligned}$$

Hence, using $\rho_i \leq 1$ for $i = 1, \ldots, m$, we obtain existence of

$$\Gamma = \lim_{s \to \infty} s^{-1} G(s)^{-1} \in \mathbb{R}^{m \times m},$$

where $\Gamma e_j = D^{-1} e_j$ if $\rho_j = 1$ and $\Gamma e_j = 0$ if $\rho_j < 1$, for all $j = 1, \ldots, m$. □

We illustrate the vector relative degree and Proposition 5.3.6 by means of an example.

Example 5.3.7.
Consider system (4.1.1) with

$$E = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \ A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}, \ B = C = I_2.$$

It can be seen that $sE - A$ is regular and

$$G(s) = C(sE - A)^{-1} B = \begin{bmatrix} s-1 & -2 \\ 0 & -3 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{s-1} & -\frac{2}{3(s-1)} \\ 0 & -\frac{1}{3} \end{bmatrix}.$$

We calculate

$$D := \lim_{s \to \infty} \operatorname{diag}(s, 1)\, G(s) = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & -\frac{1}{3} \end{bmatrix} \in \mathbf{Gl}_2(\mathbb{R}),$$

and hence $G(s)$ has vector relative degree $(1, 0)$. Proposition 5.3.6 then implies that $\mathcal{ZD}_{(4.1.1)}$ are autonomous, $[E, A, B, C]$ is right-invertible, and Γ in (5.2.3) exists. In fact, it is easy to see that the zero dynamics are asymptotically stable and

$$\Gamma = \lim_{s \to \infty} s^{-1} G(s)^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

satisfies (5.3.2): $\Gamma e_1 = D^{-1} e_1 = \left[\begin{smallmatrix} 1 & -2 \\ 0 & -3 \end{smallmatrix}\right] e_1$ and $\Gamma e_2 = 0$. Since $\Gamma = \Gamma^\top \geq 0$, the assumptions of Theorems 5.1.4 and 5.2.3 are satisfied.

We like to stress that, compared to the above, the regular system (5.2.24) from Subsection 5.2.2 does not have a vector relative degree: while its transfer function $G(s)$ is proper, the limit

$$\lim_{s \to \infty} G(s) = -\frac{1}{3} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

does not have full row rank. Nevertheless, as shown in Section 5.2.2, funnel control is feasible.

5.3.2 Strict relative degree

In this subsection we introduce the concept of strict relative degree, which is a special vector relative degree, and develop high-gain and funnel controllers for systems with positive strict relative degree.

Definition 5.3.8 (Strict relative degree)**.**
We say that a square matrix function $G(s) \in \mathbb{R}(s)^{m\times m}$ has *strict relative degree* $\operatorname{sr\,deg} G(s) = \rho \in \mathbb{Z}$ if, and only if,

$$D := \lim_{s\to\infty} s^{\rho}\, G(s) \in \mathbf{Gl}_m(\mathbb{R}).$$

The matrix D is called *high-frequency gain matrix*.

Remark 5.3.9 (Strict relative degree and high-frequency gain)**.**

(i) If $G(s) \in \mathbb{R}(s)^{m\times m}$ has vector relative degree $(\rho_1, \ldots, \rho_m) \in \mathbb{Z}^{1\times m}$, then $\rho = \rho_1 = \ldots = \rho_m$ if, and only if, $G(s)$ has strict relative degree ρ.

(ii) If $g(s) = p(s)/q(s)$, for $p(s) \in \mathbb{R}[s]$ and $q(s) \in \mathbb{R}[s]\backslash\{0\}$, is a scalar rational function, then the strict relative degree always exists and coincides with the well-known definition of relative degree:

$$\operatorname{sr\,deg} g(s) = \deg q(s) - \deg p(s).$$

(iii) An ODE system $[E, A, B, C] = [I_n, A, B, C] \in \Sigma_{n,n,m,m}$ has transfer function

$$G(s) = C(sI - A)^{-1}B = CB\, s^{-1} + CAB\, s^{-2} + CA^2B\, s^{-3} + \ldots$$

and therefore strict relative degree $\rho \in \mathbb{N}$, if, and only if,

$$\det(CA^{\rho-1}B) \neq 0 \ \text{ and, if } \rho > 1,\ \forall\, k = 0, \ldots, \rho-2 :\ CA^kB = 0\,.$$

(iv) For systems $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with regular $sE - A$, transfer function $G(s)$ as in (5.3.1) and strict relative degree $\rho \in \mathbb{N}$ we have: If $\rho = 1$, then by Proposition 5.3.6, Γ in (5.2.3) exists and we have, from the proof of Proposition 5.3.6, $\Gamma = \big(\lim_{s\to\infty} s\, G(s)\big)^{-1}$, i.e., Γ is exactly the inverse of the high-frequency gain matrix. Since, furthermore, Γ is also defined when no high-frequency gain matrix exists, we may view the definition of Γ an appropriate generalization of the high-frequency gain matrix to DAEs which do not have a strict relative degree. In particular, if $G(s)$ has proper inverse, then $\Gamma = 0$.

In order to derive high-gain and funnel controller, we first present a zero dynamics form (5.3.4) for DAE systems with positive strict relative degree. This form is a special case of the forms from Theorems 4.1.7 and 4.2.7 using the Byrnes-Isidori form for ODE systems with strictly proper transfer function derived in [127], see also [131, Sec. 5.1]. The form (5.3.4) can be viewed as a generalized Byrnes-Isidori form for regular DAE systems.

Theorem 5.3.10 (Zero dynamics form for systems with positive strict relative degree)**.** *Let* $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *be such that* $sE - A$ *is regular and the transfer function* $G(s)$ *in* (5.3.1) *has strict relative degree* $\rho \in \mathbb{N}$*. Then there exist* $W, T \in \mathbf{Gl}_n(\mathbb{R})$ *such that*

$$[E, A, B, C] \overset{W,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}]$$

with

$$\begin{aligned}
s\hat{E} - \hat{A} &\\
&= \left[\begin{array}{cccccc|cc}
sI_m & -I_m & 0 & \cdots & 0 & 0 & 0 & 0\\
0 & sI_m & -I_m & 0 & & 0 & 0 & 0\\
\vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & sI_m & -I_m & 0 & 0 & 0\\
-R_1 & -R_2 & \cdots & -R_{\rho-1} & sI_m - R_\rho & -S & 0 & 0\\
-P & 0 & \cdots & 0 & 0 & sI_\mu - Q & 0 & 0\\
\hline
0 & 0 & \cdots & 0 & 0 & 0 & sN_c - I_{n_c} & sN_{c\overline{c}}\\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & sN_{\overline{c}} - I_{n_{\overline{c}}}
\end{array}\right],\\
\hat{B} &= \left[\begin{array}{cccccc|cc} 0 & 0 & \cdots & 0 & D^\top & 0 & B_c^\top & 0 \end{array}\right]^\top,\\
\hat{C} &= \left[\begin{array}{cccccc|cc} I_m & 0 & \cdots & 0 & 0 & 0 & 0 & C_{\overline{c}} \end{array}\right],
\end{aligned} \tag{5.3.4}$$

where, for some $n_c, n_{\overline{c}} \in \mathbb{N}_0$*, and* $\mu = n - n_c - n_{\overline{c}} - \rho m$*,*

$$\begin{aligned}
&D = \lim_{s\to\infty} s^\rho\, G(s) \in \mathbf{Gl}_m(\mathbb{R}) \text{ is the high-frequency gain matrix,}\\
&S \in \mathbb{R}^{m\times\mu},\ P \in \mathbb{R}^{\mu\times m},\ Q \in \mathbb{R}^{\mu\times\mu},\ \left[R_1,\,\ldots,\,R_\rho\right] \in \mathbb{R}^{m\times\rho m},\\
&B_c \in \mathbb{R}^{n_c\times m},\ C_{\overline{c}} \in \mathbb{R}^{m\times n_{\overline{c}}},\ N_{c\overline{c}} \in \mathbb{R}^{n_c\times n_{\overline{c}}}\\
&N_c \in \mathbb{R}^{n_c\times n_c},\ N_{\overline{c}} \in \mathbb{R}^{n_{\overline{c}}\times n_{\overline{c}}} \text{ are nilpotent, and } \operatorname{rk}[\,N_c\,,\,B_c] = n_c.
\end{aligned} \tag{5.3.5}$$

The entries $R_1, \ldots, R_\rho$ *are unique, system* $[I, Q, P, S]$ *is unique up to system equivalence* $\overset{\hat{T}^{-1},\hat{T}}{\sim}$*, and the matrices* N_c *and* $N_{\overline{c}}$ *are unique up to similarity.*

If E *is invertible, then* $n_c = n_{\overline{c}} = 0$*, this means only the upper left block in* $\hat{E}$ *is present.*

The form (5.3.4) *is called* zero dynamics form *of* (4.1.1)*. The transfer function satisfies, where* $R_{\rho+1} := -I$*,*

$$G(s) = -\left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_\mu - Q)^{-1}P\right]^{-1} D. \tag{5.3.6}$$

For the proof of Theorem 5.3.10 we need the following preliminary lemma.

Lemma 5.3.11 (Decomposition for systems with strictly proper transfer function)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *be such that* $sE - A$ *is regular and the transfer function* $G(s)$ *in* (5.3.1) *is strictly proper. Then there exists* $W, T \in \mathbf{Gl}_n(\mathbb{R})$ *such that*

$$\begin{aligned}
&[E, A, B, C]\\
&\overset{W,T}{\sim} \left[\begin{bmatrix} I_{n_s} & 0 & 0\\ 0 & N_c & N_{c\overline{c}}\\ 0 & 0 & N_{\overline{c}}\end{bmatrix}, \begin{bmatrix} A_s & 0 & 0\\ 0 & I_{n_{fc}} & 0\\ 0 & 0 & I_{n_{f\overline{c}}}\end{bmatrix}, \begin{bmatrix} B_s\\ B_{fc}\\ 0\end{bmatrix}, \begin{bmatrix} C_s & 0 & C_{f\overline{c}}\end{bmatrix}\right]
\end{aligned} \tag{5.3.7}$$

for some $A_s \in \mathbb{R}^{n_s\times n_s}$*,* $B_s \in \mathbb{R}^{n_s\times m}$*,* $C_s \in \mathbb{R}^{m\times n_s}$*,* $N_c \in \mathbb{R}^{n_{fc}\times n_{fc}}$*,* $N_{c\overline{c}} \in \mathbb{R}^{n_{fc}\times n_{f\overline{c}}}$*,* $N_{\overline{c}} \in \mathbb{R}^{n_{f\overline{c}}\times n_{f\overline{c}}}$*,* $B_{fc} \in \mathbb{R}^{n_{fc}\times m}$ *and* $C_{f\overline{c}} \in \mathbb{R}^{m\times n_{f\overline{c}}}$*, where* $N_c, N_{\overline{c}}$ *are nilpotent and* $\operatorname{rk}[\,N_c\,,\,B_{fc}] = n_{fc}$*. The dimensions* $n_s, n_{fc}, n_{f\overline{c}} \in \mathbb{N}_0$ *are unique, the matrices* A_s*,* N_c *and* $N_{\overline{c}}$ *are unique up to similarity.*

Furthermore, system $[E,A,B,C]$ has strict relative degree $\rho \in \mathbb{N}$ if, and only if,

$$\det(C_s A_s^{\rho-1} B_s) \neq 0 \quad \text{and, if } \rho > 1, \quad \forall\, k = 0,\ldots,\rho-2 : \; C_s A_s^k B_s = 0\,. \tag{5.3.8}$$

Proof: *Step 1*: We show that there exist $W, T \in \mathbf{Gl}_n(\mathbb{R})$ such that (5.3.7) holds true. It follows from Theorem 2.2.5 that there exist $W_1, T_1 \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$[E,A,B,C] \overset{W,T}{\sim} \left[\begin{bmatrix} I_{n_s} & 0 \\ 0 & N \end{bmatrix}, \begin{bmatrix} A_s & 0 \\ 0 & I_{n_f} \end{bmatrix}, \begin{bmatrix} B_s \\ B_f \end{bmatrix}, \begin{bmatrix} C_s & C_f \end{bmatrix} \right],$$

for some $B_s \in \mathbb{R}^{n_f \times m}$, $B_f \in \mathbb{R}^{n_f \times m}$, $C_s \in \mathbb{R}^{m \times n_s}$, $C_f \in \mathbb{R}^{m \times n_s}$, $A_s \in \mathbb{R}^{n_s \times n_s}$ and nilpotent $N \in \mathbb{R}^{n_f \times n_f}$. Then [80, Sec. 2-1.] yields that system $[N, I_{n_f}, B_f, C_f]$ may be decomposed into controllability form so that, for some $T_2 \in \mathbf{Gl}_{n_f}(\mathbb{R})$,

$$[N, I_{n_f}, B_f, C_f] \overset{T_2^{-1}, T_2}{\sim} \left[\begin{bmatrix} N_c & N_{c\overline{c}} \\ 0 & N_{\overline{c}} \end{bmatrix}, \begin{bmatrix} I_{n_{fc}} & 0 \\ 0 & I_{n_{f\overline{c}}} \end{bmatrix}, \begin{bmatrix} B_{fc} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{fc}, & C_{f\overline{c}} \end{bmatrix} \right],$$

where $N_c \in \mathbb{R}^{n_{fc} \times n_{fc}}$, $N_{\overline{c}} \in \mathbb{R}^{n_{f\overline{c}} \times n_{f\overline{c}}}$, $N_{12} \in \mathbb{R}^{n_{fc} \times n_{f\overline{c}}}$, $B_{fc} \in \mathbb{R}^{n_{fc} \times m}$, $C_{fc} \in \mathbb{R}^{m \times n_{fc}}$, and $C_{f\overline{c}} \in \mathbb{R}^{m \times n_{f\overline{c}}}$, such that N_c, $N_{\overline{c}}$ are nilpotent and $\operatorname{rk}[\, N_c\,,\; B_{fc}\,] = n_{fc}$.

We show that $C_{fc} = 0$: Since the transfer function is invariant under system equivalence we have, using $(sN_c - I_{n_f})^{-1} = -I_{n_f} - sN - s^2N^2 - \ldots - s^{\nu-1}N^{\nu-1}$,

$$G(s) = C(sE-A)^{-1}B = C_s(sI_{n_s} - A_s)^{-1}B_s - \sum_{k=0}^{\nu-1} s^k C_{fc} N_c^k B_{fc},$$

and since $G(s)$ is strictly proper, it follows that $C_{fc}N_c^i B_{fc} = 0$ for $i = 1,\ldots,\nu-1$. The nilpotency of N_c gives $C_{fc}N_c^{\nu-1}[\, N_c\,,\; B_{fc}\,] = 0$, whence $C_{fc}N_c^{\nu-1} = 0$. Repeating this argumentation $\nu - 1$ times, we obtain $C_{fc} = 0$.

Setting $W := W_1 \begin{bmatrix} I & 0 \\ 0 & T_2^{-1} \end{bmatrix}$ and $T := \begin{bmatrix} I & 0 \\ 0 & T_2 \end{bmatrix} T_1$, we obtain (5.3.7).

Step 2: We show that the dimensions $n_s, n_{fc}, n_{f\overline{c}} \in \mathbb{N}_0$ are unique and that the matrices A_s, N_c and $N_{\overline{c}}$ are unique up to similarity: Assume that

$$\left[\begin{bmatrix} I_{n_{s1}} & 0 & 0 \\ 0 & N_{c1} & N_{c\overline{c}1} \\ 0 & 0 & N_{\overline{c}1} \end{bmatrix}, \begin{bmatrix} A_{s1} & 0 & 0 \\ 0 & I_{n_{fc1}} & 0 \\ 0 & 0 & I_{n_{f\overline{c}1}} \end{bmatrix}, \begin{bmatrix} B_{s1} \\ B_{fc1} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{s1} & 0 & C_{f\overline{c}1} \end{bmatrix}\right]$$
$$\overset{W,T}{\sim} \left[\begin{bmatrix} I_{n_{s2}} & 0 & 0 \\ 0 & N_{c2} & N_{c\overline{c}2} \\ 0 & 0 & N_{\overline{c}2} \end{bmatrix}, \begin{bmatrix} A_{s2} & 0 & 0 \\ 0 & I_{n_{fc2}} & 0 \\ 0 & 0 & I_{n_{f\overline{c}2}} \end{bmatrix}, \begin{bmatrix} B_{s2} \\ B_{fc2} \\ 0 \end{bmatrix}, \begin{bmatrix} C_{s2} & 0 & C_{f\overline{c}2} \end{bmatrix}\right].$$

Remark 2.2.6 gives that $n_{s1} = n_{s2}$ as well as the similarity of A_{s1} and A_{s2}. Remark 2.2.6 also yields the existence of $T_{11} \in \mathbb{R}^{n_{fc1} \times n_{fc2}}$, $T_{12} \in \mathbb{R}^{n_{fc1} \times n_{f\overline{c}2}}$, $T_{21} \in \mathbb{R}^{n_{f\overline{c}1} \times n_{fc2}}$, and $T_{22} \in \mathbb{R}^{n_{f\overline{c}1} \times n_{f\overline{c}2}}$ such that

$$T = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \in \mathbf{Gl}_{n_f}(\mathbb{R}), \quad \text{and} \quad \begin{bmatrix} N_{c1} & N_{c\overline{c}1} \\ 0 & N_{\overline{c}1} \end{bmatrix} \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}$$
$$= \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} N_{c2} & N_{c\overline{c}2} \\ 0 & N_{\overline{c}2} \end{bmatrix}, \begin{bmatrix} B_{fc1} \\ 0 \end{bmatrix} = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix} \begin{bmatrix} B_{fc2} \\ 0 \end{bmatrix}.$$

Therefore, $0 = T_{21}B_{fc2}$ and $T_{21}N_{c2} = N_{\overline{c}1}T_{21}$. Hence, for $k = 1, \ldots, \nu - 1$, we have $T_{21}N_{c2}^k B_{fc2} = N_{\overline{c}1}^k T_{21} B_{fc2} = 0$, and so $T_{21}N_{c2}^{\nu-1}[\, N_{c2}\ B_{fc2}\,] = 0$, whence $T_{21}N_{c2}^{\nu-1} = 0$. Repeating this argumentation $\nu - 1$ times, we obtain $T_{21} = 0$. Then $T \in \mathbf{Gl}_{n_f}(\mathbb{R})$ yields $n_{fc2} \le n_{fc1}$. By reversing the roles of the above matrices, we analogously obtain $n_{fc1} \le n_{fc2}$ and thus $n_{fc1} = n_{fc2}$, $n_{f\overline{c}1} = n_{f\overline{c}2}$. This shows that T_{11} and T_{22} are square. Together with $T \in \mathbf{Gl}_{nf}$, we obtain $T_{11} \in \mathbf{Gl}_{n_{fc}}(\mathbb{R})$ and $T_{22} \in \mathbf{Gl}_{n_{f\overline{c}}}(\mathbb{R})$. Hence N_{c1}, N_{c2} and $N_{\overline{c}1}, N_{\overline{c}2}$ are similar, respectively.

Step 3: We show that $[E, A, B, C]$ has strict relative degree $\rho > 0$ if, and only if, (5.3.8) holds. This is an immediate consequence of the fact that, due to Step 1, the transfer function has the representation $G(s) = C(sE - A)^{-1}B = C_s(sI_{n_s} - A_s)^{-1}B_s$. This completes the proof of the lemma. □

Proof of Theorem 5.3.10: We proceed in several steps.

Step 1: We show that there exist $W, T \in \mathbf{Gl}_n(\mathbb{R})$ such that $[E, A, B, C] \overset{W,T}{\sim} [\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ for $[\hat{E}, \hat{A}, \hat{B}, \hat{C}]$ as in (5.3.4). Since a positive strict relative degree implies that $G(s)$ is strictly proper, we may apply Lemma 5.3.11 to obtain (5.3.7) for some $W_1, T_1 \in \mathbf{Gl}_n(\mathbb{R})$. Furthermore, (5.3.8) holds and hence we may transform $[I, A_s, B_s, C_s]$

into Byrnes-Isidori form (see [127, Lemma 3.5]), i.e., there exists $T_2 \in \mathbf{Gl}_{n_s}(\mathbb{R})$ such that

$$[I, A_s, B_s, C_s] \overset{T_2^{-1},T_2}{\sim} \left[I, \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ D \\ 0 \end{bmatrix}, \begin{bmatrix} I_m \\ 0 \\ \\ \vdots \\ \\ 0 \end{bmatrix}^\top \right]. \tag{5.3.9}$$

Set $W := W_1 \begin{bmatrix} I & 0 \\ 0 & T_2^{-1} \end{bmatrix}$ and $T := \begin{bmatrix} I & 0 \\ 0 & T_2 \end{bmatrix} T_1$. Since $N_c, N_{\overline{c}}$ are nilpotent and $\mathrm{rk}[\, N_c\,,\, B_{fc}] = n_{fc}$, the claim follows.

Step 2: For the proof of the uniqueness statements see [31, Thm. 2.5] or [37, Thm. B.7] in combination with Lemma 5.3.11. In particular, D is uniquely determined.

Step 3: It remains to prove (5.3.6) and that $D = \lim_{s\to\infty} s^\rho\, C(sE - A)^{-1}B$.
We prove (5.3.6): Determine the solution $X(s)$ of the linear equation

$$\begin{bmatrix} sI_m & -I_m & & & \\ & \ddots & \ddots & & \\ & & sI_m & -I_m & \\ -R_1 & \ldots & -R_{\rho-1} & sI_m - R_\rho & -S \\ -P & 0 & \ldots & 0 & sI_\mu - Q \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_\rho(s) \\ X_{\rho+1}(s) \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D \\ 0 \end{bmatrix}. \tag{5.3.10}$$

A simple iterative calculation yields

$$\begin{aligned} \text{for } i = 1, \ldots \rho - 1: \quad sX_i(s) &= X_{i+1}(s), \\ -\sum_{i=1}^{\rho-1} R_i X_i(s) + (sI_m - R_\rho)X_\rho(s) - SX_{\rho+1}(s) &= D, \\ -PX_1(s) + (sI_\mu - Q)X_{\rho+1}(s) &= 0, \end{aligned}$$

and this is equivalent to

$$
\begin{aligned}
X(s) &= \left(X_1(s)^\top, sX_1(s)^\top, \dots, s^{\rho-1}X_1(s)^\top, X_{\rho+1}(s)^\top\right)^\top, \\
D &= -\sum_{i=1}^{\rho-1} R_i s^{i-1} X_1(s) + (sI_m - R_\rho)s^{\rho-1}X_1(s) - SX_{\rho+1}(s), \\
X_{\rho+1}(s) &= (sI_\mu - Q)^{-1} P X_1(s).
\end{aligned}
\tag{5.3.11}
$$

Since the transfer function is invariant under system equivalence we have

$$
\begin{aligned}
& C(sE-A)^{-1}B = \hat{C}(s\hat{E}-\hat{A})^{-1}\hat{B} \\
= \quad & \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} \left(sI_{n-n_{fc}-n_{f\overline{c}}} - \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D \\ 0 \end{bmatrix} \\
& \quad + \begin{bmatrix} 0 & C_{f\overline{c}} \end{bmatrix} \begin{bmatrix} sN_c - I_{n_{fc}} & sN_{c\overline{c}} \\ 0 & sN_{\overline{c}} - I_{n_{f\overline{c}}} \end{bmatrix}^{-1} \begin{bmatrix} B_{fc} \\ 0 \end{bmatrix}. \\
= \quad & \begin{bmatrix} I_m & 0 & \cdots & 0 \end{bmatrix} \left(sI_{n-n_{fc}-n_{f\overline{c}}} - \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 & 0 \\ 0 & 0 & I_m & 0 & & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_{\rho-1} & R_\rho & S \\ P & 0 & \cdots & 0 & 0 & Q \end{bmatrix} \right)^{-1} \begin{bmatrix} 0 \\ \vdots \\ 0 \\ D \\ 0 \end{bmatrix} \\
\overset{(5.3.10)}{=} \quad & X_1(s) \overset{(5.3.11), R_{\rho+1}=-I}{=} - \left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_\mu - Q)^{-1}P \right]^{-1} D.
\end{aligned}
$$

This proves (5.3.6). Finally,

$$
\begin{aligned}
D &= \lim_{s\to\infty} - \left[\sum_{i=1}^{\rho+1} R_i s^{i-1} + S(sI_\mu - Q)^{-1}P \right] G(s) \\
&= -\sum_{i=1}^{\rho} R_i \lim_{s\to\infty} s^{i-1}\, G(s) + \lim_{s\to\infty} s^\rho\, G(s) - \lim_{s\to\infty} S(sI_\mu - Q)^{-1}P\, G(s) \\
&= \lim_{s\to\infty} s^\rho\, G(s)
\end{aligned}
$$

and the proof of the theorem is complete. □

Remark 5.3.12 (Zero dynamics form for DAEs).
An immediate consequence of Theorem 5.3.10 is the simplified representation of system $[E, A, B, C] \in \Sigma_{n,n,m,m}$: If ν denotes the index of $sE - A$, then a trajectory satisfies

$$(x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \cap \big(\mathcal{C}^1(\mathbb{R}; \mathbb{R}^n) \times \mathcal{C}^{\nu-1}(\mathbb{R}; \mathbb{R}^m) \times \mathcal{C}^{\rho}(\mathbb{R}; \mathbb{R}^m)\big)$$

if, and only if, $Tx = \Big(y^\top, \dot{y}^\top, \ldots, y^{(\rho-1)^\top}, \eta^\top, x_c^\top, x_{\overline{c}}^\top\Big)^\top \in \mathcal{C}^1(\mathbb{R}; \mathbb{R}^n)$ fulfills

$$\boxed{\begin{array}{rcl} y^{(\rho)}(t) &=& \sum_{i=1}^{\rho} R_i y^{(i-1)}(t) + \; S\eta(t) + \; \Gamma\, u(t) \\ \dot{\eta}(t) &=& P\, y(t) + \; Q\, \eta(t) \\ x_c(t) &=& -\sum_{i=0}^{\nu-1} N_c^i B_c u^{(i)}(t) \\ x_{\overline{c}}(t) &=& 0. \end{array}} \tag{5.3.12}$$

See Figure 5.6.

High-gain control

In this subsection we consider high-gain control for DAE systems with positive strict relative degree. From the form (5.3.4) we deduce that the input u only influences the ρth derivative of the output y. Hence we need derivative output feedback in order to achieve stabilization. We consider the high-gain controller

$$u(t) = -k\, p(\tfrac{\mathrm{d}}{\mathrm{d}t})\, y(t), \tag{5.3.13}$$

where $k > 0$ and $p(s) \in \mathbb{R}[s]$ is Hurwitz, i.e., all roots of $p(s)$ are in $\mathbb{C}_-$.

We show that asymptotically stable zero dynamics are sufficient, but not necessary, for the closed-loop system (4.1.1), (5.3.13) to be asymptotically stable. For $[E, A, B, C] \in \Sigma_{n,n,m,m}$, the system (4.1.1), (5.3.13) is called *asymptotically stable* if, and only if, for all $x \in \mathcal{C}^\infty(\mathbb{R}; \mathbb{R}^n)$ the following implication holds:

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex = Ax - k\, p(\tfrac{\mathrm{d}}{\mathrm{d}t}) BCx \quad \Longrightarrow \quad \lim_{t\to\infty} x(t) = 0.$$

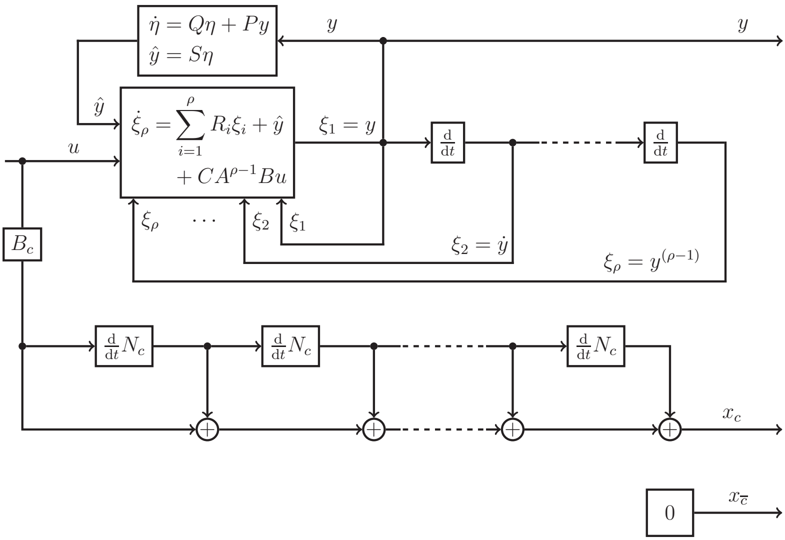


Figure 5.6: *Zero dynamics form for systems with positive strict relative degree*

Theorem 5.3.13 (High-gain control).
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ be such that $sE - A$ is regular, the transfer function $G(s)$ in (5.3.1) *has strict relative degree $\rho \in \mathbb{N}$ and the high-frequency gain matrix (cf.* (5.3.5)*) is positive definite. Let $p(s) = \sum_{i=0}^{\rho-1} p_i s^i \in \mathbb{R}[s]$ be Hurwitz and $p_{\rho-1} > 0$. Then*

$$\mathcal{ZD}_{[E,A,B,C]} \text{ is asympt. stable} \quad \Longrightarrow \quad \begin{cases} \exists\, k^* \geq 0\ \forall\, k \geq k^* : \\ \text{`(4.1.1) \& (5.3.13)' is asympt. stable.} \end{cases}$$

The converse implication is in general false even for ODE systems.

For the proof of Theorem 5.3.13 a lemma is required.

Lemma 5.3.14 (High-gain lemma).
Consider, for $D \in \mathbf{Gl}_m(\mathbb{C})$, $\tilde R \in \mathbb{C}^{m\times m}$, $\tilde S^\top, \tilde P \in \mathbb{C}^{(n-m)\times m}$, $\tilde Q \in \mathbb{C}^{(n-m)\times(n-m)}$, the parameterized matrix

$$A_\kappa := \begin{bmatrix} \tilde R - \kappa D & \tilde S \\ \tilde P & \tilde Q \end{bmatrix}, \qquad \kappa \geq 0\,.$$

Denote the spectra of D and $\tilde{Q}$ by

$$\sigma(D) = \{\gamma_1, \ldots, \gamma_m\} \subseteq \mathbb{C}\backslash\{0\} \ \textit{ and } \ \sigma(\tilde{Q}) = \{q_{m+1}, \ldots, q_n\} \subseteq \mathbb{C}\,, \ \textit{resp.}$$

Then there exist $z_1, \ldots, z_m \in \mathbb{C}$ and $\hat{\theta} > 0$ with the following property: For all $\varepsilon > 0$ and all $\theta \in (0, \hat{\theta})$ there exist $r \geq 0$ and $\kappa^* \geq 1$ such that, with a suitable enumeration of the eigenvalues $\lambda_1(A_\kappa), \ldots, \lambda_n(A_\kappa)$ of A_κ, we have, for all $\kappa \geq \kappa^*$,

(i) $B\big(z_i - \kappa\gamma_i, r + \kappa\,\theta\big) \cap B(0, 1/\varepsilon) = \emptyset$ *for* $i = 1, \ldots, m$,

(ii) $\lambda_i(A_\kappa) \in \bigcup_{j=1}^m B\big(z_j - \kappa\gamma_j, r + \kappa\,\theta\big)$ *for* $i = 1, \ldots, m$,

(iii) $\lambda_i(A_\kappa) \in \bigcup_{j=m+1}^n B\big(q_j, \varepsilon\big)$ *for* $i = m+1, \ldots, n$,

where $B(z, \varepsilon) = \{\ w \in \mathbb{C} \mid |z - w| < \varepsilon\ \}$ denotes the ball of radius ε around z in $\mathbb{C}$.

Proof:[2] Let

$$\hat{\theta} := \tfrac{1}{4} \min\big\{|\gamma_1|, \ldots, |\gamma_m|\big\} > 0 \tag{5.3.14}$$

and choose arbitrary $\theta \in (0, \hat{\theta})$. Let $U_1 \in \mathbf{Gl}_m(\mathbb{C})$, $U_2 \in \mathbf{Gl}_{n-m}(\mathbb{C})$ such that, for appropriately chosen $\delta_1, \ldots, \delta_{n-1} \in \{0, 1\}$, we have Jordan forms

$$U_1 D U_1^{-1} = \begin{bmatrix} \gamma_1 & \delta_1 & & \\ & \ddots & \ddots & \\ & & \gamma_{m-1} & \delta_{m-1} \\ & & & \gamma_m \end{bmatrix}, \quad U_2 \tilde{Q} U_2^{-1} = \begin{bmatrix} q_{m+1} & \delta_{m+1} & & \\ & \ddots & \ddots & \\ & & q_{n-1} & \delta_{n-1} \\ & & & q_n \end{bmatrix}.$$

Set

$$T_\theta := \operatorname{diag}(\theta, \theta^2, \ldots, \theta^m) \quad \text{and} \quad T_\alpha := \operatorname{diag}(\alpha, \alpha^2, \ldots, \alpha^{n-m}) \quad \text{for } \alpha > 0$$

[2] Many thanks to Achim Ilchmann (Ilmenau University of Technology) and Fabian Wirth (University of Würzburg) for the proof.

and transform A_κ to the similar matrix

$$M(\kappa,\theta,\alpha) := \begin{bmatrix} T_\theta^{-1}U_1^{-1} & 0 \\ 0 & T_\alpha U_2^{-1} \end{bmatrix} \begin{bmatrix} \tilde{R}-\kappa D & \tilde{S} \\ \tilde{P} & \tilde{Q} \end{bmatrix} \begin{bmatrix} U_1 T_\theta & 0 \\ 0 & U_2 T_\alpha^{-1} \end{bmatrix}$$

$$= \begin{bmatrix} T_\theta^{-1}U_1^{-1}\tilde{R}U_1T_\theta - \kappa \begin{bmatrix} \gamma_1 & \theta\delta_1 & & \\ & \ddots & \ddots & \\ & & \gamma_{m-1} & \theta\delta_{m-1} \\ & & & \gamma_m \end{bmatrix} & T_\theta^{-1}U_1^{-1}\tilde{S}U_2T_\alpha^{-1} \\ T_\alpha U_2^{-1}\tilde{P}U_1T_\theta & \begin{bmatrix} q_{m+1} & \delta_{m+1}/\alpha & & \\ & \ddots & \ddots & \\ & & q_{n-1} & \delta_{n-1}/\alpha \\ & & & q_n \end{bmatrix} \end{bmatrix},$$

and the off-diagonal column sums

$$\rho_j(\kappa,\theta,\alpha) := \sum_{\substack{i=1 \\ i\neq j}}^{n} |M(\kappa,\theta,\alpha)_{ij}|, \qquad j=1,\ldots,n.$$

Fix $\varepsilon > 0$. We may now choose $\alpha > 0$ sufficiently large so that the effect of the scaling matrix T_α^{-1} in the last $n-m$ columns of $M(\kappa,\theta,\alpha)$ is

$$\forall\, i = m+1,\ldots,n:\ \rho_j(\kappa,\theta,\alpha) = \rho_j(\alpha) \in [0,\varepsilon)\,.$$

Consider next the first m columns of $M(\kappa,\theta,\alpha)$. Noting that every summand in $\rho_i(\kappa,\theta,\alpha)$ which involves κ must be a product of κ and θ, we find that there exists $r = r(\alpha,\theta) \geq 0$ such that

$$\forall\, i = 1,\ldots,m\ \forall\, \kappa \geq 0:\ \rho_i(\kappa,\theta,\alpha) \leq r + \kappa\,\theta\,. \tag{5.3.15}$$

Define the diagonal entries

$$z_i := (U_1^{-1}\tilde{R}U_1)_{ii} = (T_\theta^{-1}U_1^{-1}\tilde{R}U_1T_\theta)_{ii}, \qquad i=1,\ldots,m.$$

We now show that the center of the balls $B\left(z_i - \kappa\gamma_i,\, r+\kappa\,\theta\right)$, $i = 1,\ldots,m$, tends, as $\kappa \to \infty$, to infinity at a faster pace than its radius. To this end note that using (5.3.14) gives

$$|z_i - \kappa\gamma_i| \geq \left||z_i| - 4\kappa\hat{\theta}\right|$$

and for $\kappa > (r+|z_i|)/\hat{\theta}$ we have $|z_i| < \kappa\hat{\theta}$ and hence

$$|z_i - \kappa\gamma_i| > 3\kappa\hat{\theta} - |z_i| > 2\kappa\hat{\theta} + r > \kappa\hat{\theta} + (r+\kappa\theta)\,.$$

Therefore,

$$|z_i - \kappa\gamma_i| - (\kappa\theta + r) > \kappa\hat{\theta}\,,$$

which implies that $B\,(z_i - \kappa\gamma_i,\, r + \kappa\,\theta) \cap B(0, \kappa\hat{\theta}) = \emptyset$. Choosing

$$\kappa^* > \max\left\{1/(\varepsilon\hat{\theta}), (r + |z_1|)/\hat{\theta}, \ldots, (r + |z_m|)/\hat{\theta}\right\}$$

we obtain assertion (i). Since $\gamma_i \neq 0$ for all $i = 1, \ldots, m$, we may now choose $\kappa^* \geq 1$ sufficiently large so that

$$\forall\, \kappa \geq \kappa^* : \bigcup_{i=1}^{m} B\,(z_i - \kappa\gamma_i, r + \kappa\,\theta) \;\cap \bigcup_{j=m+1}^{n} B(q_j, \varepsilon) = \emptyset\,.$$

We are now in a position to apply Gershgorin's disks, see [115, Thm 4.2.19], to deduce (ii) and (iii). This completes the proof of the lemma. □

Proof of Theorem 5.3.13: We prove "⇒". By Theorem 5.3.10, $[E, A, B, C]$ is equivalent to a system in the form (5.3.12). We introduce the 'new states'

$$\xi := \tfrac{1}{p_{\rho-1}} \cdot p\,(\tfrac{\mathrm{d}}{\mathrm{d}t})y, \quad \chi := \left(y^\top, \dot{y}^\top, \ldots, \left(y^{(\rho-2)}\right)^\top, \eta^\top\right)^\top,$$

and observe that

$$\begin{aligned} \dot{\xi}(t) &= \tilde{R}\,\xi(t) + \tilde{S}\,\chi(t) + Du(t) \\ \dot{\chi}(t) &= \tilde{P}\,\xi(t) + \tilde{Q}\,\chi(t), \end{aligned} \tag{5.3.16}$$

where $\tilde{R}, \tilde{S}$ are matrices of appropriate size and

$$\tilde{P} = [0, \ldots, 0, I_m, 0]^\top, \quad \tilde{Q} = \begin{bmatrix} \hat{A} & 0 \\ \hat{P} & Q \end{bmatrix},$$

$$\text{with} \quad \hat{A} = \begin{bmatrix} 0 & I & & \\ & \ddots & \ddots & \\ & & 0 & I \\ -\frac{p_0}{p_{\rho-1}}I & \ldots & -\frac{p_{\rho-3}}{p_{\rho-1}}I & -\frac{p_{\rho-2}}{p_{\rho-1}}I \end{bmatrix}, \quad \hat{P} = [P, 0, \ldots, 0].$$

Note that $\sigma(\hat{A}) \subseteq \mathbb{C}_-$ since $p(s)$ is Hurwitz. The feedback (5.3.13) reads in the new coordinates

$$u(t) = -k\,p\,(\tfrac{\mathrm{d}}{\mathrm{d}t})\,y(t) = -k\,p_{\rho-1}\,\xi(t), \tag{5.3.17}$$

and therefore the application of (5.3.13) to $[E, A, B, C]$ results, in terms of the new system (5.3.16), in the closed-loop system

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix}\xi(t)\\ \chi(t)\end{pmatrix} = \underbrace{\begin{bmatrix}\tilde{R} - kp_{\rho-1}D & \tilde{S}\\ \tilde{P} & \tilde{Q}\end{bmatrix}}_{=:A_k}\begin{pmatrix}\xi(t)\\ \chi(t)\end{pmatrix}. \tag{5.3.18}$$

Note that the closed-loop system (4.1.1), (5.3.13) is asymptotically stable if (5.3.18) is asymptotically stable. This is due to the fact that $p(s)$ is Hurwitz and if y decays exponentially so do all derivatives of y and, by (5.3.12), also u and all derivatives of u, which finally gives that x_c decays exponentially. The variable transformation is feasible in this case as $x \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^n)$ and hence $y = Cx \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^m)$ and $u = -k\,p(\tfrac{\mathrm{d}}{\mathrm{d}t})y \in \mathcal{C}^\infty(\mathbb{R};\mathbb{R}^m)$.
We show that (5.3.18) is asymptotically stable. Note that $\tilde{Q}$ is Hurwitz since Q is Hurwitz. Therefore, by $p_{\rho-1} > 0$, D positive definite and $\sigma(\tilde{Q}) \subseteq \mathbb{C}_-$, we may apply Lemma 5.3.14 to conclude that

$$\exists\, k^* \geq 0\ \forall\, k \geq k^* :\ \sigma(A_k) \subseteq \mathbb{C}_-.$$

This proves the claim.

To see that "⇐" does, in general not hold true, consider the counterexample (5.1.11) from the proof of Theorem 5.1.4. It is easy to see that (5.1.11) is in zero dynamics form (5.3.4) and has strict relative degree 1, hence the same arguments apply here. □

Remark 5.3.15 (High-gain stabilizability)**.**
In case of strict relative degree one, the feedback law (5.3.13) reduces to the proportional output feedback $u(t) = -k\,y(t)$, i.e. (5.1.1). If the system has higher relative degree, (5.3.13) incorporates a compensator $p(s)$ (and thus derivative feedback) to achieve a relative degree one system.
For ODE systems, the result is proved in [60] for relative degree one. By using the form (5.3.4), this result can be generalized to differential-algebraic systems with positive strict relative degree by using the same techniques.

Funnel control

In this subsection we consider funnel control for DAE systems with positive strict relative degree. The form (5.3.4) shows that the input u

only influences the ρth derivative of the output y. Hence, it is not possible to use the standard funnel controller (5.2.2) in order to achieve output feedback regulation, but a filter has to be incorporated into the controller.

Use the notation from Section 5.2.1. The control objective is output feedback regulation in the sense that the funnel controller, applied to any system $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with positive strict relative degree achieves tracking of the output of any reference signal $y_{\rm ref} \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, where ν is the index of the $sE - A$, with prespecified transient behaviour, cf. also Section 5.2.1.

For DAE systems $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with positive strict relative degree the higher degree is an obstacle; in (5.3.13) we have used derivative feedback while now we will incorporate a filter. This idea goes back to ODEs, it is shown in [126] that funnel control is feasible if a filter is incorporated in the feedback. This filter is constructed as follows

$$\dot z(t) = \underbrace{\begin{bmatrix} -I_m & I_m & 0 & \cdots & 0 & 0 \\ 0 & -I_m & I_m & \cdots & 0 & 0 \\ 0 & 0 & -I_m & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & -I_m & I_m \\ 0 & 0 & 0 & \cdots & 0 & -I_m \end{bmatrix}}_{=:F_\rho} z(t) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ I_m \end{bmatrix}}_{=:G_\rho} u(t), \quad z(0) = z^0 \tag{5.3.19}$$

with initial data $z^0 \in \mathbb{R}^{(\rho-1)m}$. The feedback law is defined recursively by the $\mathcal{C}^\infty$-functions

$$\begin{array}{rrcl} \gamma_1 : & \mathbb{R} \times \mathbb{R}^m & \to & \mathbb{R}^m, \\ & (k, e) & \mapsto & k\, e\,, \\ \gamma_2 : & \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^m & \to & \mathbb{R}^m, \\ & (k, e, z_1) & \mapsto & \gamma_1(k,e) + \|D\gamma_1(k,e)\|^2 \, k^4 \, (1 + \|z_1\|^2) \\ & & & \times (z_1 + \gamma_1(k,e)) \end{array}$$

and, for $i = 3, \ldots, \rho$,

$$\begin{aligned} \gamma_i : \mathbb{R} \times \mathbb{R}^m \times \mathbb{R}^{(i-1)m} \to \mathbb{R}^m, \quad & (k, e, (z_1, \ldots, z_{i-1})) \mapsto \\ & \gamma_{i-1}(k, e, (z_1, \ldots, z_{i-2})) + \|D\gamma_{i-1}(k, e, (z_1, \ldots, z_{i-2}))\|^2 \\ k^4 \, (1 + \|(k, e, (z_1, \ldots, z_{i-1}))\|^2) & \big(z_{i-1} + \gamma_{i-1}(k, e, (z_1, \ldots, z_{i-2}))\big) \,, \end{aligned}$$

where D denotes the derivative (Jacobian matrix). For a lengthy discussion of the intuition for the filter see [126]. Now the funnel controller (with filter (5.3.19)) for systems $[E, A, B, C] \in \Sigma_{n,m}$ with positive strict relative degree $\rho \in \mathbb{N}$ takes the form

$$\boxed{\begin{array}{rcll} u(t) & = & -\gamma_\rho\big(k(t), e(t), z(t)\big), & e(t) = y(t) - y_{\mathrm{ref}}(t)\,, \\ \dot z(t) & = & F_\rho z(t) + G_\rho u(t), & k(t) = \frac{1}{1-\varphi(t)^2\|e(t)\|^2}. \end{array}} \tag{5.3.20}$$

We will show that the assumption of asymptotically stable zero dynamics of a system (4.1.1) which has positive strict relative degree and positive definite high-frequency gain matrix implies feasibility of funnel control. In view of the fact that such systems are high-gain stabilizable (see Theorem 5.3.13), intuitively we may believe that if $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$, then the high-gain $k(t)$ forces $\|e(t)\|$ away from the funnel boundary. This is the essential property to allow for funnel control of these systems: $k(t)$ is designed in such a way that it is large if the the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$, hence avoiding contact.

Before stating the main result about funnel control we define consistency of initial values.

Definition 5.3.16 (Consistent initial value)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$, $\varphi \in \Phi^1$ and $y_{\mathrm{ref}} \in \mathcal{B}^1(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$. An initial value $x^0 \in \mathbb{R}^n$ is called *consistent* for the closed-loop system (4.1.1), (5.3.20) if, and only if, there exists a solution of the initial value problem (4.1.1), (5.3.20), $x(0) = x^0$, i.e., a function $x \in \mathcal{C}^1([0, \omega); \mathbb{R}^n)$ for some $\omega \in (0, \infty]$, such that $x(0) = x^0$ and x satisfies (4.1.1), (5.3.20) for all $t \in [0, \omega)$.

Theorem 5.3.17 (Funnel control)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ be such that $sE - A$ is regular, the transfer function $G(s)$ in (5.3.1) *has strict relative degree $\rho \in \mathbb{N}$ and the high-frequency gain matrix (cf.* (5.3.5)*) is positive definite. Suppose furthermore that $[E, A, B, C]$ has asymptotically stable zero dynamics, and let ν be the index of $sE - A$. Let $\varphi \in \Phi^{\nu+1}$ define a performance funnel $\mathcal{F}_\varphi$.*
Then, for any reference signal $y_{\mathrm{ref}} \in \mathcal{B}^{\nu+1}(\mathbb{R}_{\ge 0}; \mathbb{R}^m)$ and any consistent initial value $x^0 \in \mathbb{R}^n$, the application of the funnel controller (5.3.20) *to* (4.1.1) *yields a closed-loop initial value problem with precisely one*

maximal continuously differentiable solution $x : [0, \omega) \to \mathbb{R}^n$ *and this solution is global (i.e.* $\omega = \infty$*), and all functions* x, z, k, u *are bounded. Most importantly, the tracking error* $e = Cx - y_{\text{ref}}$ *satisfies*

$$\exists \varepsilon > 0 \; \forall t > 0 \; : \; \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon \,, \tag{5.3.21}$$

(that means e *evolves within the performance funnel* $\mathcal{F}_\varphi$ *and is uniformly bounded away from the boundary) and for the same* ε *the gain is bounded by*

$$\forall t > 0 : \; k(t) \leq \frac{1}{1 - (1 - \varphi(t)\varepsilon)^2}. \tag{5.3.22}$$

Proof: Without restriction of generality, one may consider $[E, A, B, C]$ in the form (5.3.12). Ignoring the bottom two algebraic equations in (5.3.12), existence and uniqueness of a global solution x and the bound on e follow from [126, Theorem 2]. Since γ_ρ is a $\mathcal{C}^\infty$-function, it is easy to see that u is $(\nu - 1)$-times continuously differentiable and all of these derivatives are bounded functions. Therefore, x_c and $\bar{x}_c$ in (5.3.12) are bounded functions. It remains to show the bound on k in (5.3.22): This follows from the following, which hold for all $t > 0$:

$$\begin{aligned} k(t) = \hat{k} + k(t)\varphi(t)^2\|e(t)\|^2 &\overset{(5.3.21)}{\leq} \hat{k} + k(t)\varphi(t)^2(\varphi(t)^{-1} - \varepsilon)^2 \\ &= \hat{k} + k(t)(1 - \varphi(t)\varepsilon)^2. \end{aligned}$$

□

5.3.3 Simulations

In this subsection we consider velocity control for the mechanical system depicted in Figure 5.2. The input is the relative velocity between the masses m_1 and m_2, i.e., $u(t) = \dot{z}_2(t) - \dot{z}_1(t)$. Then the mechanical system in Figure 5.2 may, analogous to position control as carried out in Subsection 5.2.2, be modeled by the second-order differential-algebraic equation

$$\begin{aligned} m_1\ddot{z}_1(t) + d_1\dot{z}_1(t) + c_1 z_1(t) - \lambda(t) &= 0 \\ m_2\ddot{z}_2(t) + d_2\dot{z}_2(t) + c_2 z_2(t) + \lambda(t) &= 0 \\ \dot{z}_2(t) - \dot{z}_1(t) &= u(t) \\ y(t) &= z_2(t). \end{aligned} \tag{5.3.23}$$

Defining the state as in (5.2.19), the model (5.3.23) may be rewritten as the linear differential-algebraic input-output system (4.1.1) for

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ -c_1 & -d_1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & -c_2 & -d_2 & -1 \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix} \qquad \text{and } E,\ B,\ C \text{ as in (5.2.20).} \tag{5.3.24}$$

We may immediately see that the pencil $sE - A$ is regular and has index $\nu = 2$; The transfer function

$$G(s) = C(sE - A)^{-1}B = \frac{m_1 s^2 + d_1 s + c_1}{(m_1 + m_2)s^3 + (d_1 + d_2)s^2 + (c_1 + c_2)s},$$

has strict relative degree 1: $\lim_{s\to\infty} s \cdot G(s) = m_1/(m_1 + m_2)$. As in Subsection 5.2.2, we may see that the zero dynamics of (5.3.24) are asymptotically stable, whence we are in the situation of Theorem 5.3.17. We choose the initial velocities

$$\dot z_1(0) = -11, \quad \dot z_2(0) = -3 \tag{5.3.25}$$

and clearly there is a unique initial constraint force $\lambda(0)$ and the initialization of (5.3.20), (5.3.24) is consistent.

Since the system has relative degree one with positive high-frequency gain $D = m_1/(m_1 + m_2) = 1/4$, all assumptions of Theorem 5.3.17 are satisfied and we may apply the funnel controller (5.3.20) with funnel boundary specified in (5.2.22) and reference signal $y_{\text{ref}} = \xi_1$ given in (5.2.21).

The simulations over the time interval $[0, 10]$, which have been performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-5}), are depicted in Figure 5.7: Figure 5.7a shows the output y tracking the reference signal y_{ref}; the error within the funnel is depicted in Figure 5.7d. Note that, due to the rather ‘academic choice’ of the example, the input u (in Figure 5.7c) and the gain function k (in Figure 5.7b) both take considerable larger values than for position control as in Subsection 5.2.2. Another reason for this behaviour is that we have kept the funnel as tight as for position control simulated in Figure 5.4, and the velocity exhibits a very ‘vivid’ behaviour which causes the error to approach the funnel boundary faster, resulting in the high values of the gain function.

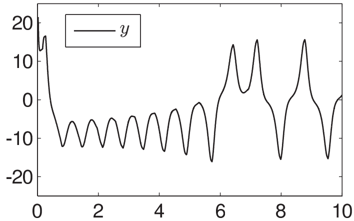


Figure a: Solution y

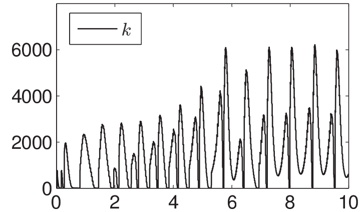


Figure b: Gain k

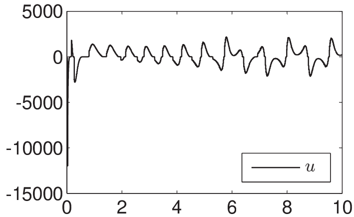


Figure c: Input u

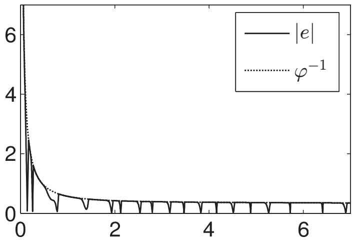


Figure d: Norm of error $|e|$ and funnel boundary φ^{-1}

Figure 5.7: *Velocity control: Simulation of the funnel controller* (5.3.20) *with funnel boundary specified in* (5.2.22) *and reference signal* $y_{\text{ref}} = \xi_1$ *given in* (5.2.21) *applied to the mechanical model* (5.3.23) *with data* (5.2.23), (5.3.25).

5.4 Notes and References

(i) Stabilization of ODE systems by high-gain control strategies came into the focus of control theory about sixty years ago, mainly impelled by applications such as stabilization of aircrafts and missiles. The advantage of high-gain control is that the system does not have to be known explicitly, only structural properties (such as asymptotically stable zero dynamics) are required. For ODEs, high-gain control strategies are well investigated, see e.g. [119, 123, 131, 132, 181]. For DAEs, high-gain stabilization techniques have been developed only recently: in contributions which are part of the present thesis.

(ii) Funnel control has been developed by Ilchmann, Ryan and

Sangwin [125] to overcome the disadvantages of high-gain control with the intuition to incorporate a time-varying gain function $k(\cdot)$ and use high values $k(t)$ only when required such that tracking with prescribed transient behavior is guaranteed (cf. Section 5.2). Nowadays, funnel controllers have been applied in a lot of technical areas, such as chemical reactor models [129], control of two-mass systems [108, 128], electrical drives [217], robotics [109] and mechatronics [110]. The simple design and moderate structural requirements make funnel controllers an eminent alternative to standard PI/PID controllers, which are often used in the engineering sciences.

(iii) The definition of vector relative degree given by Isidori [131, Sec. 5.1] is local, i.e., stated for a neighborhood of a nominal point x^0; for single-input single-output systems a global version (the uniform relative degree) is given in [131, Sec. 9.1]. An extension of the uniform relative degree concept to time-invariant nonlinear systems is given by Liberzon, Morse and Sontag [164]. For linear ODE systems, a global version of the vector relative degree has been stated by Mueller [180, 181]. The concept of strict relative degree for multi-input multi-output systems has been extended to time-varying linear and nonlinear systems by Ilchmann and Mueller [122].

6 Electrical circuits

In this chapter we consider linear differential-algebraic systems $[E, A, B, C] \in \Sigma_{n,n,m,m}$ of the form

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t)\,, \end{aligned}$$

which arise from modified nodal analysis (MNA) models of electrical circuits, cf. Section 1.5. The considered circuits may contain linear resistances, capacitances and inductances. The circuits can be described by differential-algebraic input-output systems, where the input consists of voltages of voltage sources and currents of current sources and the output consists of currents of voltage sources and voltages of current sources (cf. Section 6.8 (ii) for alternative output assignments).

As a main tool for the study of electrical circuits the notion of positive real rational functions is introduced and related to certain configurations of circuits in Section 6.1. In order to introduce the MNA modeling procedure, graph theoretical preliminaries are explained in Section 6.2 and the absence of $\mathcal{K}$-cutsets and $\mathcal{K}$-loops is characterized in terms of incidence matrices. The latter is important to derive topological criteria for asymptotic stability of electrical circuits in Section 6.3; for this result a reduction of the circuit pencil is developed. In Section 6.4 we show that, for models of electrical circuits, asymptotic stability of the zero dynamics is a structural property. That is, this property can be guaranteed if the circuit has certain interconnectivity properties. These criteria do not incorporate any parameter values. In this context, we also characterize the absence of invariant zeros in the closed right half-plane. Stabilization by high-gain output-feedback is investigated in Section 6.5. For systems with asymptotically stable zero dynamics, we prove that funnel control is feasible in Section 6.6. This result is illustrated by the simulation of a discretized transmission line in Section 6.7.

The results in Sections 6.1–6.7 stem from a joint work with Timo Reis which is submitted for publication [39].

6.1 Positive real rational functions

In this section we introduce the concept of positive realness for rational matrix functions and state a representation for this class. We also derive equivalent conditions for positive realness of matrix pencils and derive some properties of these pencils. These concepts and findings will play an important role for the analysis of MNA models.

Definition 6.1.1 (Positive real rational function)**.**
A rational matrix function $G(s) \in \mathbb{R}(s)^{m\times m}$ is called *positive real* if, and only if, $G(s)$ does not have any poles in $\mathbb{C}_+$ and, for all $\lambda \in \mathbb{C}_+$, we have

$$G(\lambda) + G^*(\lambda) \geq 0.$$

For the definition of a pole of a rational matrix function see Definition 4.3.1.

Lemma 6.1.2 (Properties of positive real functions [2, Sec. 5.1])**.**
Let $G(s) \in \mathbb{R}(s)^{m\times m}$ be positive real. Then there exist $\omega_1, \dots, \omega_k \in \mathbb{R}$, Hermitian and positive semi-definite matrices $M_1, \dots, M_k \in \mathbb{C}^{m\times m}$, $M_0, M_\infty \in \mathbb{R}^{m\times m}$ and some proper and positive real function $G_s(s) \in \mathbb{R}(s)^{m\times m}$ which does not have any poles on $i\mathbb{R}$, such that

$$G(s) = G_s(s) + sM_\infty + \frac{M_0}{s} + \sum_{j=1}^{k} \frac{M_j}{s - i\omega_j} + \frac{\overline{M}_j}{s + i\omega_j}.$$

In particular, we may characterize the positive realness of matrix pencils $sE - A \in \mathbb{R}[s]^{n\times n}$ by means of certain definiteness properties of the matrices $E, A \in \mathbb{R}^{n\times n}$.

Lemma 6.1.3 (Positive real matrix pencils)**.**
A matrix pencil $sE - A \in \mathbb{R}[s]^{n\times n}$ is positive real if, and only if, $E = E^\top \geq 0$ and $A + A^\top \leq 0$.

Proof: "$\Rightarrow$": Since $sE - A$ is positive real, Lemma 6.1.2 implies existence of some additive decomposition

$$sE - A = sM_\infty + G_s(s),$$

where $G_s(s) \in \mathbb{R}(s)^{n\times n}$ is proper and positive real, and $M_\infty \in \mathbb{R}^{n\times n}$ is symmetric and positive semi-definite. Therefore, we obtain $E = E^\top = M_\infty \geq 0$, and the constant rational function $G_p(s) = -A$ is positive real. The latter implies, by definition of positive realness, that $A + A^\top \leq 0$.

"$\Leftarrow$": Since $E = E^\top \geq 0$ and $A + A^\top \leq 0$ we have that, for all $\lambda \in \mathbb{C}_+$,

$$(\lambda E - A) + (\lambda E - A)^* = \lambda E + \overline{\lambda} E^\top - A - A^\top = 2\,\mathrm{Re}(\lambda) E - (A + A^\top) \geq 0. \tag{6.1.1}$$

Therefore, $sE - A$ is positive real. □

In the following we collect some further properties of positive real matrix pencils $sE - A$ with the additional assumption that the kernels of E and A intersect trivially. This in particular encompasses regular MNA models of passive electrical networks. Recall that $\lambda \in \mathbb{C}$ is an eigenvalue of $sE - A$ if, and only if,

$$\mathrm{rk}_{\mathbb{C}}(\lambda E - A) < \mathrm{rk}_{\mathbb{R}(s)}(sE - A).$$

Lemma 6.1.4 (Properties of positive real pencil).
Let a positive real pencil $sE - A \in \mathbb{R}[s]^{n\times n}$ be such that $\ker E \cap \ker A = \{0\}$. Then the following holds true:

(i) $sE - A$ is regular.

(ii) $(sE - A)^{-1} \in \mathbb{R}(s)^{n\times n}$ is positive real.

(iii) All eigenvalues of $sE - A$ have non-positive real part.

(iv) All eigenvalues of $sE - A$ on the imaginary axis are semi-simple.

(v) The index of $sE - A$ is at most two.

Proof: *Step 1*: To prove that (i) and (iii) hold true, we show that $\ker(\lambda E - A) = \{0\}$ for all $\lambda \in \mathbb{C}_+$. Seeking a contradiction, assume that $\lambda \in \mathbb{C}_+$ and $x \in \mathbb{C}^n \setminus \{0\}$ are such that $(\lambda E - A)x = 0$. Then we obtain

$$0 = x^*\big((\lambda E - A) + (\lambda E - A)^*\big)x \stackrel{(6.1.1)}{=} 2\,\mathrm{Re}(\lambda)x^*Ex - x^*(A + A^\top)x.$$

Since, by Lemma 6.1.3, we have $E \geq 0$, $A + A^\top \leq 0$ and $\mathrm{Re}(\lambda) > 0$, it follows $x^*Ex = x^*(A + A^\top)x = 0$, whence, in particular, $Ex = 0$. Therefore, the equation $(\lambda E - A)x = 0$ gives also rise to $Ax = 0$ and consequently, $x \in \ker E \cap \ker A = \{0\}$, a contradiction.

Step 2: We show (ii). This is a consequence of

$$\begin{aligned}
&(\lambda E - A)^{-1} + (\lambda E - A)^{-*} \\
&= (\lambda E - A)^{-1}((\lambda E - A)^* + (\lambda E - A))(\lambda E - A)^{-*} \\
&\stackrel{(6.1.1)}{=} (\lambda E - A)^{-1}(2\,\mathrm{Re}(\lambda)E - (A + A^\top))(\lambda E - A)^{-*},
\end{aligned}$$

$E \geq 0$, $A + A^\top \leq 0$ and $\mathrm{Re}(\lambda) > 0$.

Step 3: It remains to show that (iv) and (v) are valid: Since $(sE - A)^{-1}$ is positive real by (ii), Lemma 6.1.2 gives rise to the fact that all poles on the imaginary axis are of order one and, moreover, $(sE - A)^{-1} = sM + G_p(s)$, where $G_p(s) \in \mathbb{R}[s]^{n\times n}$ is proper and $M \in \mathbb{R}^{n\times n}$. This in particular means that $s^{-1}(sE - A)^{-1}$ is proper. Let $W, T \in \mathbf{Gl}_n(\mathbb{C})$ be such that $W(sE - A)T$ is in KCF as in Corollary 2.3.21. Regularity of $sE - A$ then gives rise to

$$\begin{aligned}
&(sE - A)^{-1} \\
&\quad = T^{-1}\mathrm{diag}\,(\mathcal{J}_{\rho_1}^{\lambda_1}(s)^{-1}, \ldots, \mathcal{J}_{\rho_b}^{\lambda_b}(s)^{-1}, \mathcal{N}_{\sigma_1}(s)^{-1}, \ldots, \mathcal{N}_{\sigma_c}(s)^{-1})W^{-1}.
\end{aligned} \tag{6.1.2}$$

Assuming that (iv) does not hold, i.e., there exists some $\omega \in \mathbb{R}$ such that $i\omega$ is an eigenvalue of $sE - A$ which is not semi-simple. Then there exists some $j \in \{1, \ldots, b\}$ such that $\lambda_j = i\omega$ and $\rho_j > 1$. Hence, due to

$$\mathcal{J}_{\rho_j}^{i\omega}(s)^{-1} = \sum_{l=0}^{\rho_j - 1} \frac{1}{(s - i\omega)^{l+1}} N_{\rho_j}^l,$$

the formula (6.1.2) implies that $(sE - A)^{-1}$ has a pole of order greater than one on the imaginary axis, a contradiction.

Assume that (v) does not hold, i.e., the index of $sE - A$ exceeds two. Then there exists some $j \in \{1, \ldots, c\}$ such that $\sigma_j > 2$ and

$$\mathcal{N}_{\sigma_j}(s)^{-1} = -\sum_{l=0}^{\sigma_j - 1} s^l N_{\sigma_j}^l .$$

This contradicts properness of $s^{-1}(sE - A)^{-1}$. □

6.2 Graph theoretical preliminaries

In this section we introduce the graph theoretical concepts (cf. for instance [86]) which are crucial for the modified nodal analysis of electrical circuits. We derive some characterizations for the absence of cutsets and loops in a given subgraph. These characterizations will be given in terms of algebraic properties of the incidence matrices.

Definition 6.2.1 (Graph theoretical concepts)**.**
A *graph* is a triple $\mathcal{G} = (V, E, \varphi)$ consisting of a *node set* V and a *branch set* E together with an *incidence map*

$$\varphi : E \to V \times V, \quad e \mapsto \varphi(e) = (\varphi_1(e), \varphi_2(e)),$$

where $\varphi_1(e) \neq \varphi_2(e)$ for all $e \in E$, i.e., the graph does not contain self-loops. If $\varphi(e) = (v_1, v_2)$, we call e to be *directed from* v_1 *to* v_2. v_1 is called the *initial node* and v_2 the *terminal node* of e. Two graphs $\mathcal{G}_a = (V_a, E_a, \varphi_a)$, $\mathcal{G}_b = (V_b, E_b, \varphi_b)$ are called *isomorphic*, if there exist bijective mappings $\iota_E : E_a \to E_b$, $\iota_V : V_a \to V_b$, such that $\varphi_{a,1} = \iota_V^{-1} \circ \varphi_{b,1} \circ \iota_E$ and $\varphi_{a,2} = \iota_V^{-1} \circ \varphi_{b,2} \circ \iota_E$.

Let $V' \subseteq V$ and let E' be a set of branches satisfying

$$E' \subseteq E|_{V'} := \{\, e \in E \mid \varphi_1(e) \in V' \text{ and } \varphi_2(e) \in V' \,\}.$$

Further let $\varphi|_{E'}$ be the restriction of φ to E'. Then the triple $\mathcal{K} := (V', E', \varphi|_{E'})$ is called a *subgraph of* $\mathcal{G}$. In the case where $E' = E|_{V'}$, we call $\mathcal{K}$ the *induced subgraph on* V'. If $V' = V$, then $\mathcal{K}$ is called a *spanning subgraph*. A *proper subgraph* is one with $E \neq E'$.

$\mathcal{G}$ is called *finite*, if both the node and the branch set are finite.

For each branch e, define an additional branch $-e$ being directed from the terminal to the initial node of e, that is $\varphi(-e) = (\varphi_2(e), \varphi_1(e))$

for $e \in E$. Now define the set $\tilde{E} = \{ \, e \mid e \in E \text{ or } -e \in E \, \}$. A tuple $w = (w_1, \ldots, w_r) \in \tilde{E}^r$, where for $i = 1, \ldots, r-1$,

$$v_0 := \varphi_1(v_1), \quad v_i := \varphi_2(w_i) = \varphi_1(w_{i+1})$$

is called *path from v_0 to v_r*; w is called *elementary path*, if $v_1, \ldots, v_r$ are distinct. A *loop* is an elementary path with $v_0 = v_r$. Two nodes v, v' are called *connected*, if there exists a path from v to v'. The graph itself is called connected, if any two nodes are connected. A subgraph $\mathcal{K} = (V', E', \varphi|_{E'})$ is called a *component of connectivity*, if it is connected and $\mathcal{K}^c := (V \setminus V', E \setminus E', \varphi|_{E \setminus E'})$ is a subgraph.

A spanning subgraph $\mathcal{K} = (V, E', \varphi|_{E'})$ is called a *cutset* of $\mathcal{G} = (V, E, \varphi)$, if its branch set is non-empty, $\mathcal{G} - \mathcal{K} := (V, E \setminus E', \varphi|_{E \setminus E'})$ is a disconnected subgraph and $\mathcal{G} - \mathcal{K}'$ is a connected subgraph for any proper spanning subgraph $\mathcal{K}'$ of $\mathcal{K}$.

For finite graphs we can set up special matrices which will be useful to describe Kirchhoff's laws.

Definition 6.2.2 (Incidence matrix)**.**
Let a finite graph $\mathcal{G} = (V, E, \varphi)$ with l branches $E = \{e_1, \ldots, e_l\}$ and k nodes $V = \{v_1, \ldots, v_k\}$ be given. Then the *all-node incidence matrix* of $\mathcal{G}$ is given by $\mathrm{A}_0 = (a_{ij}) \in \mathbb{R}^{k \times l}$, where

$$a_{ij} = \begin{cases} 1, & \text{if } \varphi_1(e_j) = v_i, \\ -1, & \text{if } \varphi_2(e_j) = v_i, \\ 0, & \text{otherwise.} \end{cases}$$

Since the rows of A_0 sum up to the zero row vector, one might delete an arbitrary row of A_0 to obtain a matrix A having the same rank as A_0. We call A an *incidence matrix* of $\mathcal{G}$.

This section continues with some results on the relation between properties of subgraphs and linear algebraic properties of corresponding submatrices of incidence matrices. First we declare some manners of speaking.

Definition 6.2.3.
Let $\mathcal{G}$ be a graph, $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, $\mathcal{L}$ be a subgraph of $\mathcal{G}$, and ℓ be a path of $\mathcal{G}$.

(i) $\mathcal{L}$ is called a $\mathcal{K}$-*cutset*, if $\mathcal{L}$ is a cutset of $\mathcal{K}$.

(ii) ℓ is called a $\mathcal{K}$-*loop*, if ℓ is a loop of $\mathcal{K}$.

A spanning subgraph $\mathcal{K}$ of the finite graph $\mathcal{G}$ has an incidence matrix $\mathrm{A}_{\mathcal{K}}$ which is constructed by deleting columns of the incidence matrix A of $\mathcal{G}$ corresponding to the branches of the complementary spanning subgraph $\mathcal{G}-\mathcal{K}$. By a suitable reordering of the branches, the incidence matrix reads

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{\mathcal{K}} & \mathrm{A}_{\mathcal{G}-\mathcal{K}} \end{bmatrix}. \tag{6.2.1}$$

The following lemma can be inferred form [209, Lem. 2.1 & Lem. 2.3].

Lemma 6.2.4 (Subgraphs and incidence matrices)**.**
Let $\mathcal{G}$ be a connected graph with incidence matrix $\mathrm{A} \in \mathbb{R}^{(k-1)\times l}$. Further, let $\mathcal{K}$ be a spanning subgraph. Assume that the branches of $\mathcal{G}$ are sorted in a way that (6.2.1) *is satisfied. Then the following holds true:*

(i) *The following two assertions are equivalent:*

a) *$\mathcal{G}$ does not contain $\mathcal{K}$-cutsets.*

b) $\ker \mathrm{A}_{\mathcal{G}-\mathcal{K}}^{\top} = \{0\}$.

(ii) *The following two assertions are equivalent:*

a) *$\mathcal{G}$ does not contain $\mathcal{K}$-loops.*

b) $\ker \mathrm{A}_{\mathcal{K}} = \{0\}$.

The following two auxiliary results are concerned with properties of subgraphs of subgraphs, and give some equivalent characterizations in terms of properties of their incidence matrices.

Lemma 6.2.5 (Loops in subgraphs [209, Prop. 4.5])**.**
Let $\mathcal{G}$ be a connected graph with incidence matrix $\mathrm{A} \in \mathbb{R}^{(k-1)\times l}$. Further, let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and let $\mathcal{L}$ be a spanning subgraph of $\mathcal{K}$. Assume that the branches of $\mathcal{G}$ are sorted in a way that

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{K}-\mathcal{L}} & \mathrm{A}_{\mathcal{G}-\mathcal{K}} \end{bmatrix} \quad \textit{and} \quad \mathrm{A}_{\mathcal{K}} = \begin{bmatrix} \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{K}-\mathcal{L}} \end{bmatrix}.$$

Then the following two assertions are equivalent:

a) *$\mathcal{G}$ does not contain $\mathcal{K}$-loops except for $\mathcal{L}$-loops.*

b) $\ker \mathrm{A}_{\mathcal{K}} = \ker \mathrm{A}_{\mathcal{L}} \times \{0\}$.

Lemma 6.2.6 (Cutsets in subgraphs [209, Prop. 4.4])**.**
Let $\mathcal{G}$ be a connected graph with incidence matrix $\mathrm{A} \in \mathbb{R}^{(k-1)\times l}$. Further, let $\mathcal{K}$ be a spanning subgraph of $\mathcal{G}$, and let $\mathcal{L}$ be a spanning subgraph of $\mathcal{K}$. Assume that the branches of $\mathcal{G}$ are sorted in a way that

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{K}-\mathcal{L}} & \mathrm{A}_{\mathcal{G}-\mathcal{K}} \end{bmatrix} \quad \text{and} \quad \mathrm{A}_{\mathcal{G}-\mathcal{L}} = \begin{bmatrix} \mathrm{A}_{\mathcal{K}-\mathcal{L}} & \mathrm{A}_{\mathcal{G}-\mathcal{K}} \end{bmatrix}.$$

Then the following two assertions are equivalent:

a) $\mathcal{G}$ does not contain $\mathcal{K}$-cutsets except for $\mathcal{L}$-cutsets.

b) $\ker \mathrm{A}_{\mathcal{G}-\mathcal{K}}^\top = \ker \mathrm{A}_{\mathcal{G}-\mathcal{L}}^\top$.

6.3 Circuit equations

It is well-known [83, 116] that the graph underlying an electrical circuit can be described by an incidence matrix $\mathrm{A} \in \mathbb{R}^{(k-1)\times l}$, which can be decomposed into submatrices

$$\mathrm{A} = \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} & \mathrm{A}_{\mathcal{I}} \end{bmatrix}$$

where $\mathrm{A}_{\mathcal{C}} \in \mathbb{R}^{n_e\times n_{\mathcal{C}}}, \mathrm{A}_{\mathcal{R}} \in \mathbb{R}^{n_e\times n_{\mathcal{G}}}, \mathrm{A}_{\mathcal{L}} \in \mathbb{R}^{n_e\times n_{\mathcal{L}}}, \mathrm{A}_{\mathcal{V}} \in \mathbb{R}^{n_e\times n_{\mathcal{V}}}, \mathrm{A}_{\mathcal{I}} \in \mathbb{R}^{n_e\times n_{\mathcal{I}}}$, $n_e = k-1$ and $l = n_{\mathcal{C}} + n_{\mathcal{G}} + n_{\mathcal{L}} + n_{\mathcal{V}} + n_{\mathcal{I}}$. Each submatrix is the incidence matrix of a specific subgraph of the circuit graph. $\mathrm{A}_{\mathcal{C}}$ is the incidence matrix of the subgraph consisting of all circuit nodes and all branches corresponding to capacitors. Similarly, $\mathrm{A}_{\mathcal{R}}, \mathrm{A}_{\mathcal{L}}, \mathrm{A}_{\mathcal{V}}, \mathrm{A}_{\mathcal{I}}$ are the incidence matrices corresponding to the resistor, inductor, voltage source and current source subgraphs, resp. Then using the standard MNA modeling procedure [116], which is just a clever arrangement of Kirchhoff's laws together with the characteristic equations of the devices, results in a differential-algebraic system (4.1.1) with

$$sE - A = \begin{bmatrix} s\mathrm{A}_{\mathcal{C}}\mathcal{C}\mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}}\mathcal{G}\mathrm{A}_{\mathcal{R}}^\top & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \\ -\mathrm{A}_{\mathcal{L}}^\top & s\mathcal{L} & 0 \\ -\mathrm{A}_{\mathcal{V}}^\top & 0 & 0 \end{bmatrix}, \quad B = C^\top = \begin{bmatrix} -\mathrm{A}_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I_{n_{\mathcal{V}}} \end{bmatrix}, \tag{6.3.1}$$

$$x = (\eta^\top, i_{\mathcal{L}}^\top, i_{\mathcal{V}}^\top)^\top, \quad u = (i_{\mathcal{I}}^\top, v_{\mathcal{V}}^\top)^\top, \quad y = (-v_{\mathcal{I}}^\top, -i_{\mathcal{V}}^\top)^\top, \tag{6.3.2}$$

where

$$\left.\begin{array}{l}\mathcal{C} \in \mathbb{R}^{n_\mathcal{C} \times n_\mathcal{C}}, \mathcal{G} \in \mathbb{R}^{n_\mathcal{G} \times n_\mathcal{G}}, \mathcal{L} \in \mathbb{R}^{n_\mathcal{L} \times n_\mathcal{L}}, \\ \mathrm{A}_\mathcal{C} \in \mathbb{R}^{n_e \times n_\mathcal{C}}, \mathrm{A}_\mathcal{R} \in \mathbb{R}^{n_e \times n_\mathcal{G}}, \mathrm{A}_\mathcal{L} \in \mathbb{R}^{n_e \times n_\mathcal{L}}, \mathrm{A}_\mathcal{V} \in \mathbb{R}^{n_e \times n_\mathcal{V}}, \mathrm{A}_\mathcal{I} \in \mathbb{R}^{n_e \times n_\mathcal{I}}, \\ n = n_e + n_\mathcal{L} + n_\mathcal{V}, \quad m = n_\mathcal{I} + n_\mathcal{V}. \end{array}\right\} \tag{6.3.3}$$

$\mathcal{C}$, $\mathcal{G}$ and $\mathcal{L}$ are the matrices expressing the consecutive relations of capacitances, resistances and inductances, $\eta(t)$ is the vector of node potentials, $i_\mathcal{L}(t)$, $i_\mathcal{V}(t)$, $i_\mathcal{I}(t)$ are the vectors of currents through inductances, voltage and current sources, and $v_\mathcal{V}(t)$, $v_\mathcal{I}(t)$ are the voltages of voltage and current sources.

Definition 6.3.1 (MNA model)**.**
For a given linear electrical circuit, any differential-algebraic system (4.1.1) satisfying (6.3.1)–(6.3.3), which arises from the MNA modeling procedure [116], is said to be an *MNA model* of the circuit.

It is a reasonable assumption that an electrical circuit is connected; otherwise, since the components of connectivity do not physically interact, one might consider them separately. Furthermore, in the present chapter we consider circuits with *passive* devices. These assumptions lead to the following assumptions on the MNA model (6.3.1)–(6.3.3) of the circuit (compare Lemma 6.2.4).

$$\textbf{(C1)}\quad \operatorname{rk}\begin{bmatrix}\mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{V} & \mathrm{A}_\mathcal{I}\end{bmatrix} = n_e,$$
$$\textbf{(C2)}\quad \mathcal{C} = \mathcal{C}^\top > 0, \mathcal{L} = \mathcal{L}^\top > 0, \mathcal{G} + \mathcal{G}^\top > 0.$$

It is possible that in the circuit equations (4.1.1) there are still redundant equations and superfluous variables, i.e., in general the pencil $sE - A$ arising from (6.3.1), (6.3.3) is not regular. In the following we show how this can be overcome by a simple transformation; the reduced circuit model is regular and positive real. This transformation is also important to show feasibility of funnel control in Section 6.6.

Theorem 6.3.2 (Reduction of circuit pencil)**.**
Let $sE - A \in \mathbb{R}[s]^{n \times n}$ with E, A as in (6.3.1), (6.3.3) *be given and suppose that (C1) and (C2) hold. Let $Z_{\mathcal{CRLV}}$, $Z'_{\mathcal{CRLV}}$, $\bar{Z}_\mathcal{V}$, $\bar{Z}'_\mathcal{V}$ be real matrices with full column rank such that*

$$\operatorname{im} Z_{\mathcal{CRLV}} = \ker \begin{bmatrix}\mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{V}\end{bmatrix}^\top, \quad \operatorname{im} \bar{Z}_\mathcal{V} = \ker \mathrm{A}_\mathcal{V},$$
$$\operatorname{im} Z'_{\mathcal{CRLV}} = \operatorname{im} \begin{bmatrix}\mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{V}\end{bmatrix}, \quad \operatorname{im} \bar{Z}'_\mathcal{V} = \operatorname{im} \mathrm{A}_\mathcal{V}^\top.$$

Then we have

$$T = \begin{bmatrix} Z'_{\mathcal{CRLV}} & 0 & 0 & Z_{\mathcal{CRLV}} & 0 \\ 0 & I_{n_{\mathcal{L}}} & 0 & 0 & 0 \\ 0 & 0 & \bar{Z}'_{\mathcal{V}} & 0 & \bar{Z}_{\mathcal{V}} \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}), \tag{6.3.4}$$

and

$$T^\top (sE - A) T = \begin{bmatrix} s\widetilde{E} - \widetilde{A} & 0 \\ 0 & 0 \end{bmatrix},$$

where the pencil

$$s\widetilde{E} - \widetilde{A} = \begin{bmatrix} (Z'_{\mathcal{CRLV}})^\top \left(s\mathrm{A}_{\mathcal{C}} \mathcal{C} \mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}} \mathcal{G} \mathrm{A}_{\mathcal{R}}^\top \right) Z'_{\mathcal{CRLV}} & (Z'_{\mathcal{CRLV}})^\top \mathrm{A}_{\mathcal{L}} & (Z'_{\mathcal{CRLV}})^\top \mathrm{A}_{\mathcal{V}} \bar{Z}'_{\mathcal{V}} \\ -\mathrm{A}_{\mathcal{L}}^\top Z'_{\mathcal{CRLV}} & s\mathcal{L} & 0 \\ -\bar{Z}'_{\mathcal{V}} \mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRLV}} & 0 & 0 \end{bmatrix} \tag{6.3.5}$$

is regular and satisfies $\ker \widetilde{E} \cap \ker \widetilde{A} = \{0\}$, $\widetilde{E} = \widetilde{E}^\top \geq 0$ *and* $\widetilde{A} + \widetilde{A}^\top \leq 0$.

Proof: The invertibility of T is a consequence of $\operatorname{im} Z_{\mathcal{CRLV}} \oplus \operatorname{im} Z'_{\mathcal{CRLV}} = \mathbb{R}^{n_e}$ and $\operatorname{im} \bar{Z}_{\mathcal{V}} \oplus \operatorname{im} \bar{Z}'_{\mathcal{V}} = \mathbb{R}^{n_{\mathcal{V}}}$. The properties $\widetilde{E} = \widetilde{E}^\top \geq 0$ and $\widetilde{A} + \widetilde{A}^\top \leq 0$ follow immediately from the construction of $\widetilde{E}$ and $\widetilde{A}$. To prove that $s\widetilde{E} - \widetilde{A}$ is regular, it suffices by Lemma 6.1.4 to show that $\ker \widetilde{E} \cap \ker \widetilde{A} = \{0\}$: Let $x \in \ker \widetilde{E} \cap \ker \widetilde{A}$. Partitioning according to the block structure of $\widetilde{E}$ and $\widetilde{A}$, i.e., $x = (x_1^\top, x_2^\top, x_3^\top)^\top$, and using that, by (C2), $\mathcal{C} > 0$, $\mathcal{L} > 0$ and $\mathcal{G} + \mathcal{G}^\top > 0$, we obtain from $x^\top \widetilde{E} x = x^\top \left(\widetilde{A} + \widetilde{A}^\top \right) x = 0$ that $x_2 = 0$ and

$$\begin{bmatrix} \mathrm{A}_{\mathcal{C}}^\top \\ \mathrm{A}_{\mathcal{R}}^\top \end{bmatrix} Z'_{\mathcal{CRLV}} x_1 = 0. \tag{6.3.6}$$

Furthermore, $\widetilde{A} x = 0$ gives rise to

(a) $(\bar{Z}'_{\mathcal{V}})^\top \mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRLV}} x_1 = 0$,

(b) $\mathrm{A}_{\mathcal{L}}^\top Z'_{\mathcal{CRLV}} x_1 = 0$, and

(c) $(Z'_{\mathcal{CRLV}})^\top \mathrm{A}_{\mathcal{V}} \bar{Z}'_{\mathcal{V}} x_3 = 0$.

(a) implies

$$\mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRLV}} x_1 \in \ker (\bar{Z}'_{\mathcal{V}})^\top = (\operatorname{im} \bar{Z}'_{\mathcal{V}})^\perp = (\operatorname{im} \mathrm{A}_{\mathcal{V}}^\top)^\perp,$$

whence $A_\mathcal{V}^\top Z'_{\mathcal{CRLV}} x_1 = 0$. Together with (6.3.6) and (b) this yields

$$Z'_{\mathcal{CRLV}} x_1 \in \ker \begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{V} \end{bmatrix}^\top = \operatorname{im} Z_{\mathcal{CRLV}} = (\operatorname{im} Z'_{\mathcal{CRLV}})^\perp,$$

and therefore $x_1 = 0$. By (c) we find

$$\begin{aligned} A_\mathcal{V} \bar{Z}'_\mathcal{V} x_3 \in \ker (Z'_{\mathcal{CRLV}})^\top &= (\operatorname{im} Z'_{\mathcal{CRLV}})^\perp \\ &= \ker \begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{V} \end{bmatrix}^\top \subseteq \ker A_\mathcal{V}^\top = (\operatorname{im} A_\mathcal{V})^\perp, \end{aligned}$$

and thus $A_\mathcal{V} \bar{Z}'_\mathcal{V} x_3 = 0$. From this, we obtain

$$\bar{Z}'_\mathcal{V} x_3 \in \ker A_\mathcal{V} = (\operatorname{im} A_\mathcal{V}^\top)^\perp = (\operatorname{im} \bar{Z}'_\mathcal{V})^\perp,$$

whence $x_3 = 0$. □

We may infer the following characterization of the presence of eigenvalues from Theorem 6.3.2.

Corollary 6.3.3 (Kernel and eigenvalues).
Let $sE - A \in \mathbb{R}[s]^{n \times n}$ with E, A as in (6.3.1), (6.3.3) *be given and suppose that (C1) and (C2) hold. Then*

$$\ker_{\mathbb{R}(s)} sE - A = \ker_{\mathbb{R}(s)} \begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{V} \end{bmatrix}^\top \times \{0\} \times \ker_{\mathbb{R}(s)} A_\mathcal{V}.$$

Furthermore, $\lambda \in \mathbb{C}$ is not an eigenvalue of $sE - A$ if, and only if,

$$\ker_\mathbb{C} \lambda E - A = \ker_\mathbb{C} \begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{V} \end{bmatrix}^\top \times \{0\} \times \ker_\mathbb{C} A_\mathcal{V}.$$

Proof: Using the transformation matrix T in (6.3.4) and accompanying notation from Theorem 6.3.2, we obtain (denoting the number of columns of $Z_{\mathcal{CRLV}}$ by k_1 and the number of columns of $\bar{Z}_\mathcal{V}$ by k_2) that

$$\begin{aligned} \ker_{\mathbb{R}(s)} sE - A &= T \Big(\underbrace{\ker_{\mathbb{R}(s)} (s\widetilde{E} - \widetilde{A})}_{=\{0\}} \times \mathbb{R}(s)^{k_1 + k_2} \Big) \\ &= \operatorname{im}_{\mathbb{R}(s)} Z_{\mathcal{CRLV}} \times \{0\} \times \operatorname{im}_{\mathbb{R}(s)} \bar{Z}_\mathcal{V} \\ &= \ker_{\mathbb{R}(s)} \begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{V} \end{bmatrix}^\top \times \{0\} \times \ker_{\mathbb{R}(s)} A_\mathcal{V}. \end{aligned}$$

Now let $\lambda \in \mathbb{C}$ and observe that

$$\ker_\mathbb{C} \lambda E - A = T \Big(\ker_\mathbb{C} \lambda \widetilde{E} - \widetilde{A} \times \mathbb{C}^{k_1 + k_2} \Big).$$

Recall that λ is not an eigenvalue of $sE-A$ if, and only if, $\mathrm{rk}_{\mathbb{C}}\, \lambda E-A = \mathrm{rk}_{\mathbb{R}(s)}\, sE-A$ or, equivalently, $\dim\ker_{\mathbb{C}} \lambda E - A = \dim \ker_{\mathbb{R}(s)} sE - A$. Therefore, λ is not an eigenvalue of $sE-A$ if, and only if, $\ker_{\mathbb{C}} \lambda\widetilde{E}-\widetilde{A} = \{0\}$ and this implies the last statement of the corollary. □

In the following we will use expressions like $\mathcal{VL}$-loop for a loop in the circuit graph whose branch set consists only of branches corresponding to voltage sources and/or inductors. Likewise, a $\mathcal{IC}$-cutset is a cutset in the circuit graph whose branch set consist only of branches corresponding to current sources and/or capacitors.

Corollary 6.3.4 (Regularity of circuit pencil).
Let $sE-A \in \mathbb{R}[s]^{n\times n}$ with E, A as in (6.3.1), (6.3.3) *be an MNA model of an electrical circuit and suppose that (C1) and (C2) hold. Then the following statements are equivalent:*

(i) *$sE-A$ is regular.*

(ii) $\ker \begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{V}\end{bmatrix}^\top = \{0\}$ *and* $\ker \mathrm{A}_\mathcal{V} = \{0\}$.

(iii) *The circuit neither contains $\mathcal{V}$-loops nor $\mathcal{I}$-cutsets.*

Proof: The result follows immediately from Corollary 6.3.3 and Lemma 6.2.4. □

Next we give sufficient criteria for the absence of purely imaginary eigenvalues of the pencil $sE-A$ as in (6.3.1), (6.3.3). This result can be seen as a generalization of the results in [209] to circuits which might contain $\mathcal{I}$-cutsets and/or $\mathcal{V}$-loops, i.e., where $sE-A$ is not necessarily regular.

Theorem 6.3.5 (Absence of imaginary eigenvalues).
Let $sE - A \in \mathbb{R}[s]^{n\times n}$ with E, A as in (6.3.1), (6.3.3) *be an MNA model of an electrical circuit and suppose that (C1) and (C2) hold. Furthermore, suppose that at least one of the following two assertions holds:*

(i) *The circuit neither contains $\mathcal{VL}$-loops except for $\mathcal{V}$-loops, nor $\mathcal{ICL}$-cutsets except for $\mathcal{IL}$-cutsets; equivalently*

$$\begin{aligned} &\ker \begin{bmatrix} \mathrm{A}_\mathcal{V} & \mathrm{A}_\mathcal{L}\end{bmatrix} = \ker \mathrm{A}_\mathcal{V} \times \{0\} \\ \text{and}\ \ &\ker \begin{bmatrix} \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{V}\end{bmatrix}^\top = \ker \begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{V}\end{bmatrix}^\top . \end{aligned} \tag{6.3.7}$$

(ii) *The circuit neither contains $\mathcal{IC}$-cutsets except for $\mathcal{I}$-cutsets, nor $\mathcal{VCL}$-loops except for $\mathcal{VC}$-loops; equivalently*

$$\begin{aligned} \ker \begin{bmatrix} A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} \end{bmatrix}^\top &= \ker \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} \end{bmatrix}^\top \\ \text{and } \ker \begin{bmatrix} A_{\mathcal{V}} & A_{\mathcal{C}} & A_{\mathcal{L}} \end{bmatrix} &= \ker \begin{bmatrix} A_{\mathcal{V}} & A_{\mathcal{C}} \end{bmatrix} \times \{0\}. \end{aligned} \tag{6.3.8}$$

Then all eigenvalues of $sE - A$ are contained in $\mathbb{C}_-$.

Proof: The equivalent characterizations of the absence of certain loops or cutsets in the circuit graph, resp., and kernel conditions on the element-related incidence matrices follow from Lemmas 6.2.5 and 6.2.6.

By Theorem 6.3.2 and Lemma 6.1.4 all eigenvalues of $sE - A$ are contained in $\overline{\mathbb{C}}_-$. Then, using Corollary 6.3.3, we have to show that

$$\forall\, \omega \in \mathbb{R}: \ \ker_{\mathbb{C}}(i\omega E - A) = \ker_{\mathbb{C}} \begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} \end{bmatrix}^\top \times \{0\} \times \ker_{\mathbb{C}} A_{\mathcal{V}}. \tag{6.3.9}$$

Since "$\supseteq$" does always hold true, we show "$\subseteq$". Let $\omega \in \mathbb{R}$ and $x_1 \in \mathbb{C}^{n_e}$, $x_2 \in \mathbb{C}^{n_{\mathcal{L}}}$ and $x_3 \in \mathbb{C}^{n_{\mathcal{V}}}$ be such that

$$x := (x_1^\top, x_2^\top, x_3^\top)^\top \in \ker_{\mathbb{C}}(i\omega E - A). \tag{6.3.10}$$

By the structure of $sE - A$ as in (6.3.1), relation (6.3.10) implies $A_{\mathcal{V}}^\top x_1 = 0$ and

$$\begin{aligned} 0 = x^*\big((i\omega E - A) + (i\omega E - A)^*\big)x &= -x^*(A + A^\top)x \\ &= -x_1^* A_{\mathcal{R}}(\mathcal{G} + \mathcal{G}^\top) A_{\mathcal{R}}^\top x_1, \end{aligned}$$

hence $A_{\mathcal{R}}^\top x_1 = 0$ since $\mathcal{G} + \mathcal{G}^\top > 0$ by (C2).

We show that (i) implies (6.3.9): Since $x_1 \in \ker_{\mathbb{C}} \begin{bmatrix} A_{\mathcal{R}} & A_{\mathcal{V}} \end{bmatrix}^\top$ we obtain from (6.3.7) that $x_1 \in \ker_{\mathbb{C}} A_{\mathcal{C}}^\top$. Then (6.3.10) implies $A_{\mathcal{L}} x_2 + A_{\mathcal{V}} x_3 = 0$ and by (6.3.7) we find $A_{\mathcal{V}} x_3 = 0$ and $x_2 = 0$. The latter implies that $x_1 \in \ker_{\mathbb{C}} A_{\mathcal{L}}^\top$. Altogether, we have that (6.3.9) is valid.

We show that (ii) implies (6.3.9): From (6.3.10) we have

$$A_{\mathcal{C}}(i\omega \mathcal{C} A_{\mathcal{C}}^\top x_1) + A_{\mathcal{L}} x_2 + A_{\mathcal{V}} x_3 = 0, \tag{6.3.11}$$

and by (6.3.8) we obtain $x_2 = 0$. This implies $A_{\mathcal{L}}^\top x_1 = 0$, hence $x_1 \in \ker_{\mathbb{C}} \begin{bmatrix} A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} \end{bmatrix}^\top$ which by (6.3.8) yields

$$\begin{bmatrix} A_{\mathcal{C}} & A_{\mathcal{R}} & A_{\mathcal{L}} & A_{\mathcal{V}} \end{bmatrix}^\top x_1 = 0.$$

Now, from (6.3.11) we have $A_{\mathcal{V}} x_3 = 0$ and (6.3.9) is shown. □

6.4 Zero dynamics and invariant zeros

In this section we derive topological characterizations of autonomous and asymptotically stable zero dynamics of the circuit system. The latter is done by an investigation of the invariant zeros of the system.

Using a simple transformation of the system, properties of the zero dynamics can be led back to properties of a circuit pencil where voltage sources are replaced with current sources, and vice versa. To this end, consider $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with (6.3.1), (6.3.3) and define the matrices $W, T \in \mathbf{Gl}_{n+m}(\mathbb{R})$ by

$$W = \begin{bmatrix} I_{n_e} & 0 & 0 & 0 & -\mathrm{A}_{\mathcal{V}} \\ 0 & I_{n_{\mathcal{L}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & -I_{n_{\mathcal{I}}} & 0 \\ 0 & 0 & 0 & 0 & I_{n_{\mathcal{V}}} \\ 0 & 0 & I_{n_{\mathcal{V}}} & 0 & 0 \end{bmatrix}, \quad T = \begin{bmatrix} I_{n_e} & 0 & 0 & 0 & 0 \\ 0 & I_{n_{\mathcal{L}}} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_{\mathcal{V}}} & 0 \\ 0 & 0 & I_{n_{\mathcal{I}}} & 0 & 0 \\ -\mathrm{A}_{\mathcal{V}}^\top & 0 & 0 & 0 & I_{n_{\mathcal{V}}} \end{bmatrix}.$$

Then we obtain

$$W \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} T = \begin{bmatrix} s\mathrm{A}_{\mathcal{C}}\mathcal{C}\mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}}\mathcal{G}\mathrm{A}_{\mathcal{R}}^\top & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{I}} & 0 & 0 \\ -\mathrm{A}_{\mathcal{L}}^\top & s\mathcal{L} & 0 & 0 & 0 \\ -\mathrm{A}_{\mathcal{I}}^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{n_{\mathcal{V}}} & 0 \\ 0 & 0 & 0 & 0 & I_{n_{\mathcal{V}}} \end{bmatrix}. \tag{6.4.1}$$

As desired, the upper left part is a matrix pencil which is an MNA model of a circuit in which voltage sources are replaced with current sources, and vice versa. We may now derive the following important properties, which are immediate from Corollary 6.3.3 and (6.4.1).

Corollary 6.4.1 (Kernel and eigenvalues of system pencil)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *with* (6.3.1), (6.3.3) *be an MNA model of*

an electrical circuit and suppose that (C1) and (C2) hold. Then

$$\ker_{\mathbb{R}(s)} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x_1(s) \\ 0 \\ 0 \\ x_3(s) \\ -\mathrm{A}_{\mathcal{V}}^\top x_1(s) \end{bmatrix} \;\middle|\; \begin{array}{l} x_1(s) \in \ker_{\mathbb{R}(s)} \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{I}} \end{bmatrix}^\top, \\ x_3(s) \in \ker_{\mathbb{R}(s)} \mathrm{A}_{\mathcal{I}} \end{array} \right\}.$$

Furthermore, $\lambda \in \mathbb{C}$ *is not an eigenvalue of* $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ *if, and only if,*

$$\ker_{\mathbb{C}} \begin{bmatrix} \lambda E-A & -B \\ -C & 0 \end{bmatrix} = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_3 \\ -\mathrm{A}_{\mathcal{V}}^\top x_1 \end{bmatrix} \;\middle|\; \begin{array}{l} x_1 \in \ker_{\mathbb{C}} \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{I}} \end{bmatrix}^\top, \\ x_3 \in \ker_{\mathbb{C}} \mathrm{A}_{\mathcal{I}} \end{array} \right\}.$$

We now aim to characterize autonomous zero dynamics.

Proposition 6.4.2 (Autonomous zero dynamics).
Let $[E,A,B,C] \in \Sigma_{n,n,m,m}$ *with* (6.3.1), (6.3.3) *be an MNA model of an electrical circuit and suppose that (C1) and (C2) hold. Then the following statements are equivalent:*

(i) *The zero dynamics* $\mathcal{ZD}_{[E,A,B,C]}$ *are autonomous.*

(ii) $\mathrm{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right] = n+m$.

(iii) $\ker_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right] = \{0\}$.

(iv) $\ker \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{I}} \end{bmatrix}^\top = \{0\}$ *and* $\ker \mathrm{A}_{\mathcal{I}} = \{0\}$.

(v) *The circuit neither contains* $\mathcal{I}$*-loops nor* $\mathcal{V}$*-cutsets.*

Proof: The equivalence of (i) and (ii) follows from Proposition 4.1.5 since the rank over $\mathbb{R}[s]$ and over $\mathbb{R}(s)$ coincide. (ii)⇔(iii) is clear and (iii)⇔(iv) follows from Corollary 6.4.1. The equivalence of (iv) and (v) is then a consequence of Lemma 6.2.4. □

In order to characterize asymptotic stability of the zero dynamics we need the concept of invariant zeros. An invariant zero of $[E, A, B, C] \in \Sigma_{n,n,m,m}$ is defined as an eigenvalue of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$, see e.g. [176].

Definition 6.4.3 (Invariant zeros).
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$. Then $\lambda \in \mathbb{C}$ is called an *invariant zero of* $[E, A, B, C]$ if, and only if,

$$\operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} < \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix}.$$

From Theorem 6.3.5 and (6.4.1) we get the following result on the location of invariant zeros.

Corollary 6.4.4 (Location of invariant zeros).
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *with* (6.3.1), (6.3.3) *be an MNA model of an electrical circuit and suppose that (C1) and (C2) hold. Furthermore, suppose that at least one of the following two assertions holds:*

(i) *The circuit neither contains* $\mathcal{IL}$*-loops except for* $\mathcal{I}$*-loops, nor* $\mathcal{VCL}$*-cutsets except for* $\mathcal{VL}$*-cutsets.*

(ii) *The circuit neither contains* $\mathcal{VC}$*-cutsets except for* $\mathcal{V}$*-cutsets, nor* $\mathcal{ICL}$*-loops except for* $\mathcal{IC}$*-loops.*

Then all invariant zeros of $[E, A, B, C]$ *are contained in* $\mathbb{C}_-$.

We are now in the position to characterize asymptotically stable zero dynamics.

Theorem 6.4.5 (Asymptotically stable zero dynamics).
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *with* (6.3.1), (6.3.3) *be an MNA model of an electrical circuit and suppose that (C1) and (C2) hold. Then the zero dynamics* $\mathcal{ZD}_{[E,A,B,C]}$ *are asymptotically stable if, and only if,*

a) $\mathcal{ZD}_{[E,A,B,C]}$ *are autonomous and*

b) all invariant zeros of $[E, A, B, C]$ *are contained in* $\mathbb{C}_-$.

Furthermore, suppose that at least one of the following two assertions holds:

(i) *The circuit neither contains* $\mathcal{IL}$*-loops, nor* $\mathcal{VCL}$*-cutsets except for* $\mathcal{VL}$*-cutsets with at least one inductor.*

(ii) *The circuit neither contains* $\mathcal{VC}$*-cutsets, nor* $\mathcal{ICL}$*-loops except for* $\mathcal{IC}$*-loops with at least one capacitor.*

Then the zero dynamics $\mathcal{ZD}_{[E,A,B,C]}$ *are asymptotically stable.*

Proof: *Step 1*: We show that asymptotically stable zero dynamics imply *a*) and *b*). *a*) follows from the fact that asymptotically stable zero dynamics are autonomous and *b*) follows from Lemma 4.3.9.

Step 2: We show that *a*) and *b*) imply asymptotically stable zero dynamics. By *a*) and Proposition 6.4.2 we find that $\operatorname{rk}_{\mathbb{R}(s)} \left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right] = n+m$. Then *b*) implies that

$$\forall\, \lambda \in \overline{\mathbb{C}}_+ : \ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} = \operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = n+m,$$

and therefore Lemma 4.3.9 gives asymptotic stability of the zero dynamics.

Step 3: We show that (i) or (ii) implies asymptotically stable zero dynamics. In particular, we have "The circuit neither contains $\mathcal{I}$-loops nor $\mathcal{V}$-cutsets" and hence Proposition 6.4.2 implies *a*). Furthermore, (i) or (ii) from Corollary 6.4.4 holds true and therefore *b*) is valid. This yields the assertion of the theorem. □

6.5 High-gain stabilization

In this section we consider high-gain output feedback for a system $[E, A, B, C] \in \Sigma_{n,n,m,m}$ which arises from MNA modeling of an electrical circuit. Recall that by Definition 5.1.1, roughly speaking, a system is called high-gain stabilizable if the closed-loop system

$$\tfrac{\mathrm{d}}{\mathrm{d}t} E x(t) = (A - kBC)x(t). \qquad (6.5.1)$$

is asymptotically stable for $k > 0$ large enough. In other words, there exists $\kappa > 0$ such that for all $k \geq \kappa$ the pencil $sE - (A - kBC)$ is regular and all of its eigenvalues are contained in $\mathbb{C}_-$.

We will show that for electrical circuits, i.e., $[E, A, B, C]$ with (6.3.1), (6.3.3), the high-gain need not be high; any positive k is sufficient. In order to achieve this, note that we have

$$sE - (A - kBC) = \begin{bmatrix} s\mathrm{A}_{\mathcal{C}}\mathcal{C}\mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}}\mathcal{G}\mathrm{A}_{\mathcal{R}}^\top + k\mathrm{A}_{\mathcal{I}}\mathrm{A}_{\mathcal{I}}^\top & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \\ -\mathrm{A}_{\mathcal{L}}^\top & s\mathcal{L} & 0 \\ -\mathrm{A}_{\mathcal{V}}^\top & 0 & kI_{n_{\mathcal{V}}} \end{bmatrix}. \tag{6.5.2}$$

Then, for

$$W = \begin{bmatrix} I_{n_e} & 0 & -k^{-1}\mathrm{A}_{\mathcal{V}} \\ 0 & I_{n_{\mathcal{L}}} & 0 \\ 0 & 0 & k^{-1}I_{n_{\mathcal{V}}} \end{bmatrix}, \quad T = \begin{bmatrix} I_{n_e} & 0 & 0 \\ 0 & I_{n_{\mathcal{L}}} & 0 \\ k^{-1}\mathrm{A}_{\mathcal{V}}^\top & 0 & I_{n_{\mathcal{V}}} \end{bmatrix},$$

we find that

$$\begin{aligned} &W(sE - (A - kBC))T \\ &= \begin{bmatrix} s\mathrm{A}_{\mathcal{C}}\mathcal{C}\mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}}\mathcal{G}\mathrm{A}_{\mathcal{R}}^\top + k\mathrm{A}_{\mathcal{I}}\mathrm{A}_{\mathcal{I}}^\top + k^{-1}\mathrm{A}_{\mathcal{V}}\mathrm{A}_{\mathcal{V}}^\top & \mathrm{A}_{\mathcal{L}} & 0 \\ -\mathrm{A}_{\mathcal{L}}^\top & s\mathcal{L} & 0 \\ 0 & 0 & I_{n_{\mathcal{V}}} \end{bmatrix}. \end{aligned} \tag{6.5.3}$$

The upper left part is a matrix pencil which is an MNA model of a circuit in which all current and voltage sources are replaced with resistances of values k^{-1} and k, resp. We may therefore conclude the following from Corollary 6.3.4.

Corollary 6.5.1 (Closed-loop pencil is regular)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *with* (6.3.1), (6.3.3) *be given and suppose that (C1) and (C2) hold true. Then, for all* $k > 0$*, the pencil* $sE - (A - kBC)$ *is regular.*

As a consequence of Theorem 6.3.5, we can furthermore analyze the asymptotic stability of the closed-loop system.

Theorem 6.5.2 (Asymptotic stability of closed-loop pencil)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ *with* (6.3.1), (6.3.3) *be an MNA model of an electrical circuit and suppose that (C1) and (C2) hold. Furthermore, suppose that at least one of the following two assertions holds true:*

(i) The circuit neither contains $\mathcal{L}$*-loops, nor* $\mathcal{CL}$*-cutsets except for* $\mathcal{L}$*-cutsets.*

(ii) The circuit neither contains $\mathcal{C}$-cutsets, nor $\mathcal{CL}$-loops except for $\mathcal{C}$-loops.

Then, for any $k > 0$, all eigenvalues of $sE - (A - kBC)$ are contained in $\mathbb{C}_-$.

Remark 6.5.3 (Asymptotically stable zero dynamics and high-gain)**.** Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with (6.3.1), (6.3.3) be an MNA model of an electrical circuit and suppose that (C1) and (C2) hold. Then, under one of the assumptions (i) or (ii) from Theorem 6.4.5, the respective assumption from Theorem 6.5.2 holds true, but not vice versa. Therefore, the (topological condition for) asymptotic stability of the zero dynamics implies high-gain stabilizability, but in general not the other way round, cf. Theorems 5.1.4 and 5.3.13.

6.6 Funnel control

In this section we consider funnel control for systems $[E, A, B, C] \in \Sigma_{n,n,m,m}$
with (6.3.1), (6.3.3). The aim is to achieve tracking of a reference trajectory by the output signal with prescribed transient behavior. We use the theory developed in Chapter 5 and the notation from Section 5.2.1.

The control objective is output feedback regulation so that the tracking error $e = y - y_{\text{ref}}$, where y_{ref} is the reference signal, evolves within a given funnel $\mathcal{F}_\varphi$ and all variables are bounded; here φ belongs to the class

$$\Phi := \left\{ \varphi \in \mathcal{C}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}) \cap \mathcal{B}^1(\mathbb{R}_{\geq 0}; \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi(0) = 0, \;\; \varphi(s) > 0 \\ \text{for all } s > 0 \text{ and} \\ \liminf_{s\to\infty} \varphi(s) > 0 \end{array} \right\},$$

that is, compared to Section 5.2.1, we require that the funnel boundary φ^{-1} is infinitely times continuously differentiable.

To ensure error evolution within the funnel, we use the funnel controller (5.2.2) with $\hat{k} = 1$:

$$\boxed{\begin{aligned} u(t) &= -k(t)\, e(t), \qquad \text{where} \qquad e(t) = y(t) - y_{\text{ref}}(t) \\ k(t) &= \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2}\,. \end{aligned}} \tag{6.6.1}$$

Before we state and prove feasibility of funnel control for electrical circuits, we need to define consistency of the initial value of the closed-loop system and solutions of the latter. We also define what 'feasibility of funnel control' will mean.

Definition 6.6.1 (Consistent initial value)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$, $\varphi \in \Phi$ and $y_{\text{ref}} \in \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. An initial value $x^0 \in \mathbb{R}^n$ is called *consistent* for the closed-loop system (4.1.1), (6.6.1) if, and only if, there exists a solution of the initial value problem (4.1.1), (6.6.1), $x(0) = x^0$, i.e., a function $x \in \mathcal{C}^1([0, \omega); \mathbb{R}^n)$ for some $\omega \in (0, \infty]$, such that $x(0) = x^0$ and x satisfies (4.1.1), (6.6.1) for all $t \in [0, \omega)$.

Note that, in practice, consistency of the initial state of the 'unknown' system should be satisfied as far as the DAE $[E, A, B, C]$ is the correct model.

In the following we define feasibility of funnel control for a system on a set of reference trajectories. For reference trajectories we allow signals in $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$, whereas in Section 5.2.1 signals in $\mathcal{B}^{\nu+2}(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ are allowed and $\nu \in \mathbb{N}_0$ is a number which can be calculated out of the system inversion form from Definition 4.2.6. For electrical circuits this calculation is involved and we omit it here; we restrict ourselves to the case of $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$.

Definition 6.6.2 (Feasibility of funnel control)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ and $\mathcal{S} \subseteq \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$ be a set of reference trajectories. We say that *funnel control is feasible for* $[E, A, B, C]$ *on* $\mathcal{S}$ if, and only if, for all $\varphi \in \Phi$, any reference signal $y_{\text{ref}} \in \mathcal{S}$ and any consistent initial value $x^0 \in \mathbb{R}^n$ the application of the funnel controller (6.6.1) to (4.1.1) yields a closed-loop initial-value problem that has a solution and every solution can be extended to a global solution. Furthermore, for every global solution x,

(i) x is bounded and the corresponding tracking error $e = Cx - y_{\text{ref}}$ evolves uniformly within the performance funnel $\mathcal{F}_\varphi$; more precisely,

$$\exists\, \varepsilon > 0\ \forall\, t > 0 :\ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon\,. \tag{6.6.2}$$

(ii) the corresponding gain function k given by (6.6.1) is bounded.

Remark 6.6.3 (Bound for the gain)**.**
If funnel control is feasible as stated in Definition 6.6.2, then the gain function k is bounded in the following way:

$$\forall\, t > 0:\ k(t) \le \frac{1}{1-(1-\varphi(t)\varepsilon)^2},$$

where ε is given in (6.6.2). For a proof see Step 6 of the proof of Theorem 5.2.3.

In the following we show that funnel control for systems $[E,A,B,C] \in \Sigma_{n,n,m,m}$ with (6.3.1), (6.3.3) is feasible provided that the invariant zeros have negative real part and the reference signal is sufficiently smooth and evolves in a certain subspace. The former means that the autonomous part of the zero dynamics has to be asymptotically stable, but autonomy of the whole zero dynamics is not required. As a preliminary result we derive that, for positive real systems $[E,A,B,C] \in \Sigma_{n,n,m,m}$ with asymptotically stable zero dynamics, funnel control will be feasible for any sufficiently smooth reference signal.

Proposition 6.6.4 (Funnel control for systems with stable zero dynamics)**.**
Let $[E,A,B,C] \in \Sigma_{n,n,m,m}$ be such that $E = E^\top \ge 0$, $A + A^\top \le 0$, and $B = C^\top$. Further, assume that the zero dynamics of $[E,A,B,C]$ are asymptotically stable. Then funnel control is feasible for $[E,A,B,C]$ on $\mathcal{B}^\infty(\mathbb{R}_{\ge 0};\mathbb{R}^m)$.

Proof: We aim to apply Theorem 5.2.3 for $\hat{k} = 1$ and to this end verify its assumptions.

Step 1: The zero dynamics of $[E,A,B,C]$ are asymptotically stable by assumption.

Step 2: We show that for the (left) inverse $L(s)$ of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ over $\mathbb{R}(s)$ the matrix Γ in (5.2.3) exists and satisfies $\Gamma = \Gamma^\top \ge 0$. By Lemma 6.1.3, the pencil

$$\begin{bmatrix} sE-A & -B \\ C & 0 \end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & -I_m \end{bmatrix} \begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$$

is positive real. Then $\tilde{L}(s) := L(s) \left[\begin{smallmatrix} I_n & 0 \\ 0 & -I_m \end{smallmatrix}\right]$ is the inverse of $\left[\begin{smallmatrix} sE-A & -B \\ C & 0 \end{smallmatrix}\right]$, and we have

$$\tilde{L}(\lambda) + \tilde{L}(\lambda)^* = \tilde{L}(\lambda) \left(\left[\begin{smallmatrix} \lambda E-A & -B \\ C & 0 \end{smallmatrix}\right]^* + \left[\begin{smallmatrix} \lambda E-A & -B \\ C & 0 \end{smallmatrix}\right] \right) \tilde{L}(\lambda)^* \ge 0$$

for all $\lambda \in \mathbb{C}_+$. Furthermore, since $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ does not have any invariant zeros in $\mathbb{C}_+$, $\tilde{L}(s)$ has no poles in $\mathbb{C}_+$. This shows that $\tilde{L}(s)$ is positive real. Hence, $H(s) := [0, I_m]\tilde{L}(s)[0, I_m]^\top$ is positive real and satisfies $H(s) = -[0, I_m]L(s)[0, I_m]^\top$. Now Lemma 6.1.2 yields that

$$\Gamma = \lim_{s\to\infty} s^{-1}H(s) \in \mathbb{R}^{m\times m},$$

exists and satisfies $\Gamma = \Gamma^\top \geq 0$.

Step 3: We show that $[E, A, B, C]$ is right-invertible. Since the zero dynamics of $[E, A, B, C]$ are in particular autonomous it follows from Proposition 6.4.2 (ii) that $\operatorname{rk} C = m$ and hence right-invertibility can be concluded from Remark 4.2.13.

Step 4: It remains to show that $\hat{k}$ in Theorem 5.2.3 can be chosen as $\hat{k} = 1$ and funnel control is still feasible. A careful inspection of the proof of Theorem 5.2.3 reveals that '$\hat{k}$ large enough' is needed

a) in Step 1 of the proof of Theorem 5.2.3 to show that the map $M(\cdot) = \hat{A}_{22} - k\big(I + G(\cdot)\big)$, where $G(\cdot) = G(\cdot)^\top \geq 0$ pointwise, is well-defined (i.e., pointwise invertible) for $k \geq \hat{k}$,

b) in Step 3 of the proof of Theorem 5.2.3 to show that $\hat{A}_{22} - \hat{k} \cdot k(t) I_m$ is invertible for all $t \geq 0$.

It can be deduced that both a) and b) are satisfied with $\hat{k} = 1$ if $\hat{A}_{22} - kI_m$ is negative definite for all $k > 0$, where

$$\hat{A}_{22} = [0, I_{m-m_1}]VA_{22}V^\top \begin{bmatrix} 0 \\ I_{m-m_1} \end{bmatrix},$$

m_1 and the orthogonal matrix V have been defined in Step 1 of the proof of Theorem 5.2.3, and

$$A_{22} = \lim_{s\to\infty} \left([0, I_m]L(s) \begin{bmatrix} 0 \\ I_m \end{bmatrix} + s\Gamma \right)$$

stems from the form (5.1.4). We may calculate that

$$A_{22} = \lim_{s\to\infty} (s\Gamma - H(s)) = -H_0 - \lim_{s\to\infty} H_{\rm sp}(s)$$

where, since $H(s)$ is positive real, by Lemma 6.1.2 the rational function $H_0 + H_{\mathrm{sp}}(s)$ is positive real and $\lim_{s\to\infty} H_{\mathrm{sp}}(s) = 0$. Hence, it is easy to derive that $H_0 \geq 0$ (H_0 not necessarily symmetric) and hence

$$A_{22} - kI_m = -H_0 - kI_m < 0$$

for all $k > 0$ (again $A_{22} - kI_m$ not necessarily symmetric). This implies that $\hat{A}_{22} - kI_m$ is negative definite for all $k > 0$. □

Before we prove our main result we need to know how feasibility of funnel control behaves under transformation of the system.

Lemma 6.6.5 (Funnel control under system transformation).
Let $E, A \in \mathbb{R}^{n\times n}$, $B, C^\top \in \mathbb{R}^{n\times m}$ and $\mathcal{S} \subseteq \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^m)$. Further, let $W, T \in \mathbf{Gl}_n(\mathbb{R})$, $U \in \mathcal{O}_m(\mathbb{R})$, and define

$$[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}] := [WET, WAT, WBU, U^\top CT].$$

Then funnel control is feasible for $[E, A, B, C]$ on $\mathcal{S}$ if, and only if, funnel control is feasible for $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$ on $U^\top\mathcal{S}$.

Proof: Observe that $(x, u, y) \in \mathfrak{B}_{[E,A,B,C]}$ and $y_{\mathrm{ref}} \in \mathcal{S}$ if, and only if,

$$(\widetilde{x}, \widetilde{u}, \widetilde{y}) = (T^{-1}x, U^\top u, U^\top y) \in \mathfrak{B}_{[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}]} \quad \wedge \quad U^\top y_{\mathrm{ref}} \in U^\top\mathcal{S}.$$

Then the assertion follows from the observation that, for any $\varphi \in \Phi$, and tracking errors $e = y - y_{\mathrm{ref}}$, $\tilde{e} = \tilde{y} - \tilde{y}_{\mathrm{ref}}$ we have, for all $t \geq 0$,

$$\frac{1}{1 - \varphi(t)^2\|e(t)\|^2} = \frac{1}{1 - \varphi(t)^2\|\widetilde{e}(t)\|^2}.$$ □

In the following, in order to show that funnel control is feasible for circuits where all invariant zeros are located in $\mathbb{C}_-$, but the zero dynamics are not necessarily autonomous, we derive a transformation of the circuit which decouples the 'nonautonomous part' of the zero dynamics. This part, in particular, does not affect the input-output behavior of the system.

Proposition 6.6.6 (Decoupling of circuit pencil).
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with (6.3.1), (6.3.3) be an MNA model of

an electrical circuit and suppose that (C1) and (C2) hold. Let $Z'_{\mathcal{CRLI}} \in \mathbb{R}^{n_e \times k_1}$, $Z_{\mathcal{CRLI}} \in \mathbb{R}^{n_e \times k_2}$ *with full column rank such that*

$$\begin{aligned} \operatorname{im} Z_{\mathcal{CRLI}} &= \ker \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{I}} \end{bmatrix}^\top, \\ \textit{and} \operatorname{im} Z'_{\mathcal{CRLI}} &= \operatorname{im} \begin{bmatrix} \mathrm{A}_{\mathcal{C}} & \mathrm{A}_{\mathcal{R}} & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{I}} \end{bmatrix}. \end{aligned}$$

Further, let $Z_{\mathcal{V}-\mathcal{CRLI}} \in \mathbb{R}^{n_{\mathcal{V}} \times k_3}$, $Z'_{\mathcal{V}-\mathcal{CRLV}} \in \mathbb{R}^{n_{\mathcal{V}} \times k_4}$, $\bar{Z}_{\mathcal{I}} \in \mathbb{R}^{n_{\mathcal{I}} \times k_5}$, $\bar{Z}'_{\mathcal{I}} \in \mathbb{R}^{n_{\mathcal{I}} \times k_6}$ *with orthonormal columns such that*

$$\begin{aligned} \operatorname{im} Z_{\mathcal{V}-\mathcal{CRLI}} &= \ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_{\mathcal{V}}, & \operatorname{im} \bar{Z}_{\mathcal{I}} &= \ker \mathrm{A}_{\mathcal{I}}, \\ \operatorname{im} Z'_{\mathcal{V}-\mathcal{CRLI}} &= \operatorname{im} \mathrm{A}_{\mathcal{V}}^\top Z_{\mathcal{CRLI}}, & \operatorname{im} \bar{Z}'_{\mathcal{I}} &= \operatorname{im} \mathrm{A}_{\mathcal{I}}^\top. \end{aligned}$$

Then we have

$$W^\top := T := \begin{bmatrix} Z_{\mathcal{CRLI}} & Z'_{\mathcal{CRLI}} & 0 & 0 & 0 \\ 0 & 0 & I_{n_{\mathcal{L}}} & 0 & 0 \\ 0 & 0 & 0 & Z_{\mathcal{V}-\mathcal{RCLI}} & Z'_{\mathcal{V}-\mathcal{RCLI}} \end{bmatrix} \in \mathbf{Gl}_n(\mathbb{R}) \tag{6.6.3a}$$

and

$$U := \begin{bmatrix} 0 & \bar{Z}_{\mathcal{I}} & \bar{Z}'_{\mathcal{I}} & 0 \\ Z'_{\mathcal{V}-\mathcal{RCLI}} & 0 & 0 & Z_{\mathcal{V}-\mathcal{RCLI}} \end{bmatrix} \in \mathcal{O}_m(\mathbb{R}), \tag{6.6.3b}$$

and

$$W(sE - A)T = \\ \begin{bmatrix} 0 & 0 & Z_{\mathcal{CRLI}}^\top \mathrm{A}_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRLI}} \\ 0 & s\widetilde{E}_r - \widetilde{A}_r & \begin{bmatrix} (Z'_{\mathcal{CRLI}})^\top \mathrm{A}_{\mathcal{V}} Z'_{\mathcal{V}-\mathcal{CRLI}} \\ 0 \\ 0 \end{bmatrix} \\ -(Z'_{\mathcal{V}-\mathcal{CRLI}})^\top \mathrm{A}_{\mathcal{V}}^\top Z_{\mathcal{CRLI}} & \begin{bmatrix} -(Z'_{\mathcal{V}-\mathcal{CRLI}})^\top \mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRLI}}, \, 0, \, 0 \end{bmatrix} & 0 \end{bmatrix} \tag{6.6.4}$$

and

$$WBU = \left(U^\top C T\right)^\top = \begin{bmatrix} 0 & 0 \\ 0 & \widetilde{B}_r \\ [-I_{k_4}, 0] & 0 \end{bmatrix}, \tag{6.6.5}$$

where

$$\begin{aligned} s\widetilde{E}_r - \widetilde{A}_r &= \begin{bmatrix} (Z'_{\mathcal{CRLI}})^\top \left(s\mathrm{A}_{\mathcal{C}} \mathcal{C} \mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}} \mathcal{G} \mathrm{A}_{\mathcal{R}}^\top\right) Z'_{\mathcal{CRLI}} & (Z'_{\mathcal{CRLI}})^\top \mathrm{A}_{\mathcal{L}} & Z_{\mathcal{CRLI}}^\top \mathrm{A}_{\mathcal{V}} Z_{\mathcal{V}-\mathcal{CRLI}} \\ -\mathrm{A}_{\mathcal{L}}^\top Z'_{\mathcal{CRLI}} & s\mathcal{L} & 0 \\ -Z_{\mathcal{V}-\mathcal{CRLI}}^\top \mathrm{A}_{\mathcal{V}}^\top Z'_{\mathcal{CRLI}} & 0 & 0 \end{bmatrix}, \\ \widetilde{B}_r = \widetilde{C}_r^\top &= \begin{bmatrix} -(Z'_{\mathcal{CRLI}})^\top \mathrm{A}_{\mathcal{I}} \bar{Z}'_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I_{k_3} \end{bmatrix} \end{aligned} \tag{6.6.6}$$

Furthermore, the following holds true:

(a) $k_2 = k_4$ *and* $Z_{\mathcal{CRLI}}^\top A_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}} \in \mathbf{Gl}_{k_2}(\mathbb{R})$.

(b) *The zero dynamics of the system* $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$ *are autonomous.*

(c) $\lambda \in \mathbb{C}$ *is an invariant zero of* $[E, A, B, C]$ *if, and only if,* λ *is an invariant zero of* $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$.

Proof: The invertibility of W, T and U is a consequence of

$$\begin{aligned}
&\operatorname{im} Z'_{\mathcal{CRLI}} \oplus \operatorname{im} Z_{\mathcal{CRLI}} \\
&\qquad\qquad = \operatorname{im} \begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{I} \end{bmatrix} \oplus \ker \begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{I} \end{bmatrix}^\top = \mathbb{R}^{n_e}, \\
&\operatorname{im} Z'_{\mathcal{V}-\mathcal{RCLI}} \oplus \operatorname{im} Z_{\mathcal{V}-\mathcal{RCLI}} = \operatorname{im} A_\mathcal{V}^\top Z_{\mathcal{CRLI}} \oplus \ker Z_{\mathcal{CRLI}}^\top A_\mathcal{V} = \mathbb{R}^{n_\mathcal{V}}, \\
&\operatorname{im} \bar{Z}'_\mathcal{I} \oplus \operatorname{im} \bar{Z}_\mathcal{I} = \operatorname{im} A_\mathcal{I}^\top \oplus \ker A_\mathcal{I} = \mathbb{R}^{n_\mathcal{I}}.
\end{aligned}$$

Furthermore, by choice of $Z_{\mathcal{V}-\mathcal{CRLI}}$, $Z'_{\mathcal{V}-\mathcal{CRLV}}$, $\bar{Z}_\mathcal{I}$ and $\bar{Z}'_\mathcal{I}$ the matrix U is orthogonal. The representation of the transformed system in (6.6.4), (6.6.5) and (6.6.6) is then a simple calculation.

We prove assertions (a)–(c).

(a) The assertion will be inferred from the fact that both matrices $Z_{\mathcal{CRLI}}^\top A_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}}$ and $(Z_{\mathcal{CRLI}}^\top A_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}})^\top$ have trivial kernels. To prove the first assertion, let $z \in \ker Z_{\mathcal{CRLI}}^\top A_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}}$. Then

$$Z'_{\mathcal{V}-\mathcal{CRLI}} z \in \ker Z_{\mathcal{CRLI}}^\top A_\mathcal{V} = (\operatorname{im} A_\mathcal{V}^\top Z_{\mathcal{CRLI}})^\perp = (\operatorname{im} Z'_{\mathcal{V}-\mathcal{CRLI}})^\perp.$$

Therefore, $Z'_{\mathcal{V}-\mathcal{CRLI}} z = 0$, and the full column rank of $Z'_{\mathcal{V}-\mathcal{CRLI}}$ implies $z = 0$. Now let $z \in \ker (Z'_{\mathcal{V}-\mathcal{CRLI}})^\top A_\mathcal{V}^\top Z_{\mathcal{CRLI}}$. Then

$$A_\mathcal{V}^\top Z_{\mathcal{CRLI}} z \in \ker (Z'_{\mathcal{V}-\mathcal{CRLI}})^\top = (\operatorname{im} Z'_{\mathcal{V}-\mathcal{CRLI}})^\perp = (\operatorname{im} A_\mathcal{V}^\top Z_{\mathcal{CRLI}})^\perp.$$

Thus, $Z_{\mathcal{CRLI}} z \in \ker A_\mathcal{V}^\top$ and by choice of $Z_{\mathcal{CRLI}}$ we have

$$Z_{\mathcal{CRLI}} z \in \ker \begin{bmatrix} A_\mathcal{C} & A_\mathcal{R} & A_\mathcal{L} & A_\mathcal{I} \end{bmatrix}^\top \cap \ker A_\mathcal{V}^\top \overset{(C1)}{=} \{0\},$$

Hence, we obtain $z = 0$ from the full column rank of $Z_{\mathcal{CRLI}}$.

(b) By Proposition 6.4.2 it is sufficient to show that the pencil

$$s\mathcal{E} - \mathcal{A} := \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & \widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} = \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$$

is regular. Observing that $\mathcal{E} = \mathcal{E}^\top \geq 0$ and $\mathcal{A} + \mathcal{A}^\top \leq 0$, we can use Lemma 6.1.4 to further reduce the problem to showing that $\ker \mathcal{E} \cap \ker \mathcal{A} = \{0\}$:
Let $z = (z_1, z_2, z_3, z_4, z_5) \in \ker \mathcal{E} \cap \ker \mathcal{A}$ be suitably partitioned according to the block structure of $\widetilde{E}_r$, $\widetilde{A}_r$, $\widetilde{B}_r$ and $\widetilde{C}_r$ as in (6.6.6). Then, by (C2), the equation $z^\top \mathcal{E} z = z^\top (\mathcal{A} + \mathcal{A}^\top) z = 0$ gives rise to $z_2 = 0$ and

$$z_1 \in \ker \begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} \end{bmatrix}^\top Z'_{\mathcal{CRLI}}.$$

The equation $\mathcal{A}z = 0$ further implies $z_3 = 0$ and

$$z_1 \in \ker \mathrm{A}_\mathcal{L}^\top Z'_{\mathcal{CRLI}} \;\wedge\; z_1 \in \ker (\bar{Z}'_\mathcal{I})^\top \mathrm{A}_\mathcal{I}^\top Z'_{\mathcal{CRLI}}.$$

The latter implies

$$\mathrm{A}_\mathcal{I}^\top Z'_{\mathcal{CRLI}} z_1 \in \ker (\bar{Z}'_\mathcal{I})^\top = (\operatorname{im} \bar{Z}'_\mathcal{I})^\perp = (\operatorname{im} \mathrm{A}_\mathcal{I}^\top)^\perp,$$

whence

$$z_1 \in \ker \mathrm{A}_\mathcal{I}^\top Z'_{\mathcal{CRLI}}.$$

Altogether, we have

$$\begin{aligned} Z'_{\mathcal{CRLI}} z_1 \in \ker & \begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{I} \end{bmatrix}^\top \\ & = \left(\operatorname{im} \begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{I} \end{bmatrix}\right)^\perp = (\operatorname{im} Z'_{\mathcal{CRLI}})^\perp. \end{aligned}$$

The full column rank of $Z'_{\mathcal{CRLI}}$ now implies that $z_1 = 0$. Now using that $z_1 = 0$, $z_2 = 0$ and $z_3 = 0$, we can infer from $\mathcal{A}z = 0$ that $z_5 = 0$ and

$$(Z'_{\mathcal{CRLI}})^\top \mathrm{A}_\mathcal{I} \bar{Z}'_\mathcal{I} z_4 = 0.$$

Thus,

$$\begin{aligned} \mathrm{A}_\mathcal{I} \bar{Z}'_\mathcal{I} z_4 \in \ker (Z'_{\mathcal{CRLI}})^\top & = (\operatorname{im} Z'_{\mathcal{CRLI}})^\perp \\ & = \left(\operatorname{im} \begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{I} \end{bmatrix}\right)^\perp \subseteq (\operatorname{im} \mathrm{A}_\mathcal{I})^\perp. \end{aligned}$$

Therefore, $\mathrm{A}_\mathcal{I} \bar{Z}'_\mathcal{I} z_4 = 0$ or, equivalently,

$$\bar{Z}'_\mathcal{I} z_4 \in \ker \mathrm{A}_\mathcal{I} = (\operatorname{im} \mathrm{A}_\mathcal{I}^\top)^\perp = (\operatorname{im} \bar{Z}'_\mathcal{I})^\perp.$$

This implies $\bar{Z}'_\mathcal{I} z_4 = 0$, and since $\bar{Z}'_\mathcal{I}$ has full column rank, we have that $z_4 = 0$.

(c) It can be obtained from simple row and column operations that for all $\lambda \in \mathbb{C}$ we have

$$\mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda E - A & -B \\ -C & 0 \end{bmatrix} = \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda WET - WAT & -WBU \\ -U^\top CT & 0 \end{bmatrix} = \mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda \widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} + 2k_4$$

and, similarly,

$$\mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} sE - A & -B \\ -C & 0 \end{bmatrix} = \mathrm{rk}_{\mathbb{R}(s)} \begin{bmatrix} s\widetilde{E}_r - \widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{bmatrix} + 2k_4.$$

This implies that the eigenvalues of $\left[\begin{smallmatrix} sE-A & -B \\ -C & 0 \end{smallmatrix}\right]$ coincide with those of $\left[\begin{smallmatrix} s\widetilde{E}-\widetilde{A}_r & -\widetilde{B}_r \\ -\widetilde{C}_r & 0 \end{smallmatrix}\right]$ and hence the assertion is proved.

This concludes the proof of the proposition. □

We are now in the position to prove the main result of this section.

Theorem 6.6.7 (Funnel control for circuits)**.**
Let $[E, A, B, C] \in \Sigma_{n,n,m,m}$ with (6.3.1), (6.3.3) *be an MNA model of an electrical circuit and suppose that (C1) and (C2) hold. Assume that the system $[E, A, B, C]$ does not have any invariant zeros on the imaginary axis. Let $Z_{\mathcal{CRLI}}$ be a matrix with full column rank such that*

$$\mathrm{im}\, Z_{\mathcal{CRLI}} = \ker \begin{bmatrix} \mathrm{A}_\mathcal{C} & \mathrm{A}_\mathcal{R} & \mathrm{A}_\mathcal{L} & \mathrm{A}_\mathcal{I} \end{bmatrix}^\top .$$

Then funnel control is feasible for $[E, A, B, C]$ on

$$\mathcal{B}^\infty \left(\mathbb{R}_{\geq 0}; \mathrm{im}\, \mathrm{A}_\mathcal{I}^\top \times \ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_\mathcal{V}\right).$$

Proof: *Step 1*: Use the notation from Proposition 6.6.6 and define

$$[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}] := [WET, WAT, WBU, U^\top CT].$$

Then, by Lemma 6.6.5, it suffices to prove that funnel control is feasible for $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$ on

$$\mathcal{S} := U^\top \mathcal{B}^\infty \left(\mathbb{R}_{\geq 0}; \mathrm{im}\, \mathrm{A}_\mathcal{I}^\top \times \ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_\mathcal{V}\right).$$

Step 2: We show that $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$ has asymptotically stable zero dynamics. By Proposition 6.6.6 (c), the zero dynamics of $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$ are autonomous. Furthermore, by Proposition 6.6.6 (d) and the fact that the invariant zeros of $[E, A, B, C]$ all have negative real part, we obtain from Theorem 6.4.5 that the zero dynamics of $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$ are asymptotically stable.

Step 3: We reduce the feasibility problem of funnel control to that of the
system $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$. Let

$$(\widetilde{x}, \widetilde{u}, \widetilde{y}) \in \mathfrak{B}_{[\widetilde{E},\widetilde{A},\widetilde{B},\widetilde{C}]} \quad \text{and} \quad \widetilde{y}_{\text{ref}} = U^\top \begin{pmatrix} y_{\text{ref},1} \\ y_{\text{ref},2} \end{pmatrix} \in \mathcal{S}.$$

Since

$$y_{\text{ref},1} \in \operatorname{im} \mathrm{A}_\mathcal{I}^\top = \operatorname{im} \bar{Z}'_\mathcal{I} = \left(\operatorname{im} \bar{Z}_\mathcal{I}\right)^\perp = \ker \bar{Z}_\mathcal{I}^\top$$

and

$$\begin{aligned} y_{\text{ref},2} \in \ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_\mathcal{V} &= \operatorname{im} Z_{\mathcal{V}-\mathcal{CRLI}} \\ &= (\operatorname{im} Z'_{\mathcal{V}-\mathcal{CRLI}})^\perp = \ker \left(Z'_{\mathcal{V}-\mathcal{CRLI}}\right)^\top \end{aligned}$$

we obtain that

$$\widetilde{y}_{\text{ref}} = \left(0, \;\; 0, \;\; \widetilde{y}_{\text{ref},1}, \;\; \widetilde{y}_{\text{ref},2}\right)^\top,$$

where $\widetilde{y}_{\text{ref},1} = \left(\bar{Z}'_\mathcal{I}\right)^\top y_{\text{ref},1}$ and $\widetilde{y}_{\text{ref},2} = Z_{\mathcal{V}-\mathcal{CRLI}}^\top y_{\text{ref},2}$. By suitably partitioning

$$\widetilde{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \\ x_4(t) \\ x_5(t) \end{pmatrix}, \quad \widetilde{u}(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \\ u_3(t) \\ u_4(t) \end{pmatrix}, \quad \widetilde{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{pmatrix}$$

according to the block structure of $s\widetilde{E} - \widetilde{A}$ as in (6.6.4), and $\widetilde{B}$, $\widetilde{C}$ as in (6.6.5), we obtain $Z_{\mathcal{CRLI}}^\top \mathrm{A}_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}} x_5 = 0$, whence, by Proposition 6.6.6 (b), we have $x_5 = 0$, and thus also $y_1 = 0$. Moreover, $y_2 = 0$ and

$$x_1 = -(Z_{\mathcal{CRLI}}^\top \mathrm{A}_\mathcal{V} Z'_{\mathcal{V}-\mathcal{CRLI}})^{-1} (Z'_{\mathcal{V}-\mathcal{CRLI}})^\top \mathrm{A}_\mathcal{V}^\top Z'_{\mathcal{CRLI}} x_2 - u_1,$$

and, further

$$\widetilde{x}_r(t) = \begin{pmatrix} x_2(t) \\ x_3(t) \\ x_4(t) \end{pmatrix}, \quad \widetilde{u}_r(t) = \begin{pmatrix} u_3(t) \\ u_4(t) \end{pmatrix}, \quad \widetilde{y}_r(t) = \begin{pmatrix} y_3(t) \\ y_4(t) \end{pmatrix}$$

satisfy

$$(\widetilde{x}_r, \widetilde{u}_r, \widetilde{y}_r) \in \mathfrak{B}_{[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]}.$$

Application of the funnel controller (6.6.1) then yields $\widetilde{u} = -k(\widetilde{y} - \widetilde{y}_{\mathrm{ref}})$ and hence $u_1 = 0$ and $u_2 = 0$. Therefore, funnel control is feasible for $[\widetilde{E}, \widetilde{A}, \widetilde{B}, \widetilde{C}]$ on $\mathcal{S}$ if, and only if, funnel control is feasible for $[\widetilde{E}_r, \widetilde{A}_r, \widetilde{B}_r, \widetilde{C}_r]$ on $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^{k_3+k_6})$. The latter however follows from Step 2 and Proposition 6.6.4. This concludes the proof of the theorem. □

Remark 6.6.8 (Topological criteria for funnel control)**.**
We analyze the constraints on the reference trajectories in Theorem 6.6.7.

(a) The subspace restriction

$$\forall\, t \geq 0: \ y_{\mathrm{ref}}(t) \in \operatorname{im} \mathrm{A}_{\mathcal{I}}^\top \times \ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_{\mathcal{V}} \tag{6.6.7}$$

on the reference signal can be interpreted as follows: If the circuit contains a $\mathcal{V}$-cutset, then, by Kirchhoff's current law, the currents of the voltage sources in the $\mathcal{V}$-cutset sum up to zero. Likewise, if the circuit contains an $\mathcal{I}$-loop, then Kirchhoff's voltage law implies that the voltages of the current sources in the $\mathcal{I}$-loop sum up to zero. Condition (6.6.7) therefore means that, in a sense, the reference signal has to satisfy Kirchhoff's laws pointwise, see also Figure 6.1.

(b) Invoking that

$$\begin{aligned} \ker Z_{\mathcal{CRLI}}^\top = (\operatorname{im} Z_{\mathcal{CRLI}})^\perp = \Big(\ker \big[\mathrm{A}_{\mathcal{C}} \;\; \mathrm{A}_{\mathcal{R}} \;\; \mathrm{A}_{\mathcal{L}} \;\; \mathrm{A}_{\mathcal{I}}\big]^\top\Big)^\perp \\ = \operatorname{im} \big[\mathrm{A}_{\mathcal{C}} \;\; \mathrm{A}_{\mathcal{R}} \;\; \mathrm{A}_{\mathcal{L}} \;\; \mathrm{A}_{\mathcal{I}}\big], \end{aligned}$$

we find

$$\ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_{\mathcal{V}} = \left\{\, x \in \mathbb{R}^{n_{\mathcal{V}}} \;\middle|\; \mathrm{A}_{\mathcal{V}} x \in \operatorname{im} \big[\mathrm{A}_{\mathcal{C}} \;\; \mathrm{A}_{\mathcal{R}} \;\; \mathrm{A}_{\mathcal{L}} \;\; \mathrm{A}_{\mathcal{I}}\big] \,\right\}.$$

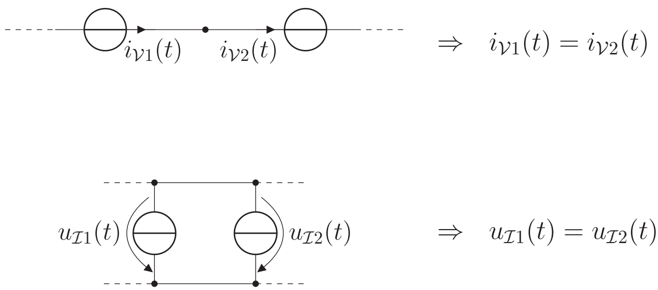


Figure 6.1: *Interpretation of condition* (6.6.7) *in terms of Kirchhoff's laws*

In particular, this space is independent of the choice of the matrix $Z_{\mathcal{CRLI}}$ with $\operatorname{im} Z_{\mathcal{CRLI}} = \ker \left[\mathrm{A}_\mathcal{C} \;\; \mathrm{A}_\mathcal{R} \;\; \mathrm{A}_\mathcal{L} \;\; \mathrm{A}_\mathcal{I}\right]^\top$.

(c) We have that $\ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_\mathcal{V} = \mathbb{R}^{n_\mathcal{V}}$ if, and only if,

$$\begin{aligned}\operatorname{im} \mathrm{A}_\mathcal{V} \subseteq \ker Z_{\mathcal{CRLI}}^\top = (\operatorname{im} Z_{\mathcal{CRLI}})^\perp &= \left(\ker \left[\mathrm{A}_\mathcal{C} \;\; \mathrm{A}_\mathcal{R} \;\; \mathrm{A}_\mathcal{L} \;\; \mathrm{A}_\mathcal{I}\right]^\top\right)^\perp \\ &= \operatorname{im} \left[\mathrm{A}_\mathcal{C} \;\; \mathrm{A}_\mathcal{R} \;\; \mathrm{A}_\mathcal{L} \;\; \mathrm{A}_\mathcal{I}\right].\end{aligned}$$

Hence, by (C1), $\ker Z_{\mathcal{CRLI}}^\top \mathrm{A}_\mathcal{V} = \mathbb{R}^{n_\mathcal{V}}$ is equivalent to

$$\operatorname{im} \left[\mathrm{A}_\mathcal{C} \;\; \mathrm{A}_\mathcal{R} \;\; \mathrm{A}_\mathcal{L} \;\; \mathrm{A}_\mathcal{I}\right] = \mathbb{R}^{n_e}.$$

The latter is, by Lemma 6.2.4, equivalent to the absence of $\mathcal{V}$-cutsets in the given electrical circuit.
Furthermore, $\operatorname{im} \mathrm{A}_\mathcal{I}^\top = \mathbb{R}^{n_\mathcal{I}}$ if, and only if, $\{0\} = \left(\operatorname{im} \mathrm{A}_\mathcal{I}^\top\right)^\perp = \ker \mathrm{A}_\mathcal{I}$. By Lemma 6.2.4 the latter is equivalent to the absence of $\mathcal{I}$-loops in the given electrical circuit.

(d) By virtue of Theorem 6.6.7 and Corollary 6.4.4, we see that funnel control is feasible for passive and connected electrical circuits (on a suitable set of reference trajectories) provided that at least one of the following two properties is satisfied:

  (i) The circuit neither contains $\mathcal{IL}$-loops except for $\mathcal{I}$-loops, nor $\mathcal{VCL}$-cutsets except for $\mathcal{VL}$-cutsets.

  (ii) The circuit neither contains $\mathcal{VC}$-cutsets except for $\mathcal{V}$-cutsets, nor $\mathcal{ICL}$-loops except for $\mathcal{IC}$-loops.

(e) By virtue of Proposition 6.6.4 and Theorem 6.4.5, we see that funnel control is feasible for passive and connected electrical circuits (on the set of *all* sufficiently smooth reference trajectories) provided that at least one of the following two properties is satisfied:

  (i) The circuit neither contains $\mathcal{IL}$-loops, nor $\mathcal{VCL}$-cutsets except for $\mathcal{VL}$-cutsets with at least one inductor.

  (ii) The circuit neither contains $\mathcal{VC}$-cutsets, nor $\mathcal{ICL}$-loops except for $\mathcal{IC}$-loops with at least one capacitor.

6.7 Simulation

For purposes of illustration we consider an example of a discretized transmission line. We derive an MNA model (6.3.1), (6.3.3) and show that the funnel controller (6.6.1) achievs tracking of a sinusoidal reference signal with prescribed transient behavior of the tracking error.

We consider a discretized transmission line as depicted in Figure 6.2, where n is the number of spacial discretization points.

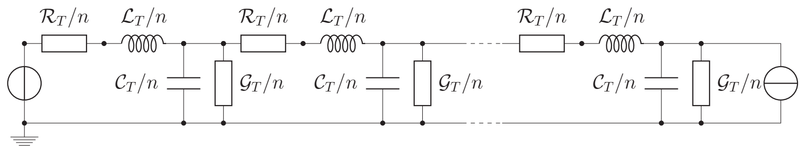


Figure 6.2: *Discretized transmission line*

The element related incidence matrices of this circuit can be calculated as

$$\begin{aligned}
A_{\mathcal{C}} &= \operatorname{diag}\left(\begin{bmatrix}0\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \ldots, \begin{bmatrix}0\\1\end{bmatrix}\right) \in \mathbb{R}^{(2n+1)\times n},\\
A_{\mathcal{R}} &= \left[\operatorname{diag}\left(\begin{bmatrix}1\\-1\end{bmatrix}, \ldots, \begin{bmatrix}1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\\0\end{bmatrix}\right),\ A_{\mathcal{C}}\right] \in \mathbb{R}^{(2n+1)\times 2n},\\
A_{\mathcal{L}} &= \operatorname{diag}\left(\begin{bmatrix}0\\1\\-1\end{bmatrix}, \begin{bmatrix}1\\-1\end{bmatrix}, \ldots, \begin{bmatrix}1\\-1\end{bmatrix}\right) \in \mathbb{R}^{(2n+1)\times n},\\
A_{\mathcal{V}} &= [1, 0, \ldots, 0]^\top \in \mathbb{R}^{(2n+1)\times 1},\\
A_{\mathcal{I}} &= [0, \ldots, 0, 1]^\top \in \mathbb{R}^{(2n+1)\times 1}.
\end{aligned}$$

The matrices expressing the consecutive relations of capacitances, resistances (and conductances, resp.) and inductances are given by

$$\mathcal{C} = \frac{\mathcal{C}_T}{n} I_n, \quad \mathcal{G} = \operatorname{diag}\left(\frac{n}{\mathcal{R}_T} I_n, \frac{\mathcal{G}_T}{n} I_n\right), \quad \mathcal{L} = \frac{\mathcal{L}_T}{n} I_n.$$

The differential-algebraic system (4.1.1) describing the discretized transmission line is then given by $[E, A, B, C]$ for the matrices in (6.3.1).

The circuit in Figure 6.2 does not contain any $\mathcal{IL}$-loops. Further, the only $\mathcal{VCL}$-cutset of the circuit is formed by the voltage source and the inductance of the left branch. We can therefore conclude from Theorem 6.4.5 that $[E, A, B, C]$ has asymptotically stable zero dynamics. Then, by Proposition 6.6.4, funnel control is feasible for $[E, A, B, C]$ on $\mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^2)$.

For the simulation we chose the parameters

$$n = 50, \quad \mathcal{C}_T = \mathcal{R}_T = \mathcal{G}_T = \mathcal{L}_T = 1,$$

and the (consistent) initial value for the closed-loop system $[E, A, B, C]$, (6.6.1) by

$$x^0 = (-1, -1.04, \underbrace{2, 1.96, \ldots, 2, 1.96}_{(n-1)\text{-times}}, \underbrace{2, \ldots, 2}_{(n+1)\text{-times}}, -2) \in \mathbb{R}^{3n+2}. \quad (6.7.1)$$

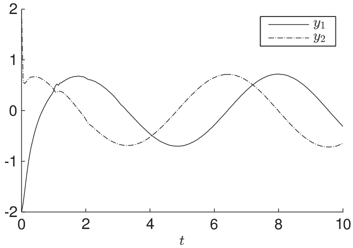


Figure a: Solution components y_1 and y_2

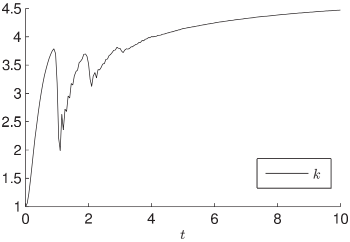


Figure b: Gain k

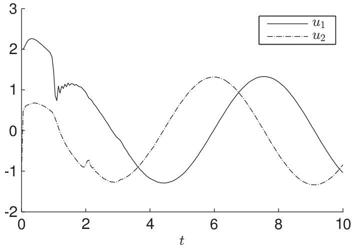


Figure c: Input components u_1 and u_2

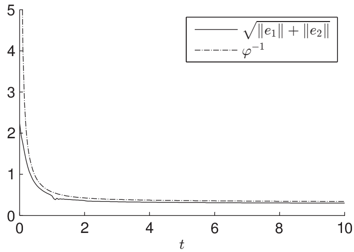


Figure d: Norm of error $\|e(\cdot)\|$ and funnel boundary $\varphi(\cdot)^{-1}$

Figure 6.3: *Simulation of the funnel controller* (6.6.1) *with funnel boundary specified in* (6.7.2) *and reference signal* $y_{\mathrm{ref}} = (\sin, \cos)^\top$ *applied to system* $[E, A, B, C]$ *with initial data* (6.7.1).

As reference signal we take $y_{\mathrm{ref}} = (\sin, \cos)^\top \in \mathcal{B}^\infty(\mathbb{R}_{\geq 0}; \mathbb{R}^2)$. The funnel $\mathcal{F}_\varphi$ is determined by the function

$$\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}, \quad t \mapsto 0.5\ te^{-t} + 2\ \arctan t\,. \tag{6.7.2}$$

Note that this prescribes an exponentially (exponent 1) decaying funnel in the transient phase $[0, T]$, where $T \approx 3$, and a tracking accuracy quantified by $\lambda = 1/\pi$ thereafter, see Figure 6.3d.

Note further that the asymptotic stability of the zero dynamics can also be verified by a numerical test which shows that all invariant zeros of $[E, A, B, C]$ have real part -1.

The simulation has been performed in MATLAB (solver: ode15s, relative tolerance: 10^{-14}, absolute tolerance: 10^{-10}). In Figure 6.3 the simulation, over the time interval $[0, 10]$, of the funnel controller (6.6.1) with funnel boundary specified in (6.7.2) and reference signal $y_{\text{ref}} = (\sin, \cos)^\top$, applied to system $[E, A, B, C]$ with initial data (6.7.1) is depicted. Figure 6.3a shows the output components y_1 and y_2 tracking the reference signal y_{ref} within the funnel shown in Figure 6.3d. Note that an action of the input components u_1 and u_2 in Figure 6.3c and the gain function k in Figure 6.3b is required only if the error $\|e(t)\|$ is close to the funnel boundary $\varphi(t)^{-1}$. It can be seen that initially the error is very close to the funnel boundary and hence the gain rises sharply. Then, at approximately $t = 1$, the distance between error and funnel boundary gets larger and the gain drops accordingly. In particular we see that the gain function k is non-monotone.

6.8 Notes and References

(i) The modified nodal analysis procedure has been developed by Ho et al. [116] at IBM, see also [93, 204, 206–208, 242]. As described in Sections 6.2 and 6.3, MNA is based on a graph theoretical consideration of the electrical circuit; the circuit topology can be directly read off from the equations arising from 6.3.1. It is a very general modeling method which can handle all basic circuit elements and, moreover and not discussed in Sections 6.1 - 6.7, other elements such as the operational amplifier, two-port elements including the gyrator, and the ideal transformer. At the same time, the MNA model takes a very compact form (cf. 6.3.1) and is hence widely used and an appropriate tool for the computational solution of circuit equations. In particular, the modeling can be done automatically and MNA is hence used in several circuit simulation programs, such as SPICE®.
For a historical overview of the development of MNA and accompanying simulation software see [196]. A very good survey of "DAEs in circuit modelling" including MNA can also be found in [208].

(ii) In the MNA model (6.3.1)–(6.3.3) of a given electrical circuit we chose the input vector to consist of the currents of current sources and voltages of voltage sources, which is a canonical choice. However, choosing the output vector to consist of the voltages of current sources and currents of voltage sources is restrictive, since one may want to measure the current through any element of the circuit or the voltage between any two points of the circuit. This can be achieved within the framework of the model (6.3.1)–(6.3.3) by adding 'virtual' sources to the circuit in the following way[1]:

1) If the current through an element X needs to be measured, then establish a series connection of X with a voltage sources (which is added to the circuit) of voltage $v = 0$. The current i through the voltage source is the same as the current through X and can hence be measured at the source. Since $v = 0$, the addition of the source does not influence the remaining circuit.

2) If the voltage between two points of the circuit needs to be measured, say two nodes with potentials η_1 and η_2, then establish a parallel connection of the circuit between these points and a current source (which is added to the circuit connecting the two nodes) of current $i = 0$. It follows that the voltage v across the voltage source satisfies $v = \eta_1 - \eta_2$, i.e., it is the same as the voltage between the two points of the circuit and can hence be measured at the source. Since $i = 0$, the addition of the source does not influence the remaining circuit.

With this procedure it can be achieved that any current or voltage in the circuit can be assigned as output, while the circuit can still be described by a model of the form (6.3.1)–(6.3.3), where some input constraints (currents and voltages of the added sources must be zero) have to be added. This can be realized by adding respective rows to B and zero rows to the pencil $sE - A$. Of course, the analysis presented in the previous sections cannot simply be carried over, but a thorough investigation with respect to the additional input constraints must be performed; this is an open problem.

(iii) It is interesting to note that the MNA procedure can also handle so called *memristive devices*. One such device, the *memristor* (memory-resistor), has been postulated by CHUA [72] as the fourth basic circuit element besides resistors, capacitors and inductors, on the basis of symmetry considerations. The memristor

[1]Many thanks to Thomas Hotz (Ilmenau University of Technology) for pointing this out to me.

provides a functional relation between electric charge and magnetic flux. The actual discovery of the memristor by Strukov et al. [225] from HP labs in 2008 (see also [249]) had a great impact in the electrical engineering community and a lot of potential applications have been reported since, see [208] and the references therein. Recent research [94, 255] reveals that the MNA method can be extended to treat memristors by introducing the magnetic flux as a new state variable. As a consequence, SPICE-like circuit simulators can be used for electrical circuits with memristors.
An interesting topic for further research is the investigation of high-gain and funnel control for nonlinear electrical circuits containing memristors.

References

[1] Adams, R. A. *Sobolev Spaces*. No. 65 in Pure and Applied Mathematics. Academic Press, New York, London, 1975.

[2] Anderson, B. D. O., and Vongpanitlerd, S. *Network Analysis and Synthesis – A Modern Systems Theory Approach*. Prentice-Hall, Englewood Cliffs, NJ, 1973.

[3] Aplevich, J. D. Minimal representations of implicit linear systems. *Automatica 21*, 3 (1985), 259–269.

[4] Aplevich, J. D. *Implicit Linear Systems*. No. 152 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1991.

[5] Arens, R. Operational calculus of linear relations. *Pacific J. Math. 9* (1961), 9–23.

[6] Armentano, V. A. Eigenvalue placement for generalized linear systems. *Syst. Control Lett. 4* (1984), 199–202.

[7] Armentano, V. A. The pencil $(sE - A)$ and controllability-observability for generalized linear systems: a geometric approach. *SIAM J. Control Optim. 24* (1986), 616–638.

[8] Ascher, U. M., and Petzold, L. R. *Computer methods for ordinary differential equations and differential-algebraic equations*. SIAM Publications, Philadelphia, 1998.

[9] Aubin, J.-P., and Cellina, A. *Differential Inclusions: Set-Valued Maps and Viability Theory*. No. 264 in Grundlehren der mathematischen Wissenschaften. Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1984.

[10] Aubin, J.-P., and Frankowska, H. *Set Valued Analysis*. Birkhäuser, Boston, 1990.

[11] Banaszuk, A., Kocięcki, M., and Lewis, F. L. Kalman Decomposition for Implicit Linear Systems. *IEEE Trans. Autom. Control 37*, 10 (1992), 1509–1514.

[12] BANASZUK, A., KOCIĘCKI, M., AND PRZYŁUSKI, K. M. On Hautus-type conditions for controllability of implicit linear discrete-time systems. *Circuits Systems Signal Process. 8*, 3 (1989), 289–298.

[13] BANASZUK, A., KOCIĘCKI, M., AND PRZYŁUSKI, K. M. Implicit linear discrete-time systems. *Math. Control Signals Syst. 3*, 3 (1990), 271–297.

[14] BANASZUK, A., KOCIĘCKI, M., AND PRZYŁUSKI, K. M. Kalman-type decompostion for implicit linear discrete-time systems, and its applications. *Int. J. Control 52*, 5 (1990), 1263–1271.

[15] BANASZUK, A., AND PRZYŁUSKI, K. M. On perturbations of controllable implicit linear systems. *IMA J. Math. Control & Information 16* (1999), 91–102.

[16] BASILE, G., AND MARRO, G. *Controlled and Conditioned Invariants in Linear System Theory.* Prentice-Hall, Englewood Cliffs, NJ, 1992.

[17] BEELEN, T. G. J., AND VAN DOOREN, P. M. An improved algorithm for the computation of Kronecker's canonical form of a singular pencil. *Lin. Alg. Appl. 105* (1988), 9–65.

[18] BELEVITCH, V. *Classical Network Theory.* Holden-Day, San Francisco, 1968.

[19] BELUR, M. N., AND TRENTELMAN, H. L. Stabilization, pole placement and regular implementability. *IEEE Trans. Autom. Control 47*, 5 (2002), 735–744.

[20] BENDER, D., AND LAUB, A. The linear quadratic optimal regulator problem for descriptor systems. *IEEE Trans. Autom. Control 32* (1987), 672–688.

[21] BENDER, D. J., AND LAUB, A. J. Controllability and observability at infinity of multivariable linear second-order models. *IEEE Trans. Autom. Control AC-30* (1985), 1234–1237.

[22] BENDER, D. J., AND LAUB, A. J. The linear-quadratic optimal regulator for descriptor systems. In *Proc. 24th IEEE Conf. Decis. Control, Ft. Lauderdale, FL* (1985), pp. 957–962.

[23] BENDER, D. J., AND LAUB, A. J. The linear-quadratic optimal regulator for descriptor systems: Discrete-time case. *Automatica 23* (1987), 71–85.

[24] BENNER, P. Advances in balancing-related model reduction for circuit simulation. In *Scientific Computing in Electrical Engineering, SCEE 2008*, J. Ross and L. Costa, Eds., vol. 14. Springer-Verlag, 2009.

[25] BENNER, P., AND BYERS, R. An arithmetic for matrix pencils: theory and new algorithms. *Numer. Math. 103* (2006), 539–573.

[26] BERGER, T. Bohl exponent for time-varying linear differential-algebraic equations. *Int. J. Control 85*, 10 (2012), 1433–1451.

[27] Berger, T. Funnel control for linear DAEs. *PAMM (Proc. Appl. Math. Mech.) 13* (2013). Submitted for publication.

[28] Berger, T. Zero dynamics and funnel control of general linear differential-algebraic systems. Submitted to *Math. Control Signals Syst.*, preprint available from the website of the author, 2013.

[29] Berger, T. Zero dynamics and stabilization for linear DAEs. Submitted to *Differential-Algebraic Equations Forum*, preprint available from the website of the author, 2013.

[30] Berger, T., Halikias, G., and Karcanias, N. Effects of dynamic and non-dynamic element changes in RC and RL networks. *Int. J. Circ. Theor. Appl.* (2013). Early View, doi: 10.1002/cta.1923.

[31] Berger, T., and Ilchmann, A. Zero dynamics of time-varying linear systems. Preprint 10-05 available online, Institute for Mathematics, Ilmenau University of Technology, 2010.

[32] Berger, T., and Ilchmann, A. On stability of time-varying linear differential-algebraic equations. *Int. J. Control 86*, 6 (2013), 1060–1076.

[33] Berger, T., and Ilchmann, A. On the standard canonical form of time-varying linear DAEs. *Quart. Appl. Math. 71*, 1 (2013), 69–87.

[34] Berger, T., Ilchmann, A., and Reis, T. Normal forms, high-gain, and funnel control for linear differential-algebraic systems. In *Control and Optimization with Differential-Algebraic Constraints*, L. T. Biegler, S. L. Campbell, and V. Mehrmann, Eds., vol. 23 of *Advances in Design and Control*. SIAM, Philadelphia, 2012, pp. 127–164.

[35] Berger, T., Ilchmann, A., and Reis, T. Zero dynamics and funnel control of linear differential-algebraic systems. *Math. Control Signals Syst. 24*, 3 (2012), 219–263.

[36] Berger, T., Ilchmann, A., and Trenn, S. The quasi-Weierstraß form for regular matrix pencils. *Lin. Alg. Appl. 436*, 10 (2012), 4052–4069.

[37] Berger, T., Ilchmann, A., and Wirth, F. Zero dynamics and stabilization for analytic linear systems. Submitted to *Acta Applicandae Mathematicae*, preprint available from the website of the authors, 2013.

[38] Berger, T., and Reis, T. Controllability of linear differential-algebraic systems - a survey. In *Surveys in Differential-Algebraic Equations I*, A. Ilchmann and T. Reis, Eds., Differential-Algebraic Equations Forum. Springer-Verlag, Berlin-Heidelberg, 2013, pp. 1–61.

[39] Berger, T., and Reis, T. Zero dynamics and funnel control for linear electrical circuits. Submitted to *SIAM J. Control Optim.*, preprint available from the website of the authors, 2013.

[40] Berger, T., and Trenn, S. The quasi-Kronecker form for matrix pencils. *SIAM J. Matrix Anal. & Appl. 33*, 2 (2012), 336–368.

[41] Berger, T., and Trenn, S. Addition to "The quasi-Kronecker form for matrix pencils". *SIAM J. Matrix Anal. & Appl. 34*, 1 (2013), 94–101.

[42] Bernhard, P. On singular implicit linear dynamical systems. *SIAM J. Control Optim. 20*, 5 (1982), 612–633.

[43] Birkhoff, G., and MacLane, S. *A Survey of Modern Algebra*, 4th ed. Macmillan Publishing Co., New York, 1977.

[44] Bonilla E., M., and Malabre, M. On the control of linear systems having internal variations. *Automatica 39* (2003), 1989–1996.

[45] Bonilla E., M., Malabre, M., and Loiseau, J.-J. Implicit systems reachability: A geometric point of view. In *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference* (Shanghai, P.R. China, 2009), pp. 4270–4275.

[46] Brenan, K. E., Campbell, S. L., and Petzold, L. R. *Numerical Solution of Initial-Value Problems in Differential-Algebraic Equations*. North-Holland, Amsterdam, 1989.

[47] Brockett, R. W. Poles, zeros, and feedback: State space interpretation. *IEEE Trans. Autom. Control 10*, 2 (1965), 129–135.

[48] Brunovský, P. A classification of linear controllable systems. *Kybernetika 3* (1970), 137–187.

[49] Bunse-Gerstner, A., Byers, R., Mehrmann, V., and Nichols, N. K. Feedback design for regularizing descriptor systems. *Lin. Alg. Appl. 299* (1999), 119–151.

[50] Bunse-Gerstner, A., Mehrmann, V., and Nichols, N. K. On derivative and proportional feedback design for descriptor systems. In *Proceedings of the International Symposium on the Mathematical Theory of Networks and Systems* (Amsterdam, Netherlands, 1989), M. A. Kaashoek et al., Eds.

[51] Bunse-Gerstner, A., Mehrmann, V., and Nichols, N. K. Regularization of descriptor systems by derivative and proportional state feedback. Report, University of Reading, Dept. of Math., Numerical Analysis Group, Reading, UK, March 1991.

[52] Byers, R., Geerts, A. H. W. T., and Mehrmann, V. Descriptor systems without controllability at infinity. *SIAM J. Control Optim. 35* (1997), 462–479.

[53] Byers, R., Kunkel, P., and Mehrmann, V. Regularization of linear descriptor systems with variable coefficients. *SIAM J. Control Optim. 35* (1997), 117–133.

[54] Byrnes, C. I., and Isidori, A. A frequency domain philosophy for nonlinear systems, with application to stabilization and to adaptive control. In *Proc. 23rd IEEE Conf. Decis. Control* (1984), vol. 1, pp. 1569–1573.

[55] Byrnes, C. I., and Isidori, A. New results and examples in nonlinear feedback stabilization. *Syst. Control Lett. 12*, 5 (1989), 437–442.

[56] Byrnes, C. I., and Isidori, A. Output regulation of nonlinear systems. *IEEE Trans. Autom. Control 35*, 2 (1990), 131–140.

[57] Byrnes, C. I., and Isidori, A. Asymptotic stabilization of minimum phase nonlinear systems. *IEEE Trans. Autom. Control 36*, 10 (1991), 1122–1137.

[58] Byrnes, C. I., and Isidori, A. Limit sets, zero dynamics, and internal models in the problem of nonlinear output regulation. *IEEE Trans. Autom. Control 48*, 10 (2003), 1712–1723.

[59] Byrnes, C. I., Lauko, I. G., Gilliam, D. S., and Shubov, V. I. Zero dynamics for relative degree one SISO distributed parameter systems. In *Proc. 37th IEEE Conf. Decis. Control* (1998), pp. 2390–2391.

[60] Byrnes, C. I., and Willems, J. C. Adaptive stabilization of multivariable linear systems. In *Proc. 23rd IEEE Conf. Decis. Control* (1984), pp. 1574–1577.

[61] Calahan, D. A. *Computer-Aided Network Design*, rev. ed. McGraw-Hill, New York, 1972.

[62] Campbell, S. L. *Singular Systems of Differential Equations I.* Pitman, New York, 1980.

[63] Campbell, S. L. *Singular Systems of Differential Equations II.* Pitman, New York, 1982.

[64] Campbell, S. L. One canonical form for higher-index linear time-varying singular systems. *Circuits Systems Signal Process. 2*, 3 (1983), 311–326.

[65] Campbell, S. L., Kunkel, P., and Mehrmann, V. Regularization of linear and nonlinear descriptor systems. In *Control and Optimization with Differential-Algebraic Constraints*, L. T. Biegler, S. L. Campbell, and V. Mehrmann, Eds., vol. 23 of *Advances in Design and Control*. SIAM, Philadelphia, 2012, pp. 17–36.

[66] Campbell, S. L., Meyer, Jr., C. D., and Rose, N. J. Applications of the Drazin inverse to linear systems of differential equations with singular constant coefficients. *SIAM J. Appl. Math. 31*, 3 (1976), 411–425.

[67] Campbell, S. L., Nichols, N. K., and Terrell, W. J. Duality, observability, and controllability for linear time-varying descriptor systems. *Circuits Systems Signal Process. 10*, 4 (1991), 455–470.

[68] Campbell, S. L., and Petzold, L. R. Canonical forms and solvable singular systems of differential equations. *SIAM J. Alg. & Disc. Meth. 4* (1983), 517–521.

[69] Chen, D., and Paden, B. Stable inversion of nonlinear non-minimum phase systems. *Int. J. Control 64*, 1 (1996), 81–97.

[70] Christodoulou, M. A., and Paraskevopoulos, P. N. Solvability, controllability, and observability of singular systems. *J. Optim. Th. & Appl. 45* (1985), 53–72.

[71] Chu, K.-w. E. The solution of the matrix equations $AXB - CXD = E$ and $(YA - DZ, YC - BZ) = (E, F)$. *Lin. Alg. Appl. 93* (1987), 93–105.

[72] Chua, L. O. Memristor - the missing circuit element. *IEEE Trans. Circ. Theory CT-18*, 5 (1971), 507–519.

[73] Cobb, J. D. *Descriptor Variable and Generalized Singularly Perturbed Systems: A Geometric Approach.* Univ. of Illinois, Dept. of Electrical Engineering, Urbana-Champaign, 1980.

[74] Cobb, J. D. Feedback and pole placement in descriptor variable systems. *Int. J. Control 33*, 6 (1981), 1135–1146.

[75] Cobb, J. D. On the solution of linear differential equations with singular coefficients. *J. Diff. Eqns. 46* (1982), 310–323.

[76] Cobb, J. D. Descriptor variable systems and optimal state regulation. *IEEE Trans. Autom. Control AC-28* (1983), 601–611.

[77] Cobb, J. D. Controllability, observability and duality in singular systems. *IEEE Trans. Autom. Control AC-29* (1984), 1076–1082.

[78] Crouch, P. E., and van der Schaft, A. J. *Variational and Hamiltonian Control Systems.* No. 101 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1986.

[79] Cuthrell, J. E., and Biegler, L. T. On the optimization of differential-algebraic process systems. *AIChE Journal 33*, 8 (1987), 1257–1270.

[80] Dai, L. *Singular Control Systems.* No. 118 in Lecture Notes in Control and Information Sciences. Springer-Verlag, Berlin, 1989.

[81] Demmel, J. W., and Kågström, B. The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: Robust software with error bounds and applications. Part I: Theory and Algorithms. *ACM Trans. Math. Soft 19* (1993), 160–174.

[82] Demmel, J. W., and Kågström, B. The generalized Schur decomposition of an arbitrary pencil $A - \lambda B$: Robust software with error bounds and applications. Part II: Software and Applications. *ACM Trans. Math. Soft 19* (1993), 175–201.

[83] Desoer, C. A., and Kuh, E. S. *Basic Circuit Theory.* McGraw-Hill, New York, 1969.

[84] Diehl, M., Bock, H. G., Schlöder, J. P., Findeisen, R., Nagy, Z., and Allgöwer, F. Real-time optimization and nonlinear model predictive control of processes governed by differential-algebraic equations. *Journal of Process Control 12* (2002), 577–585.

[85] Diehl, M., Uslu, I., Findeisen, R., Schwarzkopf, S., Allgöwer, F., Bock, H. G., Bürner, T., Gilles, E. D., Kienle, A., Schlöder, J. P., and Stein, E. Real-time optimization for large scale processes: Nonlinear model predictive control of a high purity distillation column. In *Online Optimization of Large Scale Systems: State of the Art*, M. Grötschel, S. O. Krumke, and J. Rambau, Eds. Springer-Verlag, 2001, pp. 363–384.

[86] Diestel, R. *Graph theory*, vol. 173 of *Graduate Texts in Mathematics.* Springer-Verlag, Berlin, 2005.

[87] Dieudonné, J. Sur la réduction canonique des couples des matrices. *Bull. de la Societé Mathématique de France 74* (1946), 130–146.

[88] Dorato, P. On the inverse of linear dynamical systems. *IEEE Trans. Systems Science and Cybernetics 5*, 1 (1969), 43–48.

[89] Dziurla, B., and Newcomb, R. W. *Nonregular semistate systems: examples and input-output pairing.* IEEE Press, New York, 1987.

[90] Eich-Soellner, E., and Führer, C. *Numerical Methods in Multibody Dynamics.* Teubner, Stuttgart, 1998.

[91] El-Tohami, M., Lovass-Nagy, V., and Powers, D. L. On minimal-order inverses of discrete-time descriptor systems. *Int. J. Control 41*, 4 (1985), 991–1004.

[92] Eliopoulou, H., and Karcanias, N. Properties of reachability and almost reachability subspaces of implicit systems: The extension problem. *Kybernetika 31*, 6 (1995), 530–540.

[93] Estévez Schwarz, D., and Tischendorf, C. Structural analysis for electric circuits and consequences for MNA. *Int. J. Circuit Theory Appl. 28*, 2 (2000), 131–162.

[94] Fei, W., Yu, H., Zhang, W., and Yeo, K. S. Design Exploration of Hybrid CMOS and Memristor Circuit by New Modified Nodal Analysis. *IEEE Transactions on Very Large Scale Integration (VLSI) Systems 20*, 6 (2012), 1012–1025.

[95] Fischer, G. *Lineare Algebra*, 15. ed. Vieweg, Wiesbaden, 2005.

[96] Fletcher, L. R., Kautsky, J., and Nichols, N. K. Eigenstructure assignment in descriptor systems. *IEEE Trans. Autom. Control AC-31* (1986), 1138–1141.

[97] Francis, B. A., and Wonham, W. M. The role of transmission zeros in linear multivariable regulators. *Int. J. Control 22*, 5 (1975), 657–681.

[98] Frankowska, H. On controllability and observability of implicit systems. *Syst. Control Lett. 14* (1990), 219–225.

[99] Führer, C., and Leimkuhler, B. J. Numerical solution of differential-algebraic equations for constrained mechanical motion. *Numer. Math. 59* (1991), 55–69.

[100] Gantmacher, F. R. *The Theory of Matrices (Vol. I & II).* Chelsea, New York, 1959.

[101] Geerts, A. H. W. T. Solvability conditions, consistency and weak consistency for linear differential-algebraic equations and time-invariant linear systems: The general case. *Lin. Alg. Appl. 181* (1993), 111–130.

[102] Geerts, A. H. W. T., and Mehrmann, V. Linear differential equations with constant coefficients: A distributional approach. Tech. Rep. SFB 343 90-073, Bielefeld University, Germany, Germany, 1990.

[103] Glüsing-Lüerssen, H. Feedback canonical form for singular systems. *Int. J. Control 52*, 2 (1990), 347–376.

[104] Glüsing-Lüerssen, H., and Hinrichsen, D. A Jordan control canonical form for singular systems. *Int. J. Control 48*, 5 (1988), 1769–1785.

[105] Golub, G. H., and van Loan, C. F. *Matrix Computations*, 3rd ed. No. 3 in Johns Hopkins series in the mathematical sciences. Johns Hopkins University Press, Baltimore, MD, 1996.

[106] Gresho, P. M. Incompressible fluid dynamics: Some fundamental formulation issues. *Annu. Rev. Fluid Mech. 23* (1991), 413–453.

[107] Griepentrog, E., and März, R. *Differential-algebraic equations and their numerical treatment.* No. 88 in Teubner-Texte zur Mathematik. Teubner, Leipzig, 1986.

[108] Hackl, C. M., Endisch, C., and Schröder, D. Error reference control of nonlinear two-mass flexible servo systems. In *Proceedings of the 16th Mediterranean Conference on Control and Automation, MED 2008 - Ajaccio, France, June 25-27* (2008), pp. 1047–1053.

[109] Hackl, C. M., Endisch, C., and Schröder, D. Funnel-Control in robotics: An introduction. In *Proceedings of the 16th Mediterranean Conference on Control and Automation, MED 2008 - Ajaccio, France, June 25-27* (2008), pp. 913–919.

[110] Hackl, C. M., Hofmann, A. G., and Kennel, R. M. Funnel control in mechatronics: An overview. In *Proc. 50th IEEE Conf. Decis. Control and European Control Conference ECC 2011, Orlando, USA* (2011), pp. 8000–8007.

[111] Haug, E. J. *Computer-Aided Kinematics and Dynamics of Mechanical Systems.* Allyn and Bacon, Boston, 1989.

[112] Hautus, M. L. J. Controllability and observability condition for linear autonomous systems. *Ned. Akademie. Wetenschappen, Proc. Ser. A 72* (1969), 443–448.

[113] Helmke, U., and Shayman, M. A. A canonical form for controllable singular systems. *Syst. Control Lett. 12*, 2 (1989), 111–122.

[114] Hernández, V., and Gassó, M. Explicit solution of the matrix equation $AXB - CXD = E$. *Lin. Alg. Appl. 121* (1989), 333–344.

[115] Hinrichsen, D., and Pritchard, A. J. *Mathematical Systems Theory I. Modelling, State Space Analysis, Stability and Robustness*, vol. 48 of *Texts in Applied Mathematics.* Springer-Verlag, Berlin, 2005.

[116] Ho, C.-W., Ruehli, A. E., and Brennan, P. A. The modified nodal approach to network analysis. *IEEE Trans. Circuits Syst. CAS-22*, 6 (1975), 504–509.

[117] Hou, M. Controllability and elimination of impulsive modes in descriptor systems. *IEEE Trans. Autom. Control 49*, 10 (2004), 1723–1727.

[118] Hou, M., and Müller, P. C. Observer design for descriptor systems. *IEEE Trans. Autom. Control 44*, 1 (1999), 164–168.

[119] Ilchmann, A. *Non-Identifier-Based High-Gain Adaptive Control*, vol. 189 of *Lecture Notes in Control and Information Sciences.* Springer-Verlag, London, 1993.

[120] Ilchmann, A., and Mehrmann, V. A behavioural approach to time-varying linear systems, Part 1: General theory. *SIAM J. Control Optim. 44*, 5 (2005), 1725–1747.

[121] Ilchmann, A., and Mehrmann, V. A behavioural approach to time-varying linear systems, Part 2: Descriptor systems. *SIAM J. Control Optim. 44*, 5 (2005), 1748–1765.

[122] Ilchmann, A., and Mueller, M. Time-varying linear systems: Relative degree and normal form. *IEEE Trans. Autom. Control 52*, 5 (2007), 840–851.

[123] Ilchmann, A., and Ryan, E. P. High-gain control without identification: a survey. *GAMM Mitt. 31*, 1 (2008), 115–125.

[124] Ilchmann, A., and Ryan, E. P. Performance funnels and tracking control. *Int. J. Control 82*, 10 (2009), 1828–1840.

[125] Ilchmann, A., Ryan, E. P., and Sangwin, C. J. Tracking with prescribed transient behaviour. *ESAIM: Control, Optimisation and Calculus of Variations 7* (2002), 471–493.

[126] Ilchmann, A., Ryan, E. P., and Townsend, P. Tracking control with prescribed transient behaviour for systems of known relative degree. *Syst. Control Lett. 55*, 5 (2006), 396–406.

[127] Ilchmann, A., Ryan, E. P., and Townsend, P. Tracking with prescribed transient behavior for nonlinear systems of known relative degree. *SIAM J. Control Optim. 46*, 1 (2007), 210–230.

[128] Ilchmann, A., and Schuster, H. PI-funnel control for two mass systems. *IEEE Trans. Autom. Control 54*, 4 (2009), 918–923.

[129] Ilchmann, A., and Trenn, S. Input constrained funnel control with applications to chemical reactor models. *Syst. Control Lett. 53*, 5 (2004), 361–375.

[130] Ishihara, J. Y., and Terra, M. H. Impulse controllability and observability of rectangular descriptor systems. *IEEE Trans. Autom. Control 46*, 6 (2001), 991–994.

[131] Isidori, A. *Nonlinear Control Systems*, 3rd ed. Communications and Control Engineering Series. Springer-Verlag, Berlin, 1995.

[132] Isidori, A. *Nonlinear Control Systems II.* Communications and Control Engineering Series. Springer-Verlag, London, 1999.

[133] Jaffe, S., and Karcanias, N. Matrix pencil characterization of almost (A, B)-invariant subspaces: A classification of geometric concepts. *Int. J. Control 33*, 1 (1981), 51–93.

[134] Julius, A. A., and van der Schaft, A. J. Compatibility of behavior interconnections. In *Proc. 7th European Control Conf. 2003, Cambridge, UK* (2003).

[135] Kågström, B., and Van Dooren, P. M. A generalized state-space approach for the additive decomposition of a transfer matrix. *J. Numer. Linear Algebra Appl. 1*, 2 (1992), 165–181.

[136] Kailath, T. *Linear Systems.* Prentice-Hall, Englewood Cliffs, NJ, 1980.

[137] Kalman, R. E. On the general theory of control systems. In *Proceedings of the First International Congress on Automatic Control, Moscow 1960* (London, 1961), Butterworth's, pp. 481–493.

[138] Kalman, R. E. Canonical structure of linear dynamical systems. *Proc. Nat. Acad. Sci. (USA) 48*, 4 (April 1962), 596–600.

[139] Kalman, R. E. Mathematical description of linear dynamical systems. *SIAM J. Control Optim. 1* (1963), 152–192.

[140] Karcanias, N. Regular state-space realizations of singular system control problems. In *Proc. 26th IEEE Conf. Decis. Control* (Los Angeles, CA, 1987), pp. 1144–1146.

[141] Karcanias, N., and Hayton, G. E. Generalised autonomous dynamical systems, algebraic duality and geometric theory. In *Proc. 8th IFAC World Congress (Kyoto, 1981), pt. III* (1981), pp. 13–18.

[142] Karcanias, N., and Kalogeropoulos, G. A matrix pencil approach to the study of singular systems: algebraic and geometric aspects. In *Proc. Int. Symp. on Singular Systems* (Atlanta, GA, 1987), pp. 29–33.

[143] Karcanias, N., and Kalogeropoulos, G. Geometric theory and feedback invariants of generalized linear systems: a matrix pencil approach. *Circuits Systems Signal Process. 8*, 3 (1989), 375–397.

[144] Karcanias, N., and Kouvaritakis, B. The output zeroing problem and its relationship to the invariant zero structure: a matrix pencil approach. *Int. J. Control 30*, 3 (1979), 395–415.

[145] Knobloch, H. W., and Kwakernaak, H. *Lineare Kontrolltheorie*. Springer-Verlag, Berlin, 1985.

[146] Koumboulis, F. N., and Mertzios, B. G. On Kalman's controllability and observability criteria for singular systems. *Circuits Systems Signal Process. 18*, 3 (1999), 269–290.

[147] Kouvaritakis, B., and MacFarlane, A. G. J. Geometric approach to analysis and synthesis of system zeros Part 1. Square systems. *Int. J. Control 23*, 2 (1976), 149–166.

[148] Kouvaritakis, B., and MacFarlane, A. G. J. Geometric approach to analysis and synthesis of system zeros Part 2. Non-square systems. *Int. J. Control 23*, 2 (1976), 167–181.

[149] Kronecker, L. Algebraische Reduction der Schaaren bilinearer Formen. *Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin* (1890), 1225–1237.

[150] Kučera, V., and Zagalak, P. Fundamental theorem of state feedback for singular systems. *Automatica 24*, 5 (1988), 653–658.

[151] Kuijper, M. *First-Order Representations of Linear Systems*. Birkhäuser, Boston, 1994.

[152] Kunkel, P., and Mehrmann, V. *Differential-Algebraic Equations. Analysis and Numerical Solution*. EMS Publishing House, Zürich, Switzerland, 2006.

[153] Kunkel, P., Mehrmann, V., and Rath, W. Analysis and numerical solution of control problems in descriptor form. *Math. Control Signals Syst. 14* (2001), 29–61.

[154] Lamour, R., März, R., and Tischendorf, C. *Differential Algebraic Equations: A Projector Based Analysis*, vol. 1 of *Differential-Algebraic Equations Forum*. Springer-Verlag, Heidelberg-Berlin, 2013.

[155] Lang, S. *Linear Algebra*, 2nd ed. Addison-Wesley, Reading, Massachusetts, 1970. fourth printing 1973.

[156] Lau, K., Middleton, R. H., and Braslavsky, J. H. Undershoot and settling time tradeoffs for nonminimum phase systems. *IEEE Trans. Autom. Control 48*, 8 (1990), 1389–1393.

[157] Lewis, F. L. Inversion of descriptor systems. In *Proc. American Control Conf. 1983* (San Francisco, CA, 1983), pp. 1153–1158.

[158] Lewis, F. L. A survey of linear singular systems. *IEEE Proc. Circuits, Systems and Signal Processing 5*, 1 (1986), 3–36.

[159] Lewis, F. L. A tutorial on the geometric analysis of linear time-invariant implicit systems. *Automatica 28*, 1 (1992), 119–137.

[160] Lewis, F. L., Christodoulou, M. A., and Mertzios, B. G. System inversion using orthogonal functions. *Circuits Systems Signal Process. 6*, 3 (1987), 347–362.

[161] Lewis, F. L., and Özçaldiran, K. Reachability and controllability for descriptor systems. In *Proc. 27, Midwest Symp. on Circ Syst.* (Morgantown, WV, 1984).

[162] Lewis, F. L., and Özçaldiran, K. *On the eigenstructure assignment of singular systems.* IEEE Press, New York, 1985.

[163] Lewis, F. L., and Özçaldiran, K. Geometric structure and feedback in singular systems. *IEEE Trans. Autom. Control AC-34*, 4 (1989), 450–455.

[164] Liberzon, D., Morse, A. S., and Sontag, E. D. Output-input stability and minimum-phase nonlinear systems. *IEEE Trans. Autom. Control 47*, 3 (2002), 422–436.

[165] Linh, V. H., and Mehrmann, V. Lyapunov, Bohl and Sacker-Sell spectral intervals for differential-algebraic equations. *J. of Dynamics and Differential Equations 21* (2009), 153–194.

[166] Loiseau, J.-J. Some geometric considerations about the Kronecker normal form. *Int. J. Control 42*, 6 (1985), 1411–1431.

[167] Loiseau, J.-J., Özçaldiran, K., Malabre, M., and Karcanias, N. Feedback canonical forms of singular systems. *Kybernetika 27*, 4 (1991), 289–305.

[168] Lötstedt, P., and Petzold, L. R. Numerical solution of nonlinear differential equations with algebraic constraints I: Convergence results for backward differentiation formulas. *Math. Comp. 46*, 174 (1986), 491–516.

[169] Luenberger, D. G. Dynamic equations in descriptor form. *IEEE Trans. Autom. Control AC-22* (1977), 312–321.

[170] Luenberger, D. G. Time-invariant descriptor systems. *Automatica 14* (1978), 473–480.

[171] Luenberger, D. G. *Introduction to Dynamic Systems: Theory, Models and Applications*. John Wiley and Sons Inc., New York, 1979.

[172] Luenberger, D. G. Nonlinear descriptor systems. *J. Econ. Dyn. Contr. 1* (1979), 219–242.

[173] Luenberger, D. G., and Arbel, A. Singular dynamic Leontief systems. *Econometrica 45* (1977), 991–995.

[174] Malabre, M. *More geometry about singular systems*. IEEE Press, New York, 1987.

[175] Malabre, M. Generalized linear systems: geometric and structural approaches. *Lin. Alg. Appl. 122,123,124* (1989), 591–621.

[176] Martins, N., Pellanda, P. C., and Rommes, J. Computation of transfer function dominant zeros with applications to oscillation damping control of large power systems. *IEEE Trans. Power Syst. 22*, 4 (2007), 1657–1664.

[177] Masubuchi, I. Stability and stabilization of implicit systems. In *Proc. 39th IEEE Conf. Decis. Control, Sydney, Australia* (2000), vol. 12, pp. 3636–3641.

[178] Mertzios, B. G., Christodoulou, M. A., Syrmos, B. L., and Lewis, F. L. Direct controllability and observability time domain conditions of singular systems. *IEEE Trans. Autom. Control 33*, 8 (1988), 788–791.

[179] Middleton, R. H., and Braslavsky, J. H. On the relationship between logarithmic sensitivity integrals and limiting optimal control problems. Tech. Rep. EE0009, Department of Electrical and Computer Engineering, University of Newcastle, 2000.

[180] Mueller, M. Normal form for linear systems with respect to its vector relative degree. *Lin. Alg. Appl. 430*, 4 (2009), 1292–1312.

[181] Mueller, M. *Output feedback control and robustness in the gap metric*. PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Univ.-Verl. Ilmenau, Ilmenau, 2009.

[182] Müller, P. C. Remark on the solution of linear time-invariant descriptor systems. *PAMM (Proc. Appl. Math. Mech.) 5* (2005), 175–176.

[183] Newcomb, R. W. The semistate description of nonlinear time-variable circuits. *IEEE Trans. Circuits Syst. CAS-28* (1981), 62–71.

[184] Oară, C., and Van Dooren, P. M. An improved algorithm for the computation of structural invariants of a system pencil and related geometric aspects. *Syst. Control Lett. 30*, 1 (March 1997), 39–48.

[185] Owens, D. H., and Debeljkovic, D. L. Consistency and Liapunov stability of linear descriptor systems: A geometric analysis. *IMA J. Math. Control & Information 2* (1985), 139–151.

[186] Özçaldiran, K. *Control of descriptor systems*. PhD thesis, Georgia Institute of Technology, 1985.

[187] Özçaldiran, K. A geometric characterization of the reachable and controllable subspaces of descriptor systems. *IEEE Proc. Circuits, Systems and Signal Processing 5* (1986), 37–48.

[188] Özçaldiran, K., and Haliločlu, L. Structural properties of singular systems. *Kybernetika 29*, 6 (1993), 518–546.

[189] Özçaldiran, K., and Lewis, F. L. A geometric approach to eigenstructure assignment for singular systems. *IEEE Trans. Autom. Control AC-32*, 7 (1987), 629–632.

[190] Özçaldiran, K., and Lewis, F. L. Generalized reachability subspaces for singular systems. *SIAM J. Control Optim. 27* (1989), 495–510.

[191] Özçaldiran, K., and Lewis, F. L. On the regularizability of singular systems. *IEEE Trans. Autom. Control 35*, 10 (1990), 1156.

[192] Pandolfi, L. Controllability and stabilization for linear systems of algebraic and differential equations. *J. Optim. Th. & Appl. 30* (1980), 601–620.

[193] Pandolfi, L. On the regulator problem for linear degenerate control systems. *J. Optim. Th. & Appl. 33* (1981), 241–254.

[194] Pantelides, C. C. The consistent initialization of differential-algebraic systems. *SIAM J. Sci. Stat. Comput. 9* (1988), 213–231.

[195] Park, M.-S., and Chwa, D. Orbital stabilization of inverted-pendulum systems via coupled sliding-mode control. *IEEE Trans. Ind. Electron. 56*, 9 (2009), 3556–3570.

[196] Pederson, D. O. A Historical Review of Circuit Simulation. *IEEE Trans. Circuits Syst. CAS-31*, 1 (1984), 103–111.

[197] Petzold, L. R. Numerical solution of differential-algebraic equations in mechanical systems simulation. *Physica D 60* (1992), 269–279.

[198] Polderman, J. W., and Willems, J. C. *Introduction to Mathematical Systems Theory. A Behavioral Approach*. Springer-Verlag, New York, 1998.

[199] Popov, V. M. *Hyperstability of Control Systems*. Springer-Verlag, Berlin-Heidelberg-New York, 1973. Translation based on a revised text prepared shortly after the publication of the Romanian ed., 1966.

[200] Przyłuski, K. M., and Sosnowski, A. M. Remarks on the theory of implicit linear continuous-time systems. *Kybernetika 30*, 5 (1994), 507–515.

[201] Pugh, A. C., and Ratcliffe, P. A. On the Zeros and Poles of a Rational Matrix. *Int. J. Control 30* (1979), 213–226.

[202] Rabier, P. J., and Rheinboldt, W. C. Classical and generalized solutions of time-dependent linear differential-algebraic equations. *Lin. Alg. Appl. 245* (1996), 259–293.

[203] Rath, W. *Feedback Design and Regularization for Linear Descriptor Systems with Variable Coefficients*. Dissertation, TU Chemnitz, Chemnitz, Germany, 1997.

[204] Reis, T. Circuit synthesis of passive descriptor systems - a modified nodal approach. *Int. J. Circ. Theor. Appl. 38* (2010), 44–68.

[205] Respondek, W. Right and left invertibility of nonlinear control systems. In *Nonlinear controllability and optimal control*, H. J. Sussmann, Ed. Marcel Dekker, New York, 1990, pp. 133–177.

[206] Riaza, R. Time-domain properties of reactive dual circuits. *Int. J. Circ. Theor. Appl. 34*, 3 (2006), 317–340.

[207] Riaza, R. *Differential-Algebraic Systems. Analytical Aspects and Circuit Applications*. World Scientific Publishing, Basel, 2008.

[208] Riaza, R. DAEs in Circuit Modelling: A Survey. In *Surveys in Differential-Algebraic Equations I*, A. Ilchmann and T. Reis, Eds., Differential-Algebraic Equations Forum. Springer-Verlag, Berlin-Heidelberg, 2013, pp. 97–136.

[209] Riaza, R., and Tischendorf, C. Qualitative features of matrix pencils and DAEs arising in circuit dynamics. *Dynamical Systems 22*, 2 (2007), 107–131.

[210] Rosenbrock, H. H. *State Space and Multivariable Theory*. John Wiley and Sons Inc., New York, NY, 1970.

[211] Rosenbrock, H. H. Structural properties of linear dynamical systems. *Int. J. Control 20* (1974), 191–202.

[212] Rugh, W. J. *Linear System Theory*. Information and System Sciences Series. Prentice-Hall, NJ, 1993.

[213] Rugh, W. J. *Linear System Theory*, 2nd ed. Information and System Sciences Series. Prentice-Hall, NJ, 1996.

[214] Sandovici, A., De Snoo, H., and Winkler, H. The structure of linear relations in Euclidean spaces. *Lin. Alg. Appl. 397* (2005), 141–169.

[215] Sastry, S. S., and Isidori, A. Adaptive Control of Linearizable Systems. *IEEE Trans. Autom. Control 34*, 11 (1989), 1123–1131.

[216] Schiehlen, W. Multibody system dynamics: Roots and perspectives. *Multibody System Dynamics 1* (1997), 149–188.

[217] Schuster, H., Hackl, C. M., Westermaier, C., and Schröder, D. Funnel-control for electrical drives with uncertain parameters. In *Proc. 7th Int. Power Engineering Conf. IPEC* (2005), pp. 517–522.

[218] Schwartz, L. *Théorie des Distributions*. Hermann, Paris, 1957, 1959.

[219] Shayman, M. A., and Zhou, Z. Feedback control and classification of generalized linear systems. *IEEE Trans. Autom. Control 32*, 6 (1987), 483–490.

[220] Silverman, L. M. Inversion of multivariable linear systems. *IEEE Trans. Autom. Control AC-14* (1969), 270–276.

[221] Simeon, B., Führer, C., and Rentrop, P. Differential-algebraic equations in vehicle system dynamics. *Surv. Math. Ind. 1* (1991), 1–37.

[222] Smith, H. J. S. On systems of linear indeterminate equations and congruences. *Philos. Trans. Roy. Soc. London 151* (1861–1862), 293–326. = Coll. Math. Papers, pp. 367–406.

[223] Sontag, E. D. *Mathematical Control Theory: Deterministic Finite Dimensional Systems*, 2nd ed. Springer-Verlag, New York, 1998.

[224] Sparrow, C. *The Lorenz Equations: Bifurcations, Chaos, and Strange Attractors*. No. 41 in Applied Mathematical Sciences. Springer-Verlag, 1982.

[225] Strukov, D. B., Snider, G. S., Stewart, D. R., and Williams, R. S. The missing memristor found. *Nature 453* (2008), 80–83.

[226] Tan, S. H., and Vandewalle, J. Inversion of singular systems. *IEEE Trans. Circuits Syst. CAS-35* (1988), 583–587.

[227] Trenn, S. *Distributional differential algebraic equations*. PhD thesis, Institut für Mathematik, Technische Universität Ilmenau, Universitätsverlag Ilmenau, Ilmenau, Germany, 2009.

[228] Trenn, S. Regularity of distributional differential algebraic equations. *Math. Control Signals Syst. 21*, 3 (2009), 229–264.

[229] Trenn, S. Solution concepts for linear DAEs: a survey. In *Surveys in Differential-Algebraic Equations I*, A. Ilchmann and T. Reis, Eds., Differential-Algebraic Equations Forum. Springer-Verlag, Berlin-Heidelberg, 2013, pp. 137–172.

[230] Trentelman, H. L., Stoorvogel, A. A., and Hautus, M. L. J. *Control Theory for Linear Systems*. Communications and Control Engineering. Springer-Verlag, London, 2001.

[231] Trentelman, H. L., and Willems, J. C. The behavioral approach as a paradigm for modelling interconnected systems. *European J. Control 9*, 2-3 (2003), 296–306.

[232] van der Schaft, A. J. *System Theoretic Descriptions of Physical Systems.* No. 3 in CWI Tract. CWI, Amsterdam, 1984.

[233] van der Schaft, A. J., and Schumacher, J. M. H. The complementary-slackness class of hybrid systems. *Math. Control Signals Syst. 9* (1996), 266–301.

[234] Van Dooren, P. M. The computation of Kronecker's canonical form of a singular pencil. *Lin. Alg. Appl. 27* (1979), 103–140.

[235] Van Dooren, P. M. Reducing subspaces: Definitions, properties, and algorithms. In *Matrix Pencils, Lecture Notes in Mathematics*, B. Kågström and A. Ruhe, Eds., vol. 973. Springer-Verlag, Berlin, 1983, pp. 58–73.

[236] Varga, A. Computation of Kronecker-like forms of a system pencil: applications, algorithms and software. In *Proceedings of the IEEE International Symposium on Computer-Aided Control System Design* (Dearborn, MI, USA, September 1996), pp. 77–82.

[237] Verghese, G. C. *Infinite-frequency Behavior in Generalized Dynamical Systems.* PhD thesis, Stanford University, 1978.

[238] Verghese, G. C. Further notes on singular systems. In *Proc. Joint American Contr. Conf.* (1981). Paper TA-4B.

[239] Verghese, G. C., and Kailath, T. Eigenvector chains for finite and infinite zeros of rational matrices. In *Proc. 18th Conf. Dec. and Control* (Ft. Lauderdale, FL, 1979), pp. 31–32.

[240] Verghese, G. C., and Kailath, T. Impulsive behavior in dynamical systems: structure and significance. In *Proc. 4th MTNS* (1979), P. Dewilde, Ed., pp. 162–168.

[241] Verghese, G. C., Levy, B. C., and Kailath, T. A generalized state-space for singular systems. *IEEE Trans. Autom. Control AC-26*, 4 (August 1981), 811–831.

[242] Wedepohl, L. M., and Jackson, L. Modified nodal analysis: an essential addition to electrical circuit theory and analysis. *Engineering Science and Education Journal 11*, 3 (2002), 84–92.

[243] Weierstrass, K. Zur Theorie der bilinearen und quadratischen Formen. *Berl. Monatsb.* (1868), 310–338.

[244] Wilkinson, J. H. Linear differential equations and Kronecker's canonical form. In *Recent Advances in Numerical Analysis*, C. de Boor and G. H. Golub, Eds. Academic Press, 1978, pp. 231–265.

[245] Willems, J. C. System theoretic models for the analysis of physical systems. *Ricerche di Automatica 10* (1979), 71–106.

[246] Willems, J. C. Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Autom. Control AC-36*, 3 (1991), 259–294.

[247] Willems, J. C. On interconnections, control, and feedback. *IEEE Trans. Autom. Control 42* (1997), 326–339.

[248] Willems, J. C. The behavioral approach to open and interconnected systems. *IEEE Control Systems Magazine 27*, 6 (2007), 46–99.

[249] Williams, R. S. How We Found The Missing Memristor. *IEEE Spectrum 45*, 12 (1976), 28–35.

[250] Wong, K.-T. The eigenvalue problem $\lambda Tx + Sx$. *J. Diff. Eqns. 16* (1974), 270–280.

[251] Wonham, W. M. On pole assignment in multi–input controllable linear systems. *IEEE Trans. Autom. Control AC-12* (1967), 660–665.

[252] Wonham, W. M. *Linear Multivariable Control: A Geometric Approach*, 3rd ed. Springer-Verlag, New York, 1985.

[253] Wood, J., and Zerz, E. Notes on the definition of behavioural controllability. *Syst. Control Lett. 37* (1999), 31–37.

[254] Yip, E. L., and Sincovec, R. F. Solvability, controllability and observability of continuous descriptor systems. *IEEE Trans. Autom. Control AC-26* (1981), 702–707.

[255] Yu, H., and Fei, W. A new modified nodal analysis for nano-scale memristor circuit simulation. In *Proc. IEEE Int. Symp. Circuits Syst., Paris, France* (2010), pp. 3148–3151.

[256] Zadeh, L. A., and Desoer, C. A. *Linear System Theory — The State Space Approach.* McGraw-Hill, New York, 1963.

[257] Zhao, H., and Chen, D. Stable inversion for exact and stable tracking controllers: A flight control example. In *Proc. 4th IEEE Conference on Control Applications* (1995), pp. 482–487.

[258] Zhao, H., and Chen, D. A finite energy property of stable inversion to nonminimum phase nonlinear systems. *IEEE Trans. Autom. Control 43*, 8 (1998), 1170–1174.

[259] Zhou, Z., Shayman, M. A., and Tarn, T.-J. Singular systems: A new approach in the time domain. *IEEE Trans. Autom. Control 32*, 1 (1987), 42–50.

[260] Zubova, S. P. On full controllability criteria of a descriptor system. The polynomial solution of a control problem with checkpoints. *Autom. Rem. Control 72*, 1 (2011), 23–37.

List of Symbols

$\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$	set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, set of all integers, resp.
$\ell(\alpha)$, $\lvert\alpha\rvert$	length $\ell(\alpha) = l$ and absolute value $\lvert\alpha\rvert = \sum_{i=1}^{l} \alpha_i$ of a multi-index $\alpha = (\alpha_1, \ldots, \alpha_l) \in \mathbb{N}^l$
$\mathbb{K}$	rational numbers $\mathbb{Q}$, real numbers $\mathbb{R}$ or complex numbers $\mathbb{C}$
$\mathbb{R}_{\geq 0}$ ($\mathbb{R}_{>0}$, $\mathbb{R}_{\leq 0}$, $\mathbb{R}_{<0}$)	$= [0, \infty)$ $((0, \infty)$, $(-\infty, 0]$, $(-\infty, 0))$, resp.
$\mathbb{C}_+$, $\mathbb{C}_-$	the open set of complex numbers with positive, negative real part, resp.
$R[s]$	the ring of polynomials with coefficients in a ring R and indeterminate s
$R(s)$	the quotient field of $R[s]$
$\deg p(s)$	the degree of the polynomial $p(s) \in R[s]$
$e_i^{[n]}$	ith canonical unit vector in $\mathbb{R}^n$, or e_i if the dimension is clear from the context
$R^{n \times m}$	the set of $n \times m$ matrices with entries in a ring R
I_n	$= \operatorname{diag}(1, \ldots, 1) \in \mathbb{R}^{n \times n}$, or I if the dimension is clear from the context
$0_{n,m}$	a zero matrix of size $n \times m$

N_k $= \begin{bmatrix} 0 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{k \times k}$, $k \in \mathbb{N}$

K_k $= \begin{bmatrix} 1 & 0 & & \\ & \ddots & \ddots & \\ & & 1 & 0 \end{bmatrix} \in \mathbb{R}^{(k-1) \times k}$, $k \in \mathbb{N}$

L_k $= \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & 0 & 1 \end{bmatrix} \in \mathbb{R}^{(k-1) \times k}$, $k \in \mathbb{N}$

N_α $= \operatorname{diag}(N_{\alpha_1}, \dots, N_{\alpha_k}) \in \mathbb{R}^{|\alpha| \times |\alpha|}$ for some multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$

K_α $= \operatorname{diag}(K_{\alpha_1}, \dots, K_{\alpha_k}) \in \mathbb{R}^{(|\alpha|-k) \times |\alpha|}$ for some multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$

L_α $= \operatorname{diag}(L_{\alpha_1}, \dots, L_{\alpha_k}) \in \mathbb{R}^{(|\alpha|-k) \times |\alpha|}$ for some multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$

E_α $= \operatorname{diag}(e_{\alpha_1}^{[\alpha_1]}, \dots, e_{\alpha_k}^{[\alpha_k]}) \in \mathbb{R}^{|\alpha| \times k}$ for some multi-index $\alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{N}^k$

$\mathbf{Gl}_n(R)$ the set of invertible $n \times n$ matrices with entries in a ring R

$\mathcal{O}_n(\mathbb{R})$ the group of orthogonal matrices in $\mathbb{R}^{n \times n}$

$\sigma(A)$ the spectrum of a matrix $A \in R^{n \times n}$ with entries in a ring R

$\det(A)$ the determinant of a matrix $A \in R^{n \times n}$ with entries in a ring R

$\operatorname{rk}_P A$, $\operatorname{im}_P A$, $\ker_P A$;
the rank, image and kernel of $A \in R^{n \times m}$ over a subring P of a ring R resp., or $\operatorname{rk} A$, $\operatorname{im} A$, $\ker A$ if $P = R$

$\operatorname{rk}_{\mathbb{C}}(\infty E - A)$
$= \operatorname{rk}_{\mathbb{C}} E$ for $E, A \in \mathbb{C}^{n \times m}$

$\operatorname{spec}(sE - A)$
$= \{\, \lambda \in \mathbb{C} \mid \det(\lambda E - A) = 0 \,\}$, the spectrum of a matrix pencil $sE - A \in \mathbb{C}[s]^{n \times n}$

$\mathrm{spec}_\infty(sE - A)$
: $= \{\ \lambda \in \mathbb{C} \cup \{\infty\} \mid \mathrm{rk}_{\mathbb{C}}(\lambda E - A) < n\ \}$, the augmented spectrum of a matrix pencil $sE - A \in \mathbb{C}[s]^{n\times n}$

A^* : $= \overline{A}^\top$, the conjugate transpose of $A \in \mathbb{C}^{n\times m}$

$\|x\|$: $= \sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$

$\|A\|$: $= \max\{\ \|A\,x\| \mid x \in \mathbb{R}^m,\ \|x\| = 1\ \}$, the induced matrix norm of $A \in \mathbb{R}^{n\times m}$

$A\mathcal{S}$: $= \{\ Ax \mid x \in \mathcal{S}\ \}$, the image of a set $\mathcal{S} \subseteq \mathbb{K}^m$ under $A \in \mathbb{K}^{n\times m}$

$A^{-1}\mathcal{S}$: $= \{\ x \in \mathbb{K}^m \mid Ax \in \mathcal{S}\ \}$, the preimage of the set $\mathcal{S} \subseteq \mathbb{K}^n$ under $A \in \mathbb{K}^{n\times m}$

$\mathcal{S}^\perp$: $= \{\ x \in \mathbb{K}^n \mid \forall s \in \mathcal{S}:\ x^* s = 0\ \}$, the orthogonal complement of $\mathcal{S} \subseteq \mathbb{K}^n$

$\mathcal{L}^1_{\mathrm{loc}}(I;\mathbb{R}^n)$: the set of locally Lebesgue integrable functions $f : I \to \mathbb{R}^n$, where $\int_K \|f(t)\|\,\mathrm{d}t < \infty$ for all compact $K \subseteq I$ and $I \subseteq \mathbb{R}$ is an interval

$\dot f\ (f^{(i)})$: the (ith) weak derivative of $f \in \mathcal{L}^1_{\mathrm{loc}}(I;\mathbb{R}^n)$, $i \in \mathbb{N}_0$, $I \subseteq \mathbb{R}$ an interval, see [1, Chap. 1]

$\mathcal{W}^{k,1}_{\mathrm{loc}}(I;\mathbb{R}^n)$: $= \left\{\ f \in \mathcal{L}^1_{\mathrm{loc}}(I;\mathbb{R}^n)\ \middle|\ f^{(i)} \in \mathcal{L}^1_{\mathrm{loc}}(I;\mathbb{R}^n) \text{ for } i = 0,\ldots,k\ \right\}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $I \subseteq \mathbb{R}$ an interval

$\mathcal{L}^\infty(I;\mathbb{R}^n)$: the set of essentially bounded functions $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval, see [1, Chap. 2]

$\mathcal{C}^k(I;\mathbb{R}^n)$: the set of k-times continuously differentiable functions $f : I \to \mathbb{R}^n$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $I \subseteq \mathbb{R}$ an interval

$\mathcal{B}^k(I;\mathbb{R}^n)$: $= \left\{\ f \in \mathcal{C}^k(I;\mathbb{R}^n)\ \middle|\ f^{(i)} \in \mathcal{L}^\infty(I;\mathbb{R}^n) \text{ for } i = 0,\ldots,k\ \right\}$, $k \in \mathbb{N}_0 \cup \{\infty\}$, $I \subseteq \mathbb{R}$ an interval

$f \overset{\text{a.e.}}{=} g$: means that $f, g \in \mathcal{L}^1_{\mathrm{loc}}(I;\mathbb{R}^n)$ are equal 'almost everywhere', i.e., $f(t) = g(t)$ for almost all (a.a.) $t \in I$, $I \subseteq \mathbb{R}$ an interval

$\text{ess-sup}_J\lVert f\rVert$	the essential supremum of the measurable function $f : I \to \mathbb{R}^n$ over $J \subseteq I$, $I \subseteq \mathbb{R}$ an interval
σ_τ	the τ-shift operator, i.e., for $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval, $\sigma_\tau f : I - \tau \to \mathbb{R}^n,\ t \mapsto f(t+\tau)$
ϱ	the reflection operator, i.e., for $f : I \to \mathbb{R}^n$, $I \subseteq \mathbb{R}$ an interval, $\varrho f : -I \to \mathbb{R}^n,\ t \mapsto f(-t)$
$\Sigma_{l,n}$	set of systems (3.4.1) or $[E, A]$, resp., with $E, A \in \mathbb{R}^{l\times n}$
$\Sigma_{l,n,m}$	set of systems (3.1.1) or $[E, A, B]$, resp., with $E, A \in \mathbb{R}^{l\times n}$ and $B \in \mathbb{R}^{l\times m}$
$\Sigma_{l,n,m,p}$	set of systems (4.1.1) or $[E, A, B, C]$, resp., with $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$ and $C \in \mathbb{R}^{p\times n}$
$\mathfrak{B}_{[E,A]}$	$= \big\{(x, u) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m) \mid Ex \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^l)$ and (x, u) satisfies $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t)$ for almost all $t \in \mathbb{R}\big\}$
$\mathfrak{B}_{[E,A,B,C]}$	$= \big\{(x, u, y) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \mid Ex \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^l)$ and $\frac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t),\ y(t) = Cx(t)$ for almost all $t \in \mathbb{R}\big\}$
$\mathcal{ZD}_{[E,A,B,C]}$	$= \left\{ (x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \;\middle|\; y \overset{\text{a.e.}}{=} 0 \right\}$, the zero dynamics of $[E, A, B, C] \in \Sigma_{l,n,m,p}$
$\mathcal{V}_{[E,A,B]}$	$= \big\{x^0 \in \mathbb{R}^n \mid \exists\,(x, u) \in \mathfrak{B}_{[E,A,B]} :\ x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \ \wedge\ x(0) = x^0\big\}$, the set of consistent initial states
$\mathcal{V}^{\text{diff}}_{[E,A,B]}$	$= \big\{x^0 \in \mathbb{R}^n \mid \exists\,(x, u) \in \mathfrak{B}_{[E,A,B]} :\ x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \ \wedge\ Ex(0) = Ex^0\big\}$, the set of consistent initial differential variables
$\mathcal{R}^t_{[E,A,B]}$	$= \big\{x^0 \in \mathbb{R}^n \mid \exists\,(x, u) \in \mathfrak{B}_{[E,A,B]} :\ x \in \mathcal{W}^{1,1}_{\text{loc}}(\mathbb{R}; \mathbb{R}^n) \ \wedge\ x(0) = 0 \ \wedge\ x(t) = x^0\big\}$, the reachable space at time $t \geq 0$
$\mathcal{R}_{[E,A,B]}$	$= \bigcup_{t\geq 0} \mathcal{R}^t_{[E,A,B]}$, the reachable space

$\mathcal{C}^t_{[E,A,B]}$	$= \left\{ x^0 \in \mathbb{R}^n \mid \exists\, (x,u) \in \mathfrak{B}_{[E,A,B]} : \; x \in \mathcal{W}^{1,1}_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n) \;\wedge\; x(0) = x^0 \;\wedge\; x(t) = 0 \right\}$, the controllable space at time $t \geq 0$
$\mathcal{C}_{[E,A,B]}$	$= \bigcup_{t \geq 0} \mathcal{C}^t_{[E,A,B]}$, the controllable space
$\cong$	denotes the equivalence of matrix pencils in Chapter 2
$\overset{W,T}{\sim}_{se}$	denotes the system equivalence of systems in $\Sigma_{l,n,m}$ by invertible matrices W and T in Chapter 3
$\overset{W,T,V,F}{\sim}_{fe}$	denotes the feedback equivalence of systems in $\Sigma_{l,n,m}$ by invertible matrices W, T and V and some feedback matrix F in Chapter 3
$\overset{S,T}{\sim}$	denotes the system equivalence of systems in $\Sigma_{l,n,m,p}$ by invertible matrices S and T in Chapter 4
$k_L(E,A)$	$= \inf \left\{ \mu \in \mathbb{R} \mid \exists\, M_\mu > 0 \;\forall\, x \in \mathfrak{B}_{[E,A]} \text{ for a.a. } t \geq s : \; \lVert x(t) \rVert \leq M_\mu e^{\mu(t-s)} \lVert x(s) \rVert \right\}$, the Lyapunov exponent of $[E,A] \in \Sigma_{l,n}$

Abbreviations

BIF	Byrnes-Isidori form
DAE	differential-algebraic equation
IQKF	interim quasi-Kronecker form
IQKTF	interim quasi-Kronecker triangular form
KCF	Kronecker canonical form
MNA	modified nodal analysis
ODE	ordinary differential equation
QKF	quasi-Kronecker form
QKTF	quasi-Kronecker triangular form
QWF	quasi-Weierstraß form
WCF	Weierstraß canonical form
ZDF	zero dynamics form

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