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OBSERVERS AND DYNAMIC CONTROLLERS FOR LINEAR DIFFERENTIAL-ALGEBRAIC SYSTEMS*

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Abstract. In the present paper we study state-estimation and stabilization by dynamic feedback for linear differential-algebraic systems which are not necessarily regular. We show that the observer synthesis approach for behavioral systems in [M.E. Valcher and J.C. Willems, *IEEE Trans. Automat. Control*, 44 (1999), pp. 2297–2307] can be applied to differential-algebraic systems in a closed form; i.e., the observers and dynamic controllers are again differential-algebraic systems. The concept of an (asymptotic, exact) observer is introduced, and existence is characterized. Since initialization of the observer is an important issue, we investigate regular and freely initializable observers, whose existence can be guaranteed by impulse observability of the plant. The observers are then exploited for the construction of dynamic controllers. We show that there exists a stabilizing controller if and only if the given system is both behaviorally stabilizable and behaviorally detectable.

Key words. differential-algebraic systems, descriptor systems, observer, dynamic controller

AMS subject classifications. 34A09, 93C05, 93B05, 93B07, 93B52

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NOTATION.

$\mathbb{N}$, $\mathbb{N}_0$	the set of natural numbers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$
$\ell(\alpha), \lvert\alpha\rvert$	length $\ell(\alpha) = l$ and absolute value $\lvert\alpha\rvert = \sum_{i=1}^{l} \alpha_i$ of a multi-index $\alpha = (\alpha_1, \dots, \alpha_l) \in \mathbb{N}^l$
$\mathbb{C}_+(\mathbb{C}_-)$	open set of complex numbers with positive (negative) real part, resp.,
$\mathbb{R}[s], \mathbb{R}(s)$	the ring of polynomials with coefficients in $\mathbb{R}$ and its quotient field, resp.,
$R^{n\times m}$	the set of $n \times m$ matrices with entries in a ring R
$\operatorname{im} A$, $\ker A$, $\operatorname{rk}_R A$	image, kernel, and rank of the matrix $A \in R^{n\times m}$, resp.,
$\mathbf{Gl}_n(R)$	the group of invertible matrices in $R^{n\times n}$
$\sigma(A)$	the spectrum of $A \in \mathbb{R}^{n\times n}$
$\lVert x\rVert$ =	$\sqrt{x^\top x}$, the Euclidean norm of $x \in \mathbb{R}^n$
$\overline{M}$	closure of the set M
$\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$	the set of locally Lebesgue integrable functions $f : \mathbb{R} \to \mathbb{R}^n$, where $\int_K \lVert f(t)\rVert \,\mathrm{d}t < \infty$ for all compact $K \subseteq \mathbb{R}$
$\mathcal{AC}(\mathbb{R};\mathbb{R}^n)$	the set of locally absolutely continuous functions $f : \mathbb{R} \to \mathbb{R}^n$
$\dot f$ $(f^{(i)})$	the (ith) weak derivative of $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n)$, $i \in \mathbb{N}_0$; see [1, Chap. 1]

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$f \overset{\text{a.e.}}{=} g$	the functions $f, g \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ are equal "almost everywhere"; i.e., $f(t) = g(t)$ for almost all (a.a.) $t \in \mathbb{R}$
$\operatorname{ess\,sup}_I \|f\|$	the essential supremum of the measurable function $f : \mathbb{R} \to \mathbb{R}^n$ over $I \subseteq \mathbb{R}$
$f\|_I$	the restriction of the function $f : \mathbb{R} \to \mathbb{R}^n$ to $I \subseteq \mathbb{R}$

1. Introduction. We study state-estimation and stabilization of linear time-invariant systems (the *plant*) given by differential-algebraic equations (DAEs) of the form

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) &= Ax(t) + Bu(t)\,, \\
y(t) &= Cx(t) + Du(t)\,,
\end{aligned}
\tag{1.1}
$$

where $E, A \in \mathbb{R}^{l\times n}$, $B \in \mathbb{R}^{l\times m}$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$. Systems of that type are also called *descriptor systems*. The set of systems (1.1) is denoted by $\Sigma_{l,n,m,p}$, and we write $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$. DAE systems of the form (1.1) naturally occur when modeling dynamical systems subject to algebraic constraints; for a further motivation we refer the reader to [5, 24, 42, 44, 52] and the references therein. In the present paper we put special emphasis on the nonregular case; i.e., we *do not assume* that $sE - A$ is *regular*, which would mean that $l = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$.

The functions $u : \mathbb{R} \to \mathbb{R}^m$ and $y : \mathbb{R} \to \mathbb{R}^p$ are called *input* and *output* of the system, resp. A trajectory $(x, u, y) : \mathbb{R} \to \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p$ is said to be a *solution* of (1.1) if it belongs to the *behavior* of (1.1):

$$
\mathfrak{B}_{[E,A,B,C,D]} := \left\{ (x, u, y) \in \mathcal{L}^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \;\middle|\; \begin{array}{l} Ex \in \mathcal{AC}(\mathbb{R};\mathbb{R}^l) \text{ and } (x, u, y) \\ \text{solves (1.1) for a.a. } t \in \mathbb{R} \end{array} \right\}.
$$

For the notation we refer the reader to the list that precedes this introductory section. Recall that $Ex \in \mathcal{AC}(\mathbb{R};\mathbb{R}^l)$ implies continuity of Ex (though x itself may be discontinuous). For the analysis of DAE systems in $\Sigma_{l,n,m,p}$ we assume that the states, inputs, and outputs of the system are fixed a priori by the designer. This is different from other approaches based on the behavioral setting [12, 26, 58].

In the present paper we aim to construct a stabilizing feedback controller that does not have direct access to the state of the plant but only uses information about its output. This is motivated by practice: An operator of the system has access only to the external variables of the system. The state is an internal variable which in general cannot be measured directly. We follow the classical approach: First we construct a dynamical system whose input is composed of the input u and output y of the plant. The output of the to-be-built dynamical system will be a variable z which approximates the state x in a certain sense. Such a system will be called *observer*, and we assume that it is itself a DAE,

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} E_o x_o(t) &= A_o x_o(t) + B_o \begin{pmatrix} u(t) \\ y(t) \end{pmatrix}, \\
z(t) &= C_o x_o(t) + D_o \begin{pmatrix} u(t) \\ y(t) \end{pmatrix},
\end{aligned}
\tag{1.2}
$$

with $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$. Obvious applications for observers are diagnosis and error detection [38]. In the case of feedback control, the approximate state produced by the observer is used for stabilization of the system by feedback.

For linear ODE systems (i.e., $E = I$) the above described procedure is well known (see the textbook [56]) and goes back to the work by Luenberger [47, 48]. In this case, roughly speaking, an observer is a dynamical system with the property that a zero initial observation error implies a zero observation error for all times. It can be shown that this leads to the general structure of an observer as depicted in Figure 1.

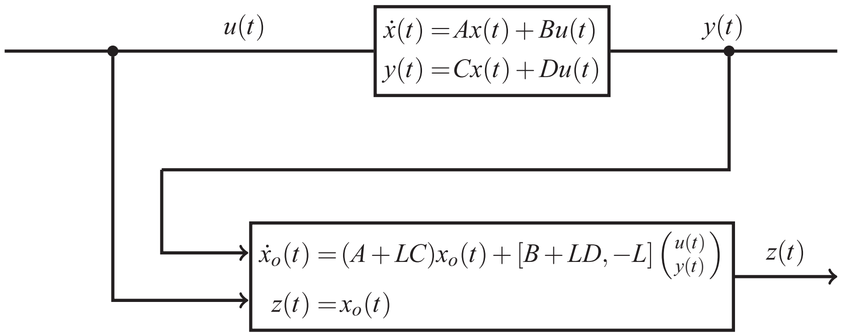

Fig. 1. *Observer for ODE systems.*

Therefore, the observer only depends on the choice of the matrix $L \in \mathbb{R}^{n\times p}$. If L is chosen such that $\sigma(A + LC) \subseteq \mathbb{C}_-$, i.e., the pair $[A, C]$ is detectable, then the observation error decays exponentially for any initial value; the observer is then called an asymptotic observer. If additionally $[A, B]$ is stabilizable, then there exists $F \in \mathbb{R}^{m\times n}$ such that $\sigma(A + BF) \subseteq \mathbb{C}_-$. Together with an asymptotic observer the feedback law $u(t) = Fz(t)$ leads to an asymptotically closed-loop system (see Figure 2) and thus solves the stabilization problem. Conversely, if there exists a stabilizing feedback controller, then $[A, B, C]$ is stabilizable and detectable.

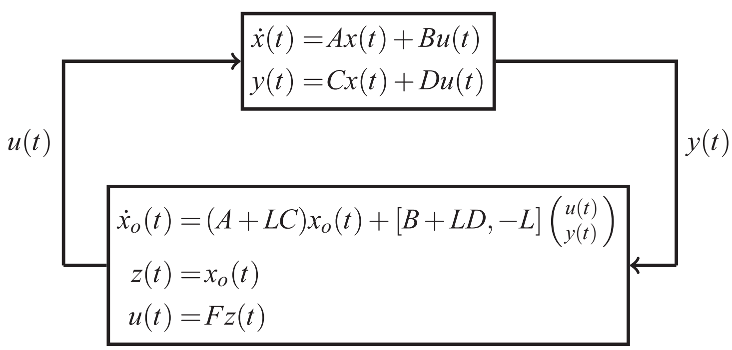

Fig. 2. *Controller for ODE systems.*

In the present paper we generalize the above theorem to the case of DAE systems; see Theorem 4.2. To this end, we introduce in subsection 3.2 an observer design, which is new even for ODE systems, since the observer is a DAE system in general. In view of implementability of the observer, we investigate the notions of regularity and free initializability. As for the design of stabilizing dynamic controllers, special care with their compatibility has to be taken.

The paper is organized as follows: In section 2 we collect the concepts used in the present paper and some results on their algebraic characterization. In section 3 we introduce the concepts of (asymptotic, exact) observers for DAE systems and characterize their existence in terms of observability and detectability. We also give sufficient criteria for DAE observers being regular or freely initializable. The observers are exploited in section 4 for the construction of stabilizing dynamic controllers for DAE systems; special care has to be taken with the compatibility of the controllers. It is shown in Theorem 4.2 that there exists a stabilizing controller if and only if the given plant is both behaviorally stabilizable and behaviorally detectable. We show that in the regular case the controller structure simplifies to some well-known results.

2. Preliminaries. We consider different notions of observability, detectability, and stabilizability for DAE systems. For the definitions of these concepts in time domain and for a detailed discussion and a comparison with the literature we refer the reader to the surveys [11, 14]; for some early references on regular DAE systems see [27, 29]. In the following we state their algebraic characterizations from [11, Cor. 4.3] and [14, Prop. 6.1 and Cor. 9.5].

PROPOSITION 2.1. *A system* $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ *is*

(i) impulse observable *if and only if* $\ker_{\mathbb{R}} E \cap A^{-1}(\operatorname{im}_{\mathbb{R}} E) \cap \ker_{\mathbb{R}} C = \{0\}$;

(ii) behaviorally observable *if and only if* $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$ *for all* $\lambda \in \mathbb{C}$;

(iii) behaviorally detectable *if and only if* $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$ *for all* $\lambda \in \overline{\mathbb{C}_+}$;

(iv) strongly detectable *if and only if* $\ker_{\mathbb{R}} E \cap A^{-1}(\operatorname{im}_{\mathbb{R}} E) \cap \ker_{\mathbb{R}} C = \{0\}$ *and* $\ker_{\mathbb{C}}(\lambda E - A) \cap \ker_{\mathbb{C}} C = \{0\}$ *for all* $\lambda \in \overline{\mathbb{C}_+}$; *and*

(v) behaviorally stabilizable *if and only if* $\operatorname{rk}_{\mathbb{R}(s)}[sE - A, B] = \operatorname{rk}_{\mathbb{C}}[\lambda E - A, B]$ *for all* $\lambda \in \overline{\mathbb{C}_+}$.

Note that behavioral detectability and behavioral stabilizability are not dual concepts; see also [14] for a comprehensive discussion of this issue.

For $E, A \in \mathbb{R}^{l \times n}$ we consider the homogeneous system

$$\frac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t); \tag{2.1}$$

the set of those systems is denoted by $\Sigma_{l,n}$. The *behavior* of (2.1) is given by

$$\mathfrak{B}_{[E,A]} := \left\{ x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n) \;\middle|\; Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l) \text{ and } x \text{ satisfies (2.1) for a.a. } t \in \mathbb{R} \right\}.$$

From [5, 11] we recall the following concepts.

DEFINITION 2.2. *A DAE* $[E, A] \in \Sigma_{l,n}$ *is called*

(a) behaviorally stable

$$:\Longleftrightarrow \;\forall\, x \in \mathfrak{B}_{[E,A]} : \; \lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|x\| = 0 \textit{ and}$$

(b) autonomous

$$:\Longleftrightarrow \;\forall\, x_1, x_2 \in \mathfrak{B}_{[E,A]} : \; x_1|_{(-\infty,0)} \overset{a.e.}{=} x_2|_{(-\infty,0)} \implies x_1 \overset{a.e.}{=} x_2.$$

For a further discussion of autonomy of DAE systems see [12, Rem. 3.3]. Here we recall the important equivalent characterization that

$$[E, A] \in \Sigma_{l,n} \text{ is autonomous} \quad\Longleftrightarrow\quad \forall\, x \in \mathfrak{B}_{[E,A]} : \; \left(Ex(0) = 0 \;\Rightarrow\; x \overset{\text{a.e.}}{=} 0 \right). \tag{2.2}$$

The following result is an immediate consequence of [11, Cor. 5.2].

LEMMA 2.3. *Let* $[E, A] \in \Sigma_{l,n}$. *Then the following hold true:*
(i) $[E, A]$ *is behaviorally stable if and only if* $\mathrm{rk}_{\mathbb{C}}(\lambda E - A) = n$ *for all* $\lambda \in \overline{\mathbb{C}_+}$.
(ii) $[E, A]$ *is autonomous if and only if* $\mathrm{rk}_{\mathbb{R}(s)}(sE - A) = n$.
(iii) $x \overset{\text{a.e.}}{=} 0$ *for all* $x \in \mathfrak{B}_{[E,A]}$ *if and only if* $\mathrm{rk}_{\mathbb{C}}(\lambda E - A) = n$ *for all* $\lambda \in \mathbb{C}$.

3. Observers.

In this section we first present rigorous definitions of the concept of an (asymptotic, exact) observer. To this end, we use the approach in [58] for the more general class of behaviors described by linear constant coefficient differential equations of possibly higher order. Thereafter, we consider observer design and characterize the existence of (asymptotic, exact) observers. In principle, we present "DAE versions" of the results in [58]. Though we treat a smaller class than [58], there is a certain benefit and novelty to our results: Observers for DAE systems can be chosen to be DAE systems themselves. Thereafter, we introduce the classes of "freely initializable" and "regular observers." The first means that the plant does not influence the set of consistent initial values of the observer. The latter means that the observer is neither under- nor overdetermined in a certain sense.

3.1. Definitions.

First of all, an observer, i.e., a dynamical system which aims to reconstruct the state, should be able to process the signals of the plant without influencing the plant itself. This is the subject of the following definition.

DEFINITION 3.1 (acceptor). *Consider a system* $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$. *Then* $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,p_o}$ *is called an* acceptor *for* $[E, A, B, C, D]$ *if for all* $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ *there exist* $x_0 \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n_o})$, $z \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{p_o})$ *such that*

$$(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}.$$

The above definition means there is a one-directed signal flow from $[E, A, B, C, D]$ to its acceptor $[E_o, A_o, B_o, C_o, D_o]$ via input and output (see Figure 3). That is, $[E, A, B, C, D]$ may influence $[E_o, A_o, B_o, C_o, D_o]$ but not vice versa. Also compare the general structure in Figure 3 with the ODE case depicted in Figure 1.

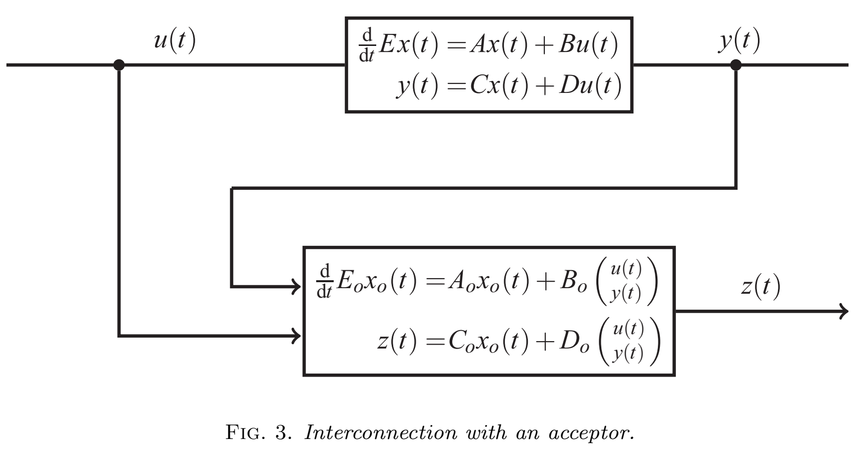

FIG. 3. *Interconnection with an acceptor.*

DEFINITION 3.2 (observer). *Consider the system* $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$. *Then* $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ *is called*

(a) *an* observer for $[E,A,B,C,D]$ *if it is an acceptor for* $[E,A,B,C,D]$ *and*

$$\forall\,(x,u,y,x_o,z)\in\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p\times\mathbb{R}^{n_o}\times\mathbb{R}^n):$$
$$\begin{pmatrix}(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\quad\wedge\quad(x_o,\left(\begin{smallmatrix}u\\y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}\\ \wedge\ Ez(0)=Ex(0)\end{pmatrix}\implies z\overset{a.e.}{=}x;$$

(b) *an* asymptotic observer for $[E,A,B,C,D]$ *if it is an observer for* $[E,A,B,C,D]$ *and*

$$\forall\,(x,u,y,x_o,z)\in\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p\times\mathbb{R}^{n_o}\times\mathbb{R}^n):$$
$$\big((x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\quad\wedge\quad(x_o,\left(\begin{smallmatrix}u\\y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}\big)$$
$$\implies\quad\lim_{t\to\infty}\operatorname{ess\,sup}_{[t,\infty)}\|z-x\|=0;\ \textit{and}$$

(c) *an* exact observer for $[E,A,B,C,D]$ *if it is an acceptor for* $[E,A,B,C,D]$ *and*

$$\forall\,(x,u,y,x_o,z)\in\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^p\times\mathbb{R}^{n_o}\times\mathbb{R}^n):$$
$$\big((x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\wedge(x_o,\left(\begin{smallmatrix}u\\y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}\big)\implies z\overset{a.e.}{=}x.$$

Remark 3.3 (observer).

(a) We have the following implications for $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$ and $[E_o,A_o,B_o,C_o,D_o]\in\Sigma_{l_o,n_o,m+p,n}$:

$[E_o,A_o,B_o,C_o,D_o]$ is an observer for $[E,A,B,C,D]$	$\Longleftarrow$	$[E_o,A_o,B_o,C_o,D_o]$ is an asymptotic observer for $[E,A,B,C,D]$	$\Longleftarrow$	$[E_o,A_o,B_o,C_o,D_o]$ is an exact observer for $[E,A,B,C,D]$

(b) The property of an observer being an acceptor is natural: As the name suggests, *an observer shall only observe and not influence the system.* The further property says that once the observer matches the state of the plant, it does not lose track.

For an asymptotic observer, the state trajectory of the plant is further attractive: Independent of the past of the observer, the *observation error*

$$e(t)=z(t)-x(t)\tag{3.1}$$

tends to zero for $t\to\infty$, whereas an exact observer matches the overall state trajectory.

(c) Our definition of an observer differs slightly from the one for behavioral systems by Valcher and Willems [58, Def. 3.1], where, adapted to our DAE setup, $[E_o,A_o,B_o,C_o,D_o]$ is called an observer for $[E,A,B,C,D]$ if

(3.2)
$$(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\ \wedge\ (x_o,\left(\begin{smallmatrix}u\\y\end{smallmatrix}\right),z)\in\mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}\ \wedge\ z|_{(-\infty,0]}\overset{\text{a.e.}}{=}x|_{(-\infty,0]}$$
$$\implies\quad z\overset{\text{a.e.}}{=}x.$$

Our definition therefore seems to be stronger at a glance. We will, however, see in Remark 3.6 that for DAE systems, our definition is equivalent to the one in [58].

(d) We stress that, different from asymptotic observers, sometimes the concept of an *estimator* is considered, which is an acceptor that satisfies the condition in Definition 3.2(b), but it is not an observer in general. For instance, the system $[1,-1,0,1,0] \in \Sigma_{1,1,1,1}$ is an estimator for the ODE system $[1,-2,0,0,0] \in \Sigma_{1,1,0,1}$ but not an observer.

3.2. Observer design. We now consider the construction of (asymptotic, exact) observers for a given system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$. In [51, p. 351] Polderman and Willems give a preeminently nice and picturesque interpretation for observers, which we would like to quote completely here:

> "*How then should we choose the equations governing a state observer?* The design that we put forward has a very appealing logic. The two central ideas are:
> 1. the observer contains a copy of the plant, called an *internal model*.
> 2. the observer is driven by the *innovations*, by the error feedback, that is, by a signal that expresses how far the actual observed output differs from what we would have expected to observe.
>
> This logic functions not unlike what happens in daily life. Suppose that we meet a friend. How do we organize our thoughts in order to deduce his or her mood, or other latent properties, from the observed manifest ones? Based on past experience, we have an "internal model" of our friend in mind, and an estimate of the "associated state" of his/her mood. This tells us what reactions to expect. When we observe an action or hear a response, then this may cause us to update the state of this internal model. If the observed reaction agrees with what we expected from our current estimate, then there is no need to change the estimate. The more the reaction differs from our expectations, the stronger is the need to update. The difference between what we actually observe and what we had expected to observe is what we call the innovations. Thus it is logical to assume that the updating algorithm for the estimate of the internal model is driven by the innovations. We may also interpret the innovations as the *surprise factor*."

We propose a new observer which thoroughly matches this exegesis: Given a plant $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$, let $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l\times k}$, and $L_y \in \mathbb{R}^{p\times k}$, and consider the following observer design, where u and y act as inputs to the observer (cf. (1.2) and Figure 3) and z, d are the states:

$$\underbrace{\begin{array}{rcl} \frac{\mathrm{d}}{\mathrm{d}t}Ez(t) &=& Az(t)+Bu(t) \\ y(t) &=& Cz(t)+Du(t) \end{array}}_{\text{internal model}} \quad \underbrace{\begin{array}{l} +L_x d(t)\,, \\ +L_y d(t) \end{array}}_{\text{innovations}} \tag{3.3}$$

or, in terms of (1.2),

$$[E_o,A_o,B_o,C_o,D_o] = \left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & L_x \\ C & L_y \end{smallmatrix}\right], \left[\begin{smallmatrix} B & 0 \\ D & -I_m \end{smallmatrix}\right], \left[\begin{smallmatrix} I_n & 0 \end{smallmatrix}\right], 0_{n,m+p}\right] \in \Sigma_{l+p,n+k,m+p,n}. \tag{3.4}$$

The observer is additively composed of an internal model, i.e., a copy of the plant (or friend), and a further term which involves the variable $d \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^k)$ and takes the role of the innovations term (or *surprise summand*). Loosely speaking, the smaller the d, the better the variables in the internal model of the plant (which is part of the observer) in coincidence with the variables in the actual plant (and the better the actual state matched by the approximate state). The only difference is that our innovations term is not an error feedback that is driven by a signal which expresses how far the actual observed output differs from what we would have expected to observe. The variable $d(t)$ is rather a measure for the correctness of the overall internal model at time t. We will show in Remark 3.9 that if $sE - A$ is square, the innovations term is indeed a feedback.

We would like to stress that (3.4) is always a DAE system in general, even if $E = I$, i.e., the plant is an ODE system. The reason is that the output equation $y(t) = Cz(t) + Du(t)$ is always part of the *state equations* of the observer in (3.3); see (3.4).

The interconnection of $[E, A, B, C, D]$ and $[E_o, A_o, B_o, C_o, D_o]$ is described by the control system

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ d(t) \end{pmatrix} = \begin{bmatrix} A & 0 & 0 & 0 \\ C & -I & 0 & 0 \\ 0 & 0 & A & L_x \\ 0 & -I & C & L_y \end{bmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ d(t) \end{pmatrix} + \begin{bmatrix} B \\ D \\ B \\ D \end{bmatrix} u(t). \tag{3.5}$$

Now considering the observation error $e(t) = z(t) - x(t)$ and multiplying (3.5) from the left with

$$W = \begin{bmatrix} I_l & 0 & 0 & 0 \\ 0 & I_p & 0 & 0 \\ -I_l & 0 & I_l & 0 \\ 0 & -I_p & 0 & I_p \end{bmatrix},$$

we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ y(t) \\ e(t) \\ d(t) \end{pmatrix} = \begin{bmatrix} A & 0 & 0 & 0 \\ C & -I & 0 & 0 \\ 0 & 0 & A & L_x \\ 0 & 0 & C & L_y \end{bmatrix}\begin{pmatrix} x(t) \\ y(t) \\ e(t) \\ d(t) \end{pmatrix} + \begin{bmatrix} B \\ D \\ 0 \\ 0 \end{bmatrix} u(t). \tag{3.6}$$

In particular, the error satisfies the DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} e(t) \\ d(t) \end{pmatrix} = \begin{bmatrix} A & L_x \\ C & L_y \end{bmatrix}\begin{pmatrix} e(t) \\ d(t) \end{pmatrix}. \tag{3.7}$$

THEOREM 3.4. *Consider the system* $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$, *and let* $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l \times k}$, *and* $L_y \in \mathbb{R}^{p \times k}$ *be such that*

$$\operatorname{rk}\begin{bmatrix} L_x \\ L_y \end{bmatrix} = k. \tag{3.8}$$

Then we have the following for the system $[E_o, A_o, B_o, C_o, D_o]$ *as in* (3.4)*:*

(a) $[E_o, A_o, B_o, C_o, D_o]$ *is an acceptor for* $[E, A, B, C, D]$.

(b) $[E_o, A_o, B_o, C_o, D_o]$ *is an observer for* $[E, A, B, C, D]$ *if and only if*

$$\operatorname{rk}_{\mathbb{R}(s)}\begin{bmatrix} -sE + A & L_x \\ C & L_y \end{bmatrix} = n + k. \tag{3.9}$$

(c) $[E_o, A_o, B_o, C_o, D_o]$ *is an asymptotic observer for* $[E, A, B, C, D]$ *if and only if*

$$\operatorname{rk}_{\mathbb{C}}\begin{bmatrix} -\lambda E + A & L_x \\ C & L_y \end{bmatrix} = n + k \quad \forall\, \lambda \in \overline{\mathbb{C}_+}. \tag{3.10}$$

(d) $[E_o, A_o, B_o, C_o, D_o]$ *is an exact observer for* $[E, A, B, C, D]$ *if and only if*

$$\operatorname{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & L_x \\ C & L_y \end{bmatrix} = n + k \quad \forall\, \lambda \in \mathbb{C}. \tag{3.11}$$

Proof.

(a) The system $[E_o, A_o, B_o, C_o, D_o]$ in (3.4) is an acceptor for $[E, A, B, C, D]$, since for all $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ we have

$$\left(\left(\begin{smallmatrix} x \\ 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), x\right) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}.$$

(b) $\Rightarrow$: Suppose that $[E_o, A_o, B_o, C_o, D_o]$ in (3.4) is an observer for $[E, A, B, C, D]$. Consider a solution $\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)$ of the DAE (3.7) with

$$\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right] \begin{pmatrix} e(0) \\ d(0) \end{pmatrix} = \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right). \tag{3.12}$$

By (3.5) and (3.6), we have

$$(0,0,0) \in \mathfrak{B}_{[E,A,B,C,D]} \quad \wedge \quad \left(\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), e\right) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}. \tag{3.13}$$

The definition of an observer together with (3.12) implies $e \overset{\text{a.e.}}{=} 0$. Then we obtain from (3.7) and (3.8) that $d \overset{\text{a.e.}}{=} 0$. We may now conclude that the DAE (3.7) is autonomous, and hence we may infer from Lemma 2.3 that (3.9) holds.

$\Leftarrow$: Assume that (3.9) is satisfied, and consider $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ and $\left(\left(\begin{smallmatrix} z \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z\right) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}$ with $Ez(0) = Ex(0)$. Then the definition of the observation error leads to $Ee(0) = 0$, and thus $\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right] \left(\begin{smallmatrix} e(o) \\ d(o) \end{smallmatrix}\right) = 0$. Again using Lemma 2.3, the assumption (3.9) gives autonomy of the DAE (3.7). Then it follows from (2.2) that $e \overset{\text{a.e.}}{=} 0$ or, equivalently, $x \overset{\text{a.e.}}{=} z$. This means that $[E_o, A_o, B_o, C_o, D_o]$ in (3.4) is an observer for $[E, A, B, C, D]$.

(c) $\Rightarrow$: Assume that $[E_o, A_o, B_o, C_o, D_o]$ in (3.4) is an asymptotic observer for $[E, A, B, C, D]$. Consider a solution $\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)$ of the DAE (3.7). Then the relations in (3.13) again hold true. The definition of an asymptotic observer gives $\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|e\| = 0$. Hence, for all solutions $\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)$ of (3.7) we have

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \left\| \left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right] \begin{pmatrix} e(t) \\ d(t) \end{pmatrix} \right\| = 0.$$

Since (3.7) is furthermore autonomous by (b), it follows from [11, Cor. 5.1] that (3.7) is behaviorally stable. Then we obtain from Lemma 2.3 that (3.10) holds true.

$\Leftarrow$: Now assume that (3.10) is satisfied and consider $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ and $\left(\left(\begin{smallmatrix} z \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z\right) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}$. By (3.6), $\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)$ satisfies (3.7). Using Lemma 2.3, we see that (3.10) implies

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \left\|\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)\right\| = 0.$$

The system $[E_o, A_o, B_o, C_o, D_o]$ in (3.4) is therefore an asymptotic observer for $[E, A, B, C, D]$.

(d) $\Rightarrow$: Assume $[E_o, A_o, B_o, C_o, D_o]$ in (3.4) is an exact observer for $[E, A, B, C, D]$. Consider a solution $\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)$ of the DAE (3.7). Then the relations in (3.13) again hold true. The definition of an exact observer yields $e \overset{\text{a.e.}}{=} 0$, whence, by (3.7) and (3.8), we have $d \overset{\text{a.e.}}{=} 0$. Hence, the solutions of the DAE (3.7) vanish almost everywhere, and we obtain from Lemma 2.3 that (3.11) holds true.

$\Leftarrow$: Now assume (3.11) is satisfied and consider $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ and $(\left(\begin{smallmatrix} z \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}$. By (3.6), $\left(\begin{smallmatrix} e \\ d \end{smallmatrix}\right)$ satisfies (3.7). Lemma 2.3 together with (3.11) then implies in particular that $e \overset{\text{a.e.}}{=} 0$, i.e., $x \overset{\text{a.e.}}{=} z$. In other words, $[E_o, A_o, B_o, C_o, D_o]$ in (3.4) is an exact observer for $[E, A, B, C, D]$. □

Note that the properties (3.9)–(3.11) in Theorem 3.4 are related to the so-called *zero dynamics* of the system $[E, A, L_x, C, L_y] \in \Sigma_{n,l,k,p}$ (see [6, 7, 8, 9] for linear DAEs). It is shown in [6] that (using the terminology of [6])

(3.9) $\iff$ the zero dynamics of $[E, A, L_x, C, L_y]$ are autonomous,

(3.10) $\iff$ the zero dynamics of $[E, A, L_x, C, L_y]$ are asymptotically stable,

(3.11) $\iff$ the zero dynamics of $[E, A, L_x, C, L_y]$ are trivial.

3.3. Existence of observers. Here the special observer $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ in (3.4) is shown to have a universal property in a certain sense: If an (asymptotic, exact) observer exists, then it can be constructed to be of the form (3.4) with $k = 0$; we would like to stress that, although the innovations are not necessary to obtain an (asymptotic, exact) observer, they will be necessary when regularity and free initializability of the observer are sought; see subsection 3.4.

THEOREM 3.5 (characterization of existence of observers). *For* $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ *the following hold true:*

(a) *There exists an observer for* $[E, A, B, C, D]$ *if and only if*

$$\operatorname{rk}_{\mathbb{R}(s)} \begin{bmatrix} -sE + A \\ C \end{bmatrix} = n. \tag{3.14}$$

(b) *There exists an asymptotic observer for* $[E, A, B, C, D]$ *if and only if*

$$\operatorname{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A \\ C \end{bmatrix} = n \quad \forall \lambda \in \overline{\mathbb{C}_+}. \tag{3.15}$$

(c) *There exists an exact observer for* $[E, A, B, C, D]$ *if and only if*

$$\operatorname{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A \\ C \end{bmatrix} = n \quad \forall \lambda \in \mathbb{C}. \tag{3.16}$$

Proof. We start with proving "$\Leftarrow$" for (a), (b), and (c) together: Consider the acceptor $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ in (3.4) with $k = 0$, $L_x = 0_{l,0}$, and $L_y = 0_{p,0}$. Then, by Theorem 3.4(a) (resp., (b), (c)), $[E_o, A_o, B_o, C_o, D_o]$ is an (asymptotic, exact) observer if (3.14) (resp., (3.15), (3.16)) holds true.

It remains to prove "$\Rightarrow$" for (a), (b), and (c):

(a) Suppose $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ is an observer for $[E, A, B, C, D]$. Consider $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ with $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$, $Ex(0) = 0$, and

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E \\ 0 \end{bmatrix} x = \begin{bmatrix} A \\ C \end{bmatrix} x. \tag{3.17}$$

Then $(x, 0, 0) \in \mathfrak{B}_{[E,A,B,C,D]}$ and

$$(0, \left(\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right), 0) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}. \tag{3.18}$$

Since $[E_o, A_o, B_o, C_o, D_o]$ is an observer for $[E, A, B, C, D]$ we obtain $x \overset{\text{a.e.}}{=} 0$. This proves that (3.17) is an autonomous DAE, whence Lemma 2.3 yields (3.14).

(b) Suppose that $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ is an asymptotic observer for $[E, A, B, C, D]$. Consider $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ with $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$ which satisfies (3.17). Then $(x, 0, 0) \in \mathfrak{B}_{[E,A,B,C,D]}$. Again consider the trivial trajectory (3.18) of the observer. The assumption that $[E_o, A_o, B_o, C_o, D_o]$ is an asymptotic observer leads to

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|x - 0\| = 0.$$

This shows that the DAE (3.17) is behaviorally stable. Then Lemma 2.3 implies (3.15).

(c) Suppose that $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ is an exact observer for $[E, A, B, C, D]$. Consider $x \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$ with $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$ which satisfies (3.17). Again we have $(x, 0, 0) \in \mathfrak{B}_{[E,A,B,C,D]}$, and we can consider the trivial trajectory (3.18) of the observer. Now using the assumption that $[E_o, A_o, B_o, C_o, D_o]$ is an exact observer, we obtain $x \overset{\text{a.e.}}{=} 0$. This shows that all solutions of the DAE (3.17) vanish almost everywhere. Then we obtain from Lemma 2.3 that (3.16) holds true. $\square$

Note that condition (3.14) is equivalent to $[\left[\begin{smallmatrix} E \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A \\ C \end{smallmatrix}\right]] \in \Sigma_{l+p,n}$ being autonomous, condition (3.15) is equivalent to $[E, A, B, C, D]$ being behaviorally detectable, and condition (3.16) is equivalent to $[E, A, B, C, D]$ being behaviorally observable.

Remark 3.6 (observers II).

(a) Recall from Remark 3.3(c) that the observer definition in [58] is slightly different from ours. Namely, it is characterized by (3.2) in the case where both the plant and observer behavior are represented by DAEs. As stated in Remark 3.3(c), an observer according to our Definition 3.2 is an observer according to [58, Def. 3.1]. Here we state that also the converse is true for the observer $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ as in (3.4): If $Ee(0) = 0$, then the autonomy of the DAE (3.7) implies that $e \overset{\text{a.e.}}{=} 0$ (in the case of an observer according to [58, Def. 3.1], autonomy of (3.7) can be shown similar to the proof of Theorem 3.4(b)) and, in particular, that $e|_{(-\infty,0]} \overset{\text{a.e.}}{=} 0$. The general reason is that, for an autonomous DAE, an initial state completely describes the future behavior. This is no longer true for the behavioral systems treated in [51, 58], since these are described by differential equations of possibly higher order.

(b) In [58], the more general framework of linear time-invariant behaviors described by equations of the form $R_1(\frac{\mathrm{d}}{\mathrm{d}t})w_1 = R_2(\frac{\mathrm{d}}{\mathrm{d}t})w_2$ is considered, where $R_1(\frac{\mathrm{d}}{\mathrm{d}t})$ and $R_2(\frac{\mathrm{d}}{\mathrm{d}t})$ are polynomial matrices in the differential operator. The solution concept coincides with ours. In the framework of [58], the function w_2 is composed of the measured variables, whereas w_1 contains the internal variables. In our case, the measured variables are input and output; w_1 takes the role of the state trajectory. Observer design in [58] consists in the construction of a system $Q(\frac{\mathrm{d}}{\mathrm{d}t})\hat{w}_1 = R(\frac{\mathrm{d}}{\mathrm{d}t})w_2$, such that $\hat{w}_1$ is an approximation of w_1 in the sense of Definition 3.2. A direct application of the results from [58] to our setup would give rise to existence of observers of the form $Q(\frac{\mathrm{d}}{\mathrm{d}t})z = R_u(\frac{\mathrm{d}}{\mathrm{d}t})u + R_y(\frac{\mathrm{d}}{\mathrm{d}t})y$. Our results guarantee that observers can indeed be chosen to be differential-algebraic systems. Further note that our criteria for existence of (asymptotic, exact) DAE observers are equivalent to those obtained for behaviors in [58, Prop. 3.2].

(c) As in [58], we consider solutions in the set of locally integrable functions. We note that an extension to solutions in the set of distributions $\mathcal{D}'(\mathbb{R};\mathbb{R}^k)$ is possible as well: By considering the *distributional behavior*

$$\mathfrak{B}^{\mathcal{D}'}_{[E,A,B,C,D]} := \left\{ (x,u,y) \in \mathcal{D}'(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p) \;\middle|\; \begin{array}{l} E\dot{x} = Ax + Bu, \\ y = Cx + Du \end{array} \right\}$$

the definition of an acceptor can be straightforwardly generalized to distributions. Within this solution concept, a DAE observer can be defined by being an acceptor together with the property

(3.19)
$$\begin{gathered}\forall\,(x,u,y,x_o,z) \in \mathcal{D}'(\mathbb{R};\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^p \times \mathbb{R}^{n_o} \times \mathbb{R}^n): \\ \begin{pmatrix} (x,u,y) \in \mathfrak{B}^{\mathcal{D}'}_{[E,A,B,C,D]} \wedge (x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}^{\mathcal{D}'}_{[E_o,A_o,B_o,C_o,D_o]} \\ \wedge\ \operatorname{supp}(x-z) \subseteq \mathbb{R}_{\geq 0} \end{pmatrix} \implies z = x,\end{gathered}$$

where $\operatorname{supp}(x-z)$ denotes the support of the distribution $x-z$; see [53, page 149]. A suitable definition of an asymptotic observer is that the observation error distribution $x-z$ is a function which tends to zero as t tends to infinity. Further, an exact observer in the distributional sense is one which enforces the error to be the zero distribution. Note that using this observer definition for distributional systems, it can be verified that the statement from Theorem 3.4 still holds. As a consequence, the existence result for observers in Theorem 3.5 is still valid in the distributional sense.

3.4. Regular and freely initializable observers. As the name suggests, an exact observer is not ideal from a practical point of view: The typical situation is that an observer will be turned on at an initial moment. If $[E_o, A_o, B_o, C_o, D_o]$ is an exact observer for $[E, A, B, C, D]$, then a consistent initialization of $[E_o, A_o, B_o, C_o, D_o]$ requires the full information about the initial value of $[E, A, B, C, D]$. As a consequence, we have a certain redundancy in the observation problem: The goal of an observer is to approximate the state trajectory x of $[E, A, B, C, D]$ by means of u and y. On the other hand, by a combination of Lemma 2.3 and Theorem 3.5(a), the state trajectory is, in case of existence of an observer, completely determined by u, y, and $Ex(0)$. That is, initialization of an exact observer already consists of the problem that needs to be solved by the observer itself.

In terms of the metaphoric explanation in [51, page 351] (see also the beginning of subsection 3.2), there is no space for innovations at all. An exact observer needs to have a complete picture of his/her friend's mood already at the beginning!

Another problem in the construction of the exact observer in the proof of Theorem 3.5 (i.e., $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ in (3.4) with $k = 0$, $L_x = 0_{l,0}$, and $L_y = 0_{p,0}$) concerns robustness issues: $[E_o, A_o, B_o, C_o, D_o]$ is no longer an acceptor if the system $[E, A, B, C, D]$ is slightly perturbed (in terms of the explanation in [51, page 351], and this may be a slightly false estimation of the character of the friend).

Further note that systems which are not freely initializable usually involve differentiation of u and y. For an observer this may be problematic when noise is present in the measurement of the output. The above findings lead to the wish for a design of observers whose initialization is not influenced by the initial state of $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ at all.

DEFINITION 3.7 (regular/freely initializable observer). *Let system* $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ *be given, and let* $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ *be an observer for* $[E, A, B, C, D]$. *Then we call* $[E_o, A_o, B_o, C_o, D_o]$

(a) regular *if* $l_o = n_o$ *and* $sE_o - A_o$ *is regular and*

(b) freely initializable *if for all* $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ *and* $x_o^0 \in \mathbb{R}^{n_o}$ *there exist* $x_o \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n_o})$, $z \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$ *such that*

$$(x_o, \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]} \quad \textit{and} \quad E_o x_o(0) = E_o x_o^0.$$

In order to study the above concepts we need to introduce the notion of an index of a regular matrix pencil: The index $\nu \in \mathbb{N}_0$ of a regular matrix pencil $sE - A \in \mathbb{R}[s]^{n\times n}$ is defined via its (quasi-)Weierstraß form [10, 42, 44]: If for some $S, T \in \mathbf{Gl}_n(\mathbb{R})$

$$S(sE - A)T = \begin{bmatrix} sI_r - J & 0 \\ 0 & sN - I_{n-r} \end{bmatrix}, \quad \text{then} \quad \nu := \begin{cases} 0 & \text{if } r = n, \\ \min\left\{ k \in \mathbb{N} \;\middle|\; N^k = 0 \right\} & \text{if } r < n, \end{cases}$$

where N is nilpotent. The index is independent of the choice of S, T and can be computed via the Wong sequences corresponding to $sE - A$ as shown in [10].

Next we give sufficient conditions for the existence of regular and freely initializable observers. In particular, it will turn out that an observer exists if and only if a regular observer exists.

THEOREM 3.8 (existence of regular and freely initializable observers). *Let a system* $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ *be given.*

(a) *If* (3.14) *holds true (equivalently, an observer exists; see Theorem* 3.5(a)*), then there exist* $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l\times k}$, *and* $L_y \in \mathbb{R}^{p\times k}$ *such that* $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ *as in* (3.4) *is a regular observer for* $[E, A, B, C, D]$.

(b) *If* (3.15) *holds true (equivalently, an asymptotic observer exists; see Theorem* 3.5(b)*), then there exist* $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l\times k}$, *and* $L_y \in \mathbb{R}^{p\times k}$ *such that* $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ *as in* (3.4) *is a regular asymptotic observer for* $[E, A, B, C, D]$.

(c) *If* $[E, A, B, C, D]$ *is impulse observable, then there exist* $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l\times k}$, *and* $L_y \in \mathbb{R}^{p\times k}$ *such that the pencil*

$$\begin{bmatrix} -sE + A & L_x \\ C & L_y \end{bmatrix} \tag{3.20}$$

is square and regular and its index is at most one.

In this case, the observer $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ *as in* (3.4) *is regular and freely initializable.*

(d) *If* $[E, A, B, C, D]$ *is strongly detectable, then there exist* $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l\times k}$, *and* $L_y \in \mathbb{R}^{p\times k}$ *such that the pencil* (3.20) *is square and regular, and its index is at most one and satisfies* (3.10).

In this case, $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ *as in* (3.4) *is a regular and freely initializable asymptotic observer for* $[E, A, B, C, D]$.

Proof.

(a) We show the existence of $k \in \mathbb{N}_0$, $L_x \in \mathbb{R}^{l\times k}$, and $L_y \in \mathbb{R}^{p\times k}$ such that the pencil in (3.20) is square and regular. For the proof we introduce the following notation: For $j \in \mathbb{N}$ let

$$N_j = \begin{bmatrix} 0 & & \\ 1 & \ddots & \\ & \ddots & \ddots \\ & & 1 \;\; 0 \end{bmatrix} \in \mathbb{R}^{j\times j}, \quad K_j = \begin{bmatrix} 1 \;\; 0 & & \\ & \ddots & \ddots & \\ & & 1 \;\; 0 \end{bmatrix}, \; L_j = \begin{bmatrix} 0 \;\; 1 & & \\ & \ddots & \ddots & \\ & & 0 \;\; 1 \end{bmatrix} \in \mathbb{R}^{(j-1)\times j}.$$

Further, let $e_i^{[j]} \in \mathbb{R}^j$ be the ith canonical unit vector, and, for some multi-index $\alpha = (\alpha_1, \ldots, \alpha_r) \in \mathbb{N}^r$, we define

$$\begin{aligned} N_\alpha &= \mathrm{diag}(N_{\alpha_1}, \ldots, N_{\alpha_r}) \in \mathbb{R}^{|\alpha| \times |\alpha|}, \\ K_\alpha &= \mathrm{diag}(K_{\alpha_1}, \ldots, K_{\alpha_r}) \in \mathbb{R}^{(|\alpha| - \ell(\alpha)) \times |\alpha|}, \\ L_\alpha &= \mathrm{diag}(L_{\alpha_1}, \ldots, L_{\alpha_r}) \in \mathbb{R}^{(|\alpha| - \ell(\alpha)) \times |\alpha|}, \\ E_\alpha &= \mathrm{diag}(e_{\alpha_1}^{[\alpha_1]}, \ldots, e_{\alpha_r}^{[\alpha_r]}) \in \mathbb{R}^{|\alpha| \times \ell(\alpha)}. \end{aligned}$$

By [14, Thm. 4.4] there exist $S \in \mathbf{Gl}_l(\mathbb{R}), T \in \mathbf{Gl}_n(\mathbb{R}), V \in \mathbf{Gl}_p(\mathbb{R})$, and $L \in \mathbb{R}^{l \times p}$ such that

$$\begin{aligned} &[SET\,,\, SAT - LCT\,,\, VCT] \\ &= \left[\begin{bmatrix} I_{|\alpha|} & 0 & 0 & 0 & 0 & 0 \\ 0 & K_\beta^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & L_\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & K_\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & N_\kappa^\top & 0 \\ 0 & 0 & 0 & 0 & 0 & I_{n_{\overline{\sigma}}} \end{bmatrix}, \begin{bmatrix} N_\alpha & 0 & 0 & 0 & 0 & 0 \\ 0 & L_\beta^\top & 0 & 0 & 0 & 0 \\ 0 & 0 & K_\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & L_\epsilon & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{|\kappa|} & 0 \\ 0 & 0 & 0 & 0 & 0 & A_{\overline{\sigma}} \end{bmatrix}, \begin{bmatrix} E_\alpha^\top & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \right] \end{aligned} \tag{3.21}$$

for some multi-indices $\alpha, \beta, \gamma, \epsilon, \kappa$ and a matrix $A_{\overline{\sigma}} \in \mathbb{R}^{n_{\overline{\sigma}} \times n_{\overline{\sigma}}}$. Observe that, for all $F_\alpha \in \mathbb{R}^{\ell(\alpha) \times |\alpha|}$, the system

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} I_{|\alpha|} & 0 \\ 0 & 0 \end{bmatrix} z(t) = \begin{bmatrix} N_\alpha & F_\alpha^\top \\ E_\alpha^\top & -I_{\ell(\alpha)} \end{bmatrix} z(t) \tag{3.22}$$

is regular and has index at most one, since

$$\deg\det \begin{bmatrix} -sI_{|\alpha|} + N_\alpha & F_\alpha^\top \\ E_\alpha^\top & -I_{\ell(\alpha)} \end{bmatrix} = |\alpha| = \mathrm{rk} \begin{bmatrix} I_{|\alpha|} & 0 \\ 0 & 0 \end{bmatrix}.$$

We choose $F_\alpha = 0$ here. Furthermore, for

$$a_j = [a_{j0}, \ldots, a_{j\beta_j - 2}, 1]^\top \in \mathbb{R}^{\beta_j}$$

with the property that the polynomials

$$p_j(s) = s^{\beta_j} + a_{j\beta_j - 1} s^{\beta_j - 1} + \cdots + a_{j0} \in \mathbb{R}[s]$$

are Hurwitz for $j = 1, \ldots, \ell(\beta)$, the choice

$$B_\beta = \mathrm{diag}(a_1, \ldots, a_{\ell(\beta)}) \in \mathbb{R}^{|\beta| \times \ell(\beta)}$$

leads to the system

$$\frac{\mathrm{d}}{\mathrm{d}t} [K_\beta^\top, 0] \begin{pmatrix} z(t) \\ u(t) \end{pmatrix} = [L_\beta^\top, B_\beta] \begin{pmatrix} z(t) \\ u(t) \end{pmatrix}. \tag{3.23}$$

We see that the input u is uniquely determined by $u = -E_{\beta-1}^\top z$, where $\beta - 1 = (\beta_1 - 1, \ldots, \beta_{\ell(\beta)} - 1)$ and if $\beta_j = 1$ for some j, then the respective x-component does not exist and the equation simply reads as $u_j = 0$. With $B_{\beta-1} = \mathrm{diag}(\tilde{a}_1, \ldots, \tilde{a}_{\ell(\beta)})$, where $\tilde{a}_j = [a_{j0}, \ldots, a_{j\beta_j - 2}]^\top$, a permutation of rows in (3.23) and insertion of u give

$$\begin{aligned} \dot{z}(t) &= (N_{\beta-1} - B_{\beta-1} E_{\beta-1}^\top) z(t), \\ u(t) &= E_{\beta-1}^\top z(t). \end{aligned}$$

It is now clear that the pencil $s[K_\beta^\top, 0] - [L_\beta^\top, B_\beta]$ in system (3.23) is regular and has index at most one. Furthermore, the characteristic polynomial of $N_{\beta-1} + B_{\beta-1}E_{\beta-1}^\top$ (which is a block diagonalization of companion matrices) is given by

$$\det\big(sI - (N_{\beta-1} + B_{\beta-1}E_{\beta-1}^\top)\big) = \prod_{j=1}^{\ell(\beta)} p_j(s),$$

which is Hurwitz, since all $p_j(s)$ are Hurwitz. Therefore, (3.23) is also behaviorally stable. Finally, observe that

$$\forall\, \lambda \in \mathbb{C}:\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda L_\gamma + K_\gamma & 0 \\ E_\gamma^\top & 0 \\ 0 & -\lambda N_\kappa^\top + I_{|\kappa|} \end{bmatrix} = |\gamma| + |\kappa|,$$

and hence the above pencil is square and regular and satisfies (3.10). Now, assumption (3.14) implies $\ell(\varepsilon) = 0$, and hence the choice

(3.24)

$$k = \ell(\alpha) + \ell(\beta) + (p - \ell(\alpha) - \ell(\gamma)),$$

$$\tilde{L}_x = \begin{bmatrix} F_\alpha^\top & 0 & 0 \\ 0 & B_\beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{R}^{l\times k}, \quad \tilde{L}_y = \begin{bmatrix} -I_{\ell(\alpha)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I_{p-\ell(\alpha)-\ell(\gamma)} \end{bmatrix} \in \mathbb{R}^{p\times k}$$

leads to

$$\begin{aligned} n + k &= \big(|\alpha| + |\beta| - \ell(\beta) + |\gamma| + |\kappa| + n_{\overline{o}}\big) + \big(\ell(\alpha) + \ell(\beta) + (p - \ell(\alpha) - \ell(\gamma))\big) \\ &= \big(|\alpha| + |\beta| + |\gamma| - \ell(\gamma) + |\kappa| + n_{\overline{o}}\big) + p = l + p, \end{aligned}$$

by which the pencil

$$\begin{aligned} &\begin{bmatrix} -sSET + SAT - LCT & \tilde{L}_x \\ VCT & \tilde{L}_y \end{bmatrix} \\ &= P_1 \begin{bmatrix} -sI_{|\alpha|}+N_\alpha & F_\alpha^\top & 0 & 0 & 0 & 0 & 0 & 0 \\ E_\alpha^\top & -I_{\ell(\alpha)} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -sK_\beta^\top + L_\beta^\top & B_\beta & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -sL_\gamma + K_\gamma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & E_\gamma^\top & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -sN_\kappa^\top + I_{|\kappa|} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -sI_{n_{\overline{o}}} + A_{\overline{o}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & I_{p-\ell(\alpha)-\ell(\gamma)} \end{bmatrix} P_2 \end{aligned}$$

is square and regular, where P_1, P_2 are appropriate block permutation matrices. Therefore, with

$$L_x = S^{-1}\tilde{L}_x + LV^{-1}\tilde{L}_y, \quad L_y = V^{-1}\tilde{L}_y, \tag{3.25}$$

the pencil (3.20) is square and regular. We can further conclude from Theorem 3.4(b) that $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ as in (3.4) is an observer for $[E, A, B, C, D]$.

(b) If (3.15) is true, then $[E, A, B, C, D]$ is behaviorally detectable and by [14, Cor. 9.3] we find $\ell(\varepsilon) = 0$ and $\sigma(A_{\overline{o}}) \subseteq \mathbb{C}_-$ in (3.21). By [56, Thm. 4.20], there exists $F_\alpha \in \mathbb{R}^{\ell(\alpha)\times|\alpha|}$ such that, using the same notation as in (a), $\sigma(N_\alpha^\top + E_\alpha F_\alpha) \subseteq \mathbb{C}_-$. Therefore, system (3.22) is additionally behaviorally stable. Then, using the same choice as in (3.24) and (3.25), it follows from the fact that the systems (3.22) and (3.23) are regular and behaviorally stable that the pencil in (3.20) is square and regular and satisfies (3.10). Then, by Theorem 3.4(c), $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ as in (3.4) is an asymptotic observer for $[E, A, B, C, D]$.

(c) Impulse observability implies, invoking [14, Lem. 5.1], that $|\gamma| = \ell(\gamma)$, $\ell(\varepsilon) = 0$ and $|\kappa| = \ell(\kappa)$ in (3.21). Then, using the same choice as in (3.24) and (3.25), it follows from the fact that the systems (3.22) and (3.23) are regular and of index at most one that the pencil in (3.20) is square and regular and its index is at most one. Next we prove that $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ as in (3.4) is a freely initializable observer for $[E, A, B, C, D]$:
Let $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ and $z^0 \in \mathbb{R}^n$. Since the pencil (3.20) is square and regular and its index is at most one, there exist $e \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$ and $d \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^k)$ such that $Ee \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$, $Ee(0) = E(z^0 - x(0))$, and the DAE (3.7) is satisfied for a.a. $t \in \mathbb{R}$. Now consider $z := x + e \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$. Then, by $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$, we obtain $Ez \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$ and

$$E_o \begin{pmatrix} z(0) \\ d(0) \end{pmatrix} = \begin{pmatrix} Ex(0) + Ee(0) \\ 0 \end{pmatrix} = E_o z^0.$$

By $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ and (3.7) we obtain that (3.6) holds true. Hence, the DAE (3.5) is satisfied. In particular, we have

$$\left(\left(\begin{smallmatrix} z \\ d \end{smallmatrix}\right), \left(\begin{smallmatrix} u \\ y \end{smallmatrix}\right), z \right) \in \mathfrak{B}_{[E_o,A_o,B_o,C_o,D_o]}.$$

(d) Strong detectability implies, invoking [14, Cor. 9.3], that $|\gamma| = \ell(\gamma)$, $\ell(\varepsilon) = 0$, $|\kappa| = \ell(\kappa)$, and $\sigma(A_{\overline{\sigma}}) \subseteq \mathbb{C}_-$ in (3.21). As in (b) we may choose $F_\alpha \in \mathbb{R}^{\ell(\alpha)\times|\alpha|}$ such that $\sigma(N_\alpha^\top + E_\alpha F_\alpha) \subseteq \mathbb{C}_-$, and hence system (3.22) is behaviorally stable. Then, using the same choice as in (3.24) and (3.25), it follows from the fact that the systems (3.22) and (3.23) are regular, of index at most one and behaviorally stable, that the pencil in (3.20) is square and regular, and its index is at most one and it satisfies (3.10). Then, by Theorem 3.4(c), $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l+p,n+k,m+p,n}$ as in (3.4) is an asymptotic observer for $[E, A, B, C, D]$. Regularity and free initializability of $[E_o, A_o, B_o, C_o, D_o]$ follow from (c). ◻

It has recently been shown in [13] that the converse implications in Theorem 3.8(c) and (d) are not true in general. Instead of impulse observability (and, additionally, behavioral detectability) it is necessary and sufficient for existence of an (asymptotic) regular and freely initializable observer that the completely controllable part of the plant in terms of a Kalman controllability decomposition (see [15]) is impulse observable (and, additionally, the plant is behaviorally detectable).

Remark 3.9 (regular observers for square systems). Let a system $[E, A, B, C, D] \in \Sigma_{n,n,m,p}$ be given.

(a) If $sE - A \in \mathbb{R}[s]^{n\times n}$ is regular, then $[E, A, B, C, D]$ has property (3.14), whence a regular observer exists by Theorem 3.8(a).

(b) By [14, Thm. 9.8], the following hold true:

(i) If $[E, A, B, C, D]$ is impulse observable, then there exists some $L \in \mathbb{R}^{n\times p}$ such that $sE - (A + LC)$ is regular and its index is at most one.

(ii) If $[E, A, B, C, D]$ is behaviorally detectable, then there exists some $L \in \mathbb{R}^{n\times p}$ such that $sE-(A+LC)$ is regular and $[E, A+LC]$ is behaviorally stable.

(iii) If $[E, A, B, C, D]$ is strongly detectable, then there exists some $L \in \mathbb{R}^{n\times p}$ such that $sE-(A+LC)$ is regular, its index is at most one, and $[E, A+LC]$ is behaviorally stable.

As a consequence, if $sE-A$ is square, we may conclude from

$$\begin{bmatrix} I_n & L \\ 0 & I_p \end{bmatrix} \begin{bmatrix} -sE+A & -L \\ C & I_p \end{bmatrix} \begin{bmatrix} I_n & 0 \\ -C & I_p \end{bmatrix} = \begin{bmatrix} -sE+A+LC & 0 \\ 0 & I_p \end{bmatrix} \tag{3.26}$$

that we can make the choice $L_y = I_p$, $L_x = -L$ for the matrices in Theorem 3.8 and (3.20) is square and regular. Therefore, we have

$$d(t) = Cz(t) + Du(t) - y(t)$$

in the observer realization (3.3). Inserting this into the first equation in (3.3) we can eliminate the auxiliary variable d, and we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} Ez(t) = (A+LC)z(t) + (B+LD)u(t) - Ly(t). \tag{3.27}$$

Hence, we find that regular (asymptotic, freely initializable) observers for square systems can always be chosen of the form (3.27), i.e.,

$$[E_o, A_o, B_o, C_o, D_o] = [E, A+LC, [B+LD\ ,\ -L]\ , I_n, 0_{n,m+p}]\,. \tag{3.28}$$

3.5. Notes and references. Observers for differential-algebraic systems have been considered in various publications. The existing results (as well as ours) all rely on the principal idea by Luenberger in the seminal works [47, 48] for systems governed by ODEs. It has been first observed by Wang and Dai [59, 60] that the classical Luenberger observer straightforwardly generalizes to DAE systems $[E, A, B, C, D] \in \Sigma_{n,n,m,p}$ with regular $sE-A$ (this is a special case of Remark 3.9). Further aspects of observer design for $[E, A, B, C, D] \in \Sigma_{n,n,m,p}$ with regular $sE-A$ have been presented in [35, 37, 49, 54, 65]. These results have been applied to models for mechanical multibody systems in [40].

Articles [34, 39] treat observer design for general DAE systems $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ with the property

$$\operatorname{rk} \begin{bmatrix} E & A \\ 0 & C \\ 0 & E \end{bmatrix} = n + \operatorname{rk} E;$$

i.e., impulse observable systems are considered. It has been proved in [34] that systems with this property admit observers which can be realized by ODEs. This corresponds to our result in Theorem 3.8(c), where we have proved that observers $[E_o, A_o, B_o, C_o, D_o] \in \Sigma_{l_o,n_o,m+p,n}$ can be constructed with $l_o = n_o$ and a regular pencil $sE_o - A_o$ whose index is at most one. By resolving the algebraic constraints, this observer can indeed be reformulated as an ODE.

The results for the regular case have been generalized in [19, 20, 21, 22] to input-output systems which are governed by DAEs with variable coefficients. Observer design for classes of nonlinear DAEs has been treated in [2, 23, 25, 33, 43, 66]. In particular, the article [43] gives criteria for the existence of observers with index at most one.

Numerical aspects of observer design for DAEs are presented in [16, 17, 36, 46, 50].

4. Dynamic controllers. In the present section we consider the problem of *dynamic compensation*, that is, a suitable interconnection with a controller system which only uses the knowledge of the output to stabilize a given plant $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$. We will consider design of stabilizing controllers and present equivalent conditions for their existence.

4.1. Definitions. We use the concept of control in the behavioral sense which has its origin in the works by Willems, Polderman, and Trentelman (see [4, 51, 57, 63, 64]), where differential behaviors and their stabilization via *control by interconnection* are considered. The latter means a systematic addition of some further (differential) equations such that a desired behavior is achieved; see Figure 4.

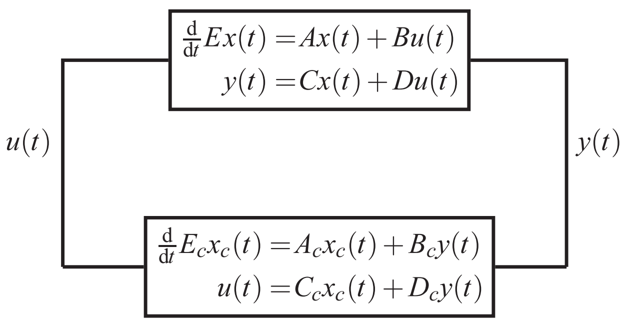

FIG. 4. *Interconnection with a controller.*

Note that if $y = x$, one could make the extreme choice $E_c = 0, A_c = 0, B_c = I, C_c = 0, D_c = 0$ for the controller, which would result in an interconnected system where each trajectory vanishes. This, however, is not suitable from a practical point of view, since in this interconnection, the space of consistent initial differential variables is a proper subset of the initial differential variables which are consistent with the original system $[E, A, B, C, D]$. Consequently, the interconnected system does not have the causality property; that is, the implementation of the controller at a certain time $t \in \mathbb{R}$ is not possible, since this causes jumps in the differential variables. To avoid this, we introduce the concept of *compatibility.* In order to define compatibility we need to introduce the space of consistent initial differential variables for $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$:

$$\mathcal{V}^{\mathrm{diff}}_{[E,A,B,C,D]} = \left\{ \; x^0 \in \mathbb{R}^n \;\middle|\; \exists\, (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} : \; Ex(0) = Ex^0 \; \right\}.$$

DEFINITION 4.1 (compatible/stabilizing/freely initializable controller). *Let a system* $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ *be given. Then we call a system* $[E_c, A_c, B_c, C_c, D_c] \in \Sigma_{l_c,n_c,p,m}$

(a) *a* compatible controller *for* $[E, A, B, C, D]$ *if*

$$\forall\, x^0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B,C,D]} \;\exists\, (x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]} \;\exists\, x_c \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{n_c}) :$$
$$Ex(0) = Ex^0 \;\wedge\; (x_c, y, u) \in \mathfrak{B}_{[E_c,A_c,B_c,C_c,D_c]};$$

(b) *a* stabilizing controller *if it is a compatible controller and*

$$\begin{aligned}&\forall\,(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\ \forall\,x_c\in\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n_c}):\\&\left((x_c,y,u)\in\mathfrak{B}_{[E_c,A_c,B_c,C_c,D_c]}\implies\lim_{t\to\infty}\operatorname{ess\,sup}_{[t,\infty)}\left\|\begin{pmatrix}x\\u\\y\\x_c\end{pmatrix}\right\|=0\right);\ \textit{and}\end{aligned}$$

(c) *a* freely initializable controller *if*

$$\begin{aligned}&\forall\,x^0\in\mathcal{V}^{\mathrm{diff}}_{[E,A,B,C,D]}\ \forall\,x_c^0\in\mathbb{R}^{n_c}\ \exists\,(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\ \exists\,x_c\in\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n_c}):\\&\qquad Ex(0)=Ex^0\ \wedge\ E_cx_c(0)=E_cx_c^0\ \wedge\ (x_c,y,u)\in\mathfrak{B}_{[E_c,A_c,B_c,C_c,D_c]}.\end{aligned}$$

Note that the above definition of compatibility is a slight modification of the concept introduced by Julius and van der Schaft in [41], where an interconnection is called compatible if any trajectory of the system without control law can be concatenated with a trajectory of the interconnected system. This certainly implies that the space of initial differential variables of the interconnected system cannot be smaller than the corresponding set for the nominal system. The above compatibility definition also generalizes the compatibility concept introduced in [5, 11] for DAE control systems.

We would like to stress that any freely initializable controller is, in particular, compatible.

4.2. Controller design and existence. In the following we show that the existence of a stabilizing controller is equivalent to behavioral stabilizability and behavioral detectability. We also investigate when a stabilizing controller is freely initializable. Our main result, Theorem 4.2, can be viewed as a "DAE version" of [4, Thm. 6] derived for the class of linear behaviors (the same class as considered in [58]). The benefit and novelty of Theorem 4.2 are that we show that stabilizing controllers for DAE systems can be chosen to be DAE systems themselves.

THEOREM 4.2 (stabilizing controllers). *Let $[E,A,B,C,D]\in\Sigma_{l,n,m,p}$ be given. Then there exists a stabilizing controller for $[E,A,B,C,D]$ if and only if $[E,A,B,C,D]$ is both behaviorally stabilizable and behaviorally detectable.*

Proof of necessity in Theorem 4.2. Let $[E_c,A_c,B_c,C_c,D_c]\in\Sigma_{l_c,n_c,p,m}$ be a stabilizing controller for $[E,A,B,C,D]$.

Step 1. We prove that $[E,A,B,C,D]$ is behaviorally stabilizable. Let $(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}$; then $x^0\in\mathcal{V}^{\mathrm{diff}}_{[E,A,B,C,D]}$. By compatibility of $[E_c,A_c,B_c,C_c,D_c]$ for $[E,A,B,C,D]$, there exists some $(\tilde{x},\tilde{u},\tilde{y})\in\mathfrak{B}_{[E,A,B,C,D]}$ with $E\tilde{x}(0)=Ex(0)$ and some $x_c\in\mathcal{L}^1_{\mathrm{loc}}(\mathbb{R};\mathbb{R}^{n_c})$ such that $(x_c,\tilde{y},\tilde{u})\in\mathfrak{B}_{[E_c,A_c,B_c,C_c,D_c]}$. Since $[E_c,A_c,B_c,C_c,D_c]$ is stabilizing we further obtain that

$$\lim_{t\to\infty}\operatorname{ess\,sup}_{[t,\infty)}\|\tilde{x}\|=0.$$

We have shown that

$$\begin{aligned}&\forall\,(x,u,y)\in\mathfrak{B}_{[E,A,B,C,D]}\ \exists\,(\tilde{x},\tilde{u},\tilde{y})\in\mathfrak{B}_{[E,A,B,C,D]}:\\&\qquad E\tilde{x}(0)=Ex(0)\ \wedge\ \lim_{t\to\infty}\operatorname{ess\,sup}_{[t,\infty)}\|\tilde{x}\|=0.\end{aligned}$$

Using the same arguments as in, for instance, [11, Rem. 3.7] it can be shown that the above property is equivalent to behavioral stabilizability of $[E,A,B,C,D]$.

Step 2. We prove that $[E,A,B,C,D]$ is behaviorally detectable. Let $(x,0,0) \in \mathfrak{B}_{[E,A,B,C,D]}$. Then, by using $(0,0,0) \in \mathfrak{B}_{[E_c,A_c,B_c,C_c,D_c]}$ and the property that $[E_c,A_c,B_c,C_c,D_c]$ is a stabilizing controller, we obtain

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|x\| = 0.$$

This proves that $[E,A,B,C,D]$ is behaviorally detectable. □

The proof of sufficiency in Theorem 4.2 is based on a construction of a suitable controller for a given behaviorally stabilizable and behaviorally detectable system $[E,A,B,C,D] \in \Sigma_{l,n,m,p}$: If full information on the state is available (i.e., $C = I_n$), then a stabilizing controller can be constructed with $E_c = 0$ (i.e., it is actually no longer dynamic); see [11, Thm. 5.4]. To this end, let $K_x \in \mathbb{R}^{l_c\times n}$ and $K_u \in \mathbb{R}^{l_c\times m}$ be such that the DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0\\ 0 & 0\end{bmatrix}\begin{pmatrix} x(t)\\ u(t)\end{pmatrix} = \begin{bmatrix} A & B\\ K_x & K_u\end{bmatrix}\begin{pmatrix} x(t)\\ u(t)\end{pmatrix}$$

is behaviorally stable.

For the general case, we use the basic strategy in the classical approach [48]: We couple the plant with an observer to approximate the state. The approximate state is then used (as if it were the state) to determine an input which stabilizes the system.

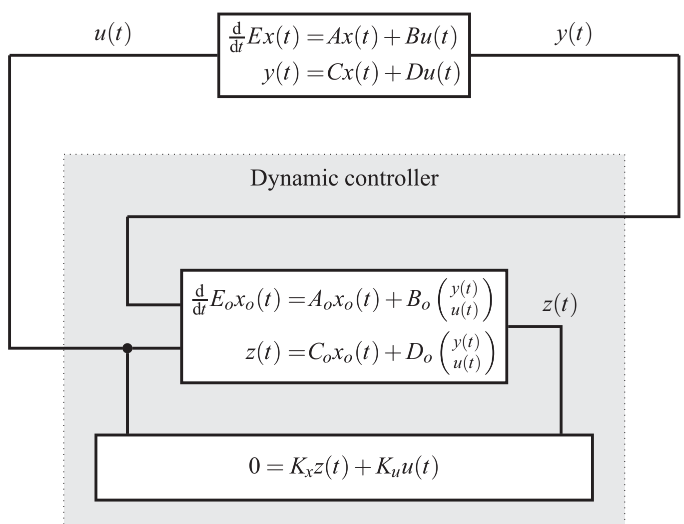

FIG. 5. *Controller structure.*

More precisely, we add the static relation

$$K_x z(t) + K_u u(t) = 0 \tag{4.1}$$

to the model of the plant coupled with an asymptotic observer of the form (3.4), the output of which is the approximate state z; see Figure 5. Then we obtain the closed-loop system described by the DAE

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ d(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & 0 & 0 & 0 & B \\ C & -I_p & 0 & 0 & D \\ 0 & 0 & A & L_x & B \\ 0 & -I_p & C & L_y & D \\ 0 & 0 & K_x & 0 & K_u \end{bmatrix}\begin{pmatrix} x(t) \\ y(t) \\ z(t) \\ d(t) \\ u(t) \end{pmatrix}. \tag{4.2}$$

By using

$$\begin{bmatrix} I_l & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_{l_c} \\ -I_l & 0 & I_l & 0 & 0 \\ 0 & -I_p & 0 & I_p & 0 \\ 0 & I_p & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -sE+A & 0 & 0 & 0 & B \\ C & -I & 0 & 0 & D \\ 0 & 0 & -sE+A & L_x & B \\ 0 & -I & C & L_y & D \\ 0 & 0 & K_x & 0 & K_u \end{bmatrix}\begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I_p \\ I_n & 0 & I_n & 0 & 0 \\ 0 & 0 & 0 & I_k & 0 \\ 0 & I_m & 0 & 0 & 0 \end{bmatrix}$$
$$= \begin{bmatrix} -sE+A & B & 0 & 0 & 0 \\ K_x & K_u & K_x & 0 & 0 \\ 0 & 0 & -sE+A & L_x & 0 \\ 0 & 0 & C & L_y & 0 \\ C & D & 0 & 0 & -I_p \end{bmatrix},$$

we obtain that x, u, y, z, d solve (3.5) if and only if $y(t) = Cx(t) + Du(t)$ and, using the observation error $e(t) = z(t) - x(t)$, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \\ e(t) \\ d(t) \end{pmatrix} = \begin{bmatrix} A & B & 0 & 0 \\ K_x & K_u & K_x & 0 \\ 0 & 0 & A & L_x \\ 0 & 0 & C & L_y \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \\ e(t) \\ d(t) \end{pmatrix}. \tag{4.3}$$

Next we analyze the properties of the previously introduced controller. To this end, for given $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$, consider

$$\begin{aligned} &[E_c, A_c, B_c, C_c, D_c] \\ &= \left[\begin{bmatrix} E & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} A & L_x & B \\ C & L_y & D \\ K_x & 0 & K_u \end{bmatrix}, \begin{bmatrix} 0 \\ -I_p \\ 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & I_m \end{bmatrix}, 0_{m,p}\right] \in \Sigma_{l+p+l_c, n+k+m, p, m}. \end{aligned} \tag{4.4}$$

By an interconnection of this system with $[E, A, B, C, D]$ as depicted in Figure 4, we see that the state of the controller contains a copy of the input u. The closed-loop system is therefore described by the DAE (4.2).

In the following we analyze the properties of the controller (4.4) in terms of the following properties of $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ and the matrices $K_x \in \mathbb{R}^{l_c \times n}$, $K_u \in \mathbb{R}^{l_c \times m}$, $L_x \in \mathbb{R}^{n \times k}$, $L_y \in \mathbb{R}^{p \times k}$:

(C1) $[0, K_u, K_x, I_m, 0]$ is a compatible controller for $[E, A, B, I_n, 0]$.

(C2) The DAE $\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} A & B \\ K_x & K_u \end{smallmatrix}\right]\right]$ is behaviorally stable.

(C3) For all $f \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^{l_c})$, $x^0 \in \mathcal{V}^{\mathrm{diff}}_{[E,A,B,C,D]}$, there exist $x \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^n)$, $u \in \mathcal{L}^1_{\mathrm{loc}}(\mathbb{R}; \mathbb{R}^m)$ with $Ex \in \mathcal{AC}(\mathbb{R}; \mathbb{R}^l)$, $Ex(0) = Ex^0$ and, for a.a. $t \in \mathbb{R}$,

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix}\begin{pmatrix} x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} 0 \\ f(t) \end{pmatrix}.$$

(C4) $[E_o, A_o, B_o, C_o, D_o]$ as in (3.4) (for some $k \in \mathbb{N}_0$) with (3.8) is an asymptotic observer for $[E, A, B, C, D]$.

(C5) $[E_o, A_o, B_o, C_o, D_o]$ as in (3.4) (for some $k \in \mathbb{N}_0$) with (3.8) is a freely initializable observer for $[E, A, B, C, D]$.

Note that (C4) is equivalent to behavioral detectability of $[E, A, B, C, D]$ and (C5) implies strong detectability of $[E, A, B, C, D]$.

Before we present the main result on properties of the controller (4.4), we show that for a behaviorally stabilizable system we can always find $K_x \in \mathbb{R}^{l_c \times n}$, $K_u \in \mathbb{R}^{l_c \times m}$ with the properties (C1)–(C3).

LEMMA 4.3. *Let $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ be behaviorally stabilizable. Then there exist $l_c \in \mathbb{N}_0$, $K_x \in \mathbb{R}^{l_c \times n}$, and $K_u \in \mathbb{R}^{l_c \times m}$ with the properties* (C1), (C2)*, and* (C3).

Proof. Properties (C1) and (C2) are an immediate consequence of [5, Thm. 3.4.10]. Property (C3) follows from the fact that in the construction used in the proof of [5, Thm. 3.4.10] the inhomogeneity f is only applied to a regular subsystem which has index at most one, and thus solutions exist for all inhomogeneities of this form and all consistent initial values. $\square$

THEOREM 4.4. *Let $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$, and let $k, l_c \in \mathbb{N}_0$ and $K_x \in \mathbb{R}^{l_c \times n}$, $K_u \in \mathbb{R}^{l_c \times m}$, $L_x \in \mathbb{R}^{n \times k}$, and $L_y \in \mathbb{R}^{p \times k}$. Then for $[E_c, A_c, B_c, C_c, D_c]$ we have the following as in* (4.4)*:*

(a) *If* (C1) *holds, then $[E_c, A_c, B_c, C_c, D_c]$ is a compatible controller for $[E, A, B, C, D]$.*

(b) *If* (C1), (C2)*, and* (C4) *hold, then $[E_c, A_c, B_c, C_c, D_c]$ is a stabilizing controller for $[E, A, B, C, D]$.*

(c) *If* (C1)–(C5) *hold, then $[E_c, A_c, B_c, C_c, D_c]$ is a freely initializable stabilizing controller for $[E, A, B, C, D]$.*

Proof.

(a) Assume $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}$; then by (C1) there exists $(x, u) \in \mathfrak{B}_{\left[\left[\begin{smallmatrix} E & 0 \\ 0 & 0\end{smallmatrix}\right], \left[\begin{smallmatrix} A & B \\ K_x & K_u\end{smallmatrix}\right]\right]}$ with $Ex(0) = Ex^0$. Then we obtain that the DAE (4.2) is satisfied for $y = Cx + Du$, $z = x$, and $d = 0$. Therefore, we have $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$, $Ex(0) = Ex^0$, and $(x_c, y, u) \in \mathfrak{B}_{[E_c,A_c,B_c,C_c,D_c]}$ for

$$x_c = \begin{pmatrix} x \\ 0 \\ u \end{pmatrix}.$$

This shows compatibility of the controller $[E_c, A_c, B_c, C_c, D_c]$.

(b) Assume (C1), (C2), and (C4) hold true. Compatibility of $[E_c, A_c, B_c, C_c, D_c]$ is a consequence of statement (a). Next we show, using (C2) and (C4), that the closed-loop system is behaviorally stable. By (C2) together with Lemma 2.3 and (C4) together with Theorem 3.4(c), we have

$$\forall \lambda \in \overline{\mathbb{C}_+}: \quad \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & B \\ K_x & K_u \end{bmatrix} = n + m \ \wedge\ \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & L_x \\ C & L_y \end{bmatrix} = n + k,$$

and thus

$$\forall \lambda \in \overline{\mathbb{C}_+}: \quad \operatorname{rk}_{\mathbb{C}} \begin{bmatrix} -\lambda E + A & B & 0 & 0 \\ K_x & K_u & K_x & 0 \\ 0 & 0 & -\lambda E + A & L_x \\ 0 & 0 & C & L_y \end{bmatrix} = 2n + m + k.$$

Then Lemma 2.3 implies that the DAE (4.3) is behaviorally stable. Now, let $(x, u, y) \in \mathfrak{B}_{[E,A,B,C,D]}$ and $x_c \in \mathcal{L}^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^{n_c})$ such that $(x_c, y, u) \in$

$\mathfrak{B}_{[E_c,A_c,B_c,C_c,D_c]}$. Write

$$x_c = \begin{pmatrix} z \\ d \\ u \end{pmatrix}$$

according to the decomposition in (4.4). Then, for $e = z - x$, (x, u, d, e) solves (4.3), and hence, by behavioral stability,

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \left\| \begin{pmatrix} x \\ u \\ d \\ e \end{pmatrix} \right\| = 0.$$

Therefore, invoking also $y = Cx + Du$, we further find that

$$\lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|y\| = 0 \quad \wedge \quad \lim_{t\to\infty} \operatorname{ess\,sup}_{[t,\infty)} \|x_c\| = 0.$$

This shows that $[E_c, A_c, B_c, C_c, D_c]$ is a stabilizing controller.

(c) Assume that (C1)–(C5) hold true. By (b), we obtain that $[E_c, A_c, B_c, C_c, D_c]$ is a stabilizing controller for $[E, A, B, C, D]$. To prove statement (c), it therefore suffices to show that for all $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}$ and $z^0 \in \mathbb{R}^n$, there exists a solution of the DAE (4.2) with $Ex(0) = x^0$ and $Ez(0) = Ez^0$.
Assume that $x^0 \in \mathcal{V}^{\text{diff}}_{[E,A,B,C,D]}$ and $z^0 \in \mathbb{R}^n$. By (C5), $[E_o, A_o, B_o, C_o, D_o]$ as in (3.4) is a freely initializable observer for $[E, A, B, C, D]$. Thus, by an application of the definition of freely initializable observers to the trivial trajectory $(0, 0, 0) \in \mathcal{B}_{[E,A,B,C,D]}$, we obtain that there exists some solution (e, d) of the DAE (3.7) with $Ee(0) = E(z^0 - x^0)$. Moreover, by (C3), there exists a solution (x, u) of the DAE

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} = \begin{bmatrix} A & B \\ K_x & K_u \end{bmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} + \begin{pmatrix} 0 \\ K_x e(t) \end{pmatrix}, \quad Ex(0) = Ex^0.$$

Hence, the DAE (4.3) is solved by (x, u, e, d). Therefore, x, u, $z = x + e$, d, and $y = Cx + Du$ satisfy (4.2) with $Ex(0) = Ex^0$ and

$$Ez(0) = Ex(0) + Ee(0) = Ex^0 + E(z^0 - x^0) = Ez^0.$$

This proves the desired result. ☐

We are now in the position to finish the proof of Theorem 4.2.

Proof of sufficiency in Theorem 4.2. The assertion follows from Theorem 4.4(b) together with Lemma 4.3 and Theorem 3.4(c). ☐

Remark 4.5 (controllers).

(a) Assume that $[E, A, B, C, D] \in \Sigma_{l,n,m,p}$ is behaviorally stabilizable. By the Kalman decomposition from [14, Thm. 13.1] (see also [3]) there exist $W \in \mathbf{Gl}_l(\mathbb{R})$, $T \in \mathbf{Gl}_n(\mathbb{R})$ such that

$$W(sE - A)T = \begin{bmatrix} sE_{11} - A_{11} & sE_{12} - A_{12} \\ 0 & sE_{22} - A_{22} \end{bmatrix}, \quad WB = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}, \quad CT = \begin{bmatrix} 0 & C_2 \end{bmatrix},$$

where $[E_{22}, A_{22}, B_2, C_2, D] \in \Sigma_{l_o,n_o,m,p}$ is completely observable (in the sense of [14]) and behaviorally stabilizable, and where $\operatorname{rk} \left[\begin{smallmatrix} E_{22} \\ C_2 \end{smallmatrix}\right] = n_o$. Therefore, $[E_{22}, A_{22}, B_2, C_2, D]$ is, in particular, strongly detectable. From Theorem 3.8(d), Lemma 4.3, and Theorem 4.4(c) we may then conclude that there exists a freely initializable stabilizing controller $[E_{c2}, A_{c2}, B_{c2}, C_{c2}, D_{c2}] \in \Sigma_{l_{c2},n_{c2},p,m}$ for $[E_{22}, A_{22}, B_2, C_2, D]$.

(b) As we have pointed out below Figure 4, compatibility of a controller means that it is possible to turn on the controller at $t = 0$. Consequently, the following definition can be used for compatibility of controllers, where the distributional solution framework as introduced in Remark 3.6(c) is used: We call a controller *compatible in the distributional sense* if

$$\forall\,(x,u,y) \in \mathfrak{B}^{\mathcal{D}'}_{[E,A,B,C,D]}\ \exists\,(\bar{x},\bar{u},\bar{y}) \in \mathfrak{B}^{\mathcal{D}'}_{[E,A,B,C,D]}\ \exists\,x_c \in \mathcal{D}'(\mathbb{R};\mathbb{R}^{n_c}) :$$
$$\operatorname{supp}(x-\bar{x}, u-\bar{u}, y-\bar{y}) \subseteq \mathbb{R}_{\geq 0}\ \wedge\ (x_c,\bar{y},\bar{u}) \in \mathfrak{B}^{\mathcal{D}'}_{[E_c,A_c,B_c,C_c,D_c]}.$$

A controller, in the distributional sense, is defined to be stabilizing if the restriction of $(\bar{x},\bar{u},\bar{y})$ to $\mathbb{R}_{>0}$ is a function tending to zero as t tends to infinity. We note that again the controller (4.4) can be used to stabilize the system. In particular, the criteria for the existence of a stabilizing controller in the distributional sense are the same as those in the function sense in Theorem 4.2.

Remark 4.6 (controllers for regular systems). Let a behaviorally stabilizable and behaviorally detectable system $[E,A,B,C,D] \in \Sigma_{n,n,m,p}$ be given such that $sE-A \in \mathbb{R}[s]^{n\times n}$ is regular. Then, by Remark 3.9, we can make the choice $L_y = I_p$, $L_x = -L$ in the observer realization (3.4). Elimination of the variable d gives rise to an asymptotic observer (3.27). If $[E,A,B,C,D]$ is additionally impulse observable, then, invoking Remark 3.9, L can be chosen such that (3.27) is a regular and freely initializable observer. By regularity of $sE-A$ and behavioral stabilizability of $[E,A,B,C,D]$, there exists some $F \in \mathbb{R}^{m\times n}$ such that $\operatorname{rk}_{\mathbb{C}}\big(\lambda E - (A+BF)\big) = n$ for all $\lambda \in \overline{\mathbb{C}_+}$ (in particular, $sE-(A+BF)$ is regular). By using

$$\begin{bmatrix} I_n & -B \\ 0 & I_p \end{bmatrix} \begin{bmatrix} -sE+A & B \\ -F & I_m \end{bmatrix} \begin{bmatrix} I_n & 0 \\ F & I_p \end{bmatrix} = \begin{bmatrix} -sE+A+BF & 0 \\ 0 & I_p \end{bmatrix}$$

together with regularity of $sE-(A+BF)$, we now obtain that (C1)–(C3) hold true for $K_x = -F$ and $K_u = I_m$. In other words, we add the feedback relation $u(t) = Fz(t)$ to the observer (3.27). Altogether, this means that $u(t) = Fz(t)$ and

$$d(t) = Cz(t) + Du(t) - y(t) = (C+DF)z(t) - y(t),$$

and thus we can eliminate the variables d and the copy of the input in the controller realization (4.4) to obtain the following simplified controller realization:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}Ez(t) &= (A+LC+BF+LDF)z(t) - Ly(t), \\ u(t) &= Fz(t); \end{aligned}$$

see Figure 6.

4.3. Notes and references. Dynamic controllers are also called *dynamic compensators* or *output regulators* in the literature. Research on generalization of Luenberger's ideas for ODEs to the DAE case started in the 1980's.

Dai and Wang used the following approach for strongly stabilizable and strongly detectable systems $[E,A,B,C,0] \in \Sigma_{n,n,m,p}$ with regular $sE-A \in \mathbb{R}[s]^{n\times n}$ (see [28,

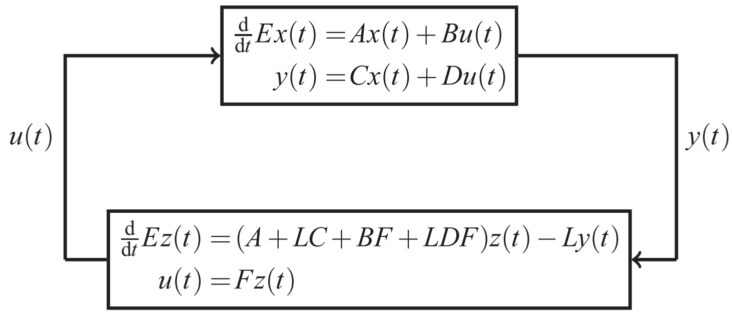

FIG. 6. *Controller for regular systems.*

30, 61, 62]): First, it is shown that there exists a proportional output feedback $u(t) = Ky(t) + v(t)$ for some $K \in \mathbb{R}^{m\times p}$, such that for the closed-loop system

$$[E_K, A_K, B_K, C_K, D_K] = [E, A + BKC, B, C, 0]$$

we have that $sE_K - A_K$ is regular and its index is at most one. Thereafter, a realization of this system by an ODE is considered and an ordinary stabilizing controller $[I_{n_c}, A_c, B_c, 0] \in \Sigma_{n_c,n_c,p,m}$ according to Luenberger's approach is applied. A stabilizing controller is then given by $[I_{n_c}, A_c, B_c, K]$.

The more direct approach for regular systems as described in Remark 4.6 has been presented in [18, 31, 32, 45, 55, 67].

To the best of our knowledge, controller design for systems with singular $sE - A$ has not been studied before.

5. Conclusions. In this paper we have studied existence and design of observers for linear time-invariant differential-algebraic systems which are not necessarily regular. To this end, we followed the definition of (asymptotic, exact) observers for behavioral systems from [58]. In particular, the novel observer design (3.3), (3.4) is again a DAE system. Existence of observers has been characterized in terms of behavioral detectability and observability of the plant, resp. In view of implementability of the observer, we have investigated its regularity and free initializability; these properties can be guaranteed provided that the plant is impulse observable.

We have used the results on observers for design of stabilizing controllers for DAE systems. Existence of stabilizing controllers was proved to be equivalent to behavioral stabilizability and behavioral detectability of the plant, which generalizes well-known results. The existence and design of compatible and freely initializable controllers was studied as well.

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