Disturbance decoupling for descriptor systems by behavioral feedback

Thomas Berger

Fachbereich Mathematik, Universität Hamburg

Braunschweig, March 10, 2016

$$\boxed{\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} E x(t) &= A x(t) + B u(t), \\ y(t) &= C x(t) \end{aligned}}$$

$$[E, A, B, C]$$

$$E, A \in \mathbb{R}^{\ell \times n},\ B \in \mathbb{R}^{\ell \times m},\ C \in \mathbb{R}^{p \times n}$$

$$\mathfrak{B}_{[E,A,B,C]} = \{\ (x, u, y)\ \mid [E, A, B, C] \text{ is satisfied }\ \}$$

Some basic notions

$sE - A$ is **regular**, if $\ell = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$

generalized Wong sequences:

$$\mathcal{V}_0 := \ker C, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i + \operatorname{im} B) \cap \ker C, \ i \geq 0,$$
$$\mathcal{V}_i \searrow \mathcal{V}^*_{[E,A,B,C]}$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i + \operatorname{im} B) \cap \ker C, \ i \geq 0,$$
$$\mathcal{W}_i \nearrow \mathcal{W}^*_{[E,A,B,C]}$$

$\underline{E = I}$:

- $(\mathcal{V}_i)$ is *invariant subspace algorithm* [Wonham '85]
- $(\mathcal{W}_i)$ is *controllability subspace algorithm* [Wonham '85]

Some basic notions

$sE - A$ is **regular**, if $\ell = n$ and $\det(sE - A) \in \mathbb{R}[s] \setminus \{0\}$

generalized Wong sequences:

$$\mathcal{V}_0 := \ker C, \qquad \mathcal{V}_{i+1} := A^{-1}(E\mathcal{V}_i + \operatorname{im} B) \cap \ker C, \ i \geq 0,$$
$$\mathcal{V}_i \searrow \mathcal{V}^*_{[E,A,B,C]}$$
$$\mathcal{W}_0 := \{0\}, \qquad \mathcal{W}_{i+1} := E^{-1}(A\mathcal{W}_i + \operatorname{im} B) \cap \ker C, \ i \geq 0,$$
$$\mathcal{W}_i \nearrow \mathcal{W}^*_{[E,A,B,C]}$$

$\underline{E = I}$:

- $(\mathcal{V}_i)$ is *invariant subspace algorithm* [Wonham '85]
- $(\mathcal{W}_i)$ is *controllability subspace algorithm* [Wonham '85]

$$\tfrac{\mathrm{d}}{\mathrm{d}t} Ex(t) = Ax(t) + Bu(t) + Qd(t), \quad d \in \mathcal{C}^\infty$$
$$y(t) = Cx(t)$$

$sE - A$ regular: disturbance decoupled $\iff C(sE-A)^{-1}Q = 0$

$$\Phi_{[E,A,B,C]} : \mathcal{C}^\infty \to \mathcal{P}(\mathcal{C}^\infty),$$
$$u \mapsto \left\{ y \in \mathcal{C}^\infty \;\middle|\; \exists x \in \mathcal{C}^\infty : \ (x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \right\}$$

Def. ($B = 0$): $[E, A, Q, C]$ is DD $:\iff$

$$\forall d_1, d_2 \in \mathcal{C}^\infty : \ \Phi_{[E,A,Q,C]}(d_1) = \Phi_{[E,A,Q,C]}(d_2)$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}Ex(t) = Ax(t) + Bu(t) + Qd(t), \quad d \in \mathcal{C}^\infty$$
$$y(t) = Cx(t)$$

$sE - A$ regular: disturbance decoupled $\iff C(sE-A)^{-1}Q = 0$

$$\Phi_{[E,A,B,C]} : \mathcal{C}^\infty \to \mathcal{P}(\mathcal{C}^\infty),$$
$$u \mapsto \left\{ \, y \in \mathcal{C}^\infty \;\middle|\; \exists\, x \in \mathcal{C}^\infty : \; (x,u,y) \in \mathfrak{B}_{[E,A,B,C]} \, \right\}$$

Def. $(B = 0)$: $[E, A, Q, C]$ is DD $:\iff$

$$\forall\, d_1, d_2 \in \mathcal{C}^\infty : \; \Phi_{[E,A,Q,C]}(d_1) = \Phi_{[E,A,Q,C]}(d_2)$$

Thm.:

$$[E, A, Q, C] \text{ is DD} \quad \Longleftrightarrow \quad \operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,0,C]} + A\mathcal{W}^*_{[E,A,0,C]}$$

Thm. [Wonham '85]:

$$\exists\, F: \ [I, A+BF, Q, C] \text{ is DD} \quad \Longleftrightarrow \quad \operatorname{im} Q \subseteq \mathcal{V}^*_{[I,A,B,C]}$$

Thm. [Banaszuk et al. '90]:

$\exists\, F: \ [E, A+BF, Q, C]$ is DD $\Longleftrightarrow$

- $\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B$,
- $\dim\left(E\mathcal{R}^* + \operatorname{im} Q\right) \le \dim \mathcal{R}^*$
  where $\mathcal{R}^* = \mathcal{V}^*_{[E,A,[B,Q],C]} \cap \mathcal{W}^*_{[E,A,[B,Q],C]}$

Thm.:

$$[E, A, Q, C] \text{ is DD} \iff \operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,0,C]} + A\mathcal{W}^*_{[E,A,0,C]}$$

Thm. [Wonham '85]:

$$\exists\, F : \ [I, A + BF, Q, C] \text{ is DD} \iff \operatorname{im} Q \subseteq \mathcal{V}^*_{[I,A,B,C]}$$

Thm. [Banaszuk et al. '90]:

$\exists\, F : \ [E, A + BF, Q, C]$ is DD $\iff$

- $\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B$,
- $\dim (E\mathcal{R}^* + \operatorname{im} Q) \leq \dim \mathcal{R}^*$
  where $\mathcal{R}^* = \mathcal{V}^*_{[E,A,[B,Q],C]} \cap \mathcal{W}^*_{[E,A,[B,Q],C]}$

Thm.:

$$[E, A, Q, C] \text{ is DD} \iff \operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,0,C]} + A\mathcal{W}^*_{[E,A,0,C]}$$

Thm. [Wonham '85]:

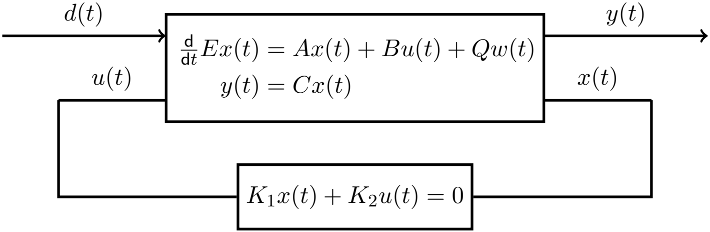
$$\exists\, F : [I, A + BF, Q, C] \text{ is DD} \iff \operatorname{im} Q \subseteq \mathcal{V}^*_{[I,A,B,C]}$$

Thm. [Banaszuk et al. '90]:

$\exists\, F : [E, A + BF, Q, C]$ is DD $\iff$

- $\operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B$,
- $\dim\left(E\mathcal{R}^* + \operatorname{im} Q\right) \leq \dim \mathcal{R}^*$
  where $\mathcal{R}^* = \mathcal{V}^*_{[E,A,[B,Q],C]} \cap \mathcal{W}^*_{[E,A,[B,Q],C]}$

Behavioral feedback



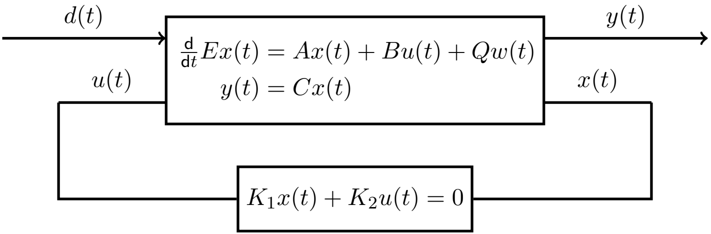
closed-loop system:

$$[E^K, A^K, Q^K, C^K] = \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_1 & K_2 \end{bmatrix}, \begin{bmatrix} Q \\ 0 \end{bmatrix}, [C, 0] \right]$$

Def.: $K = [K_1, K_2]$ is *compatible* for $[E, A, B, C]$, if

$$\forall\, (x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \;\; \exists\, (\tilde{x}, \tilde{u}) \in \mathfrak{B}_{[E^K, A^K, 0, 0]} : \;\; Ex(0) = E\tilde{x}(0)$$

Behavioral feedback



closed-loop system:

$$[E^K, A^K, Q^K, C^K] = \left[\begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A & B \\ K_1 & K_2 \end{bmatrix}, \begin{bmatrix} Q \\ 0 \end{bmatrix}, [C, 0] \right]$$

Def.: $K = [K_1, K_2]$ is *compatible* for $[E, A, B, C]$, if

$$\forall\, (x, u, y) \in \mathfrak{B}_{[E,A,B,C]} \;\; \exists\, (\tilde{x}, \tilde{u}) \in \mathfrak{B}_{[E^K, A^K, 0, 0]} : \;\; Ex(0) = E\tilde{x}(0)$$

Thm.:

$$\begin{aligned} & \exists \text{ compatible } K = [K_1, K_2]\text{: } [E^K, A^K, Q^K, C^K] \text{ is DD} \\ \iff & \operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B \end{aligned}$$

Still a lot of freedom in the choice of K!

Thm.:

$$\exists \text{ compatible } K = [K_1, K_2]\colon\ [E^K, A^K, Q^K, C^K] \text{ is DD}$$
$$\iff \operatorname{im} Q \subseteq E\mathcal{V}^*_{[E,A,B,C]} + A\mathcal{W}^*_{[E,A,B,C]} + \operatorname{im} B$$

Still a lot of freedom in the choice of K!

Open problem due to [Lebret '94]

Disturbance decoupling does not guarantee disturbance rejection!

Example:

$$\dot{x}_1(t) = -x_2(t) + d(t), \quad y(t) = x_1(t)$$

is DD since $\Phi(d_1) = \Phi(d_2) = \mathcal{C}^\infty$ for all $d_1, d_2 \in \mathcal{C}^\infty$

but: y still depends on d

$$y(t) = x_1(0) + \int_0^t d(s) - x_2(s)\,\mathrm{d}s$$

Open problem due to [Lebret '94]

Disturbance decoupling does not guarantee disturbance rejection!

Example:

$$\dot{x}_1(t) = -x_2(t) + d(t), \quad y(t) = x_1(t)$$

is DD since $\Phi(d_1) = \Phi(d_2) = \mathcal{C}^\infty$ for all $d_1, d_2 \in \mathcal{C}^\infty$

but: y still depends on d

$$y(t) = x_1(0) + \int_0^t d(s) - x_2(s) \, \mathrm{d}s$$

Open problem due to [Lebret '94]

Disturbance decoupling does not guarantee disturbance rejection!

Example:

$$\dot{x}_1(t) = -x_2(t) + d(t), \quad y(t) = x_1(t)$$

is DD since $\Phi(d_1) = \Phi(d_2) = \mathcal{C}^\infty$ for all $d_1, d_2 \in \mathcal{C}^\infty$

but: y still depends on d

$$y(t) = x_1(0) + \int_0^t d(s) - x_2(s) \, \mathrm{d}s$$

additional assumption: $\frac{\mathrm{d}}{\mathrm{d}t} E^K z(t) = A^K z(t)$ has unique solutions

i.e., $[E^K, A^K]$ is autonomous $\iff \ker_{\mathbb{R}[s]}(sE^K - A^K) = \{0\}$

Aim: Characterize existence of compatible K s.t.

- $[E^K, A^K, Q^K, C^K]$ is DD
- $[E^K, A^K]$ is autonomous

additional assumption: $\frac{\mathrm{d}}{\mathrm{d}t} E^K z(t) = A^K z(t)$ has unique solutions

i.e., $[E^K, A^K]$ is autonomous $\iff \ker_{\mathbb{R}[s]}(sE^K - A^K) = \{0\}$

Aim: Characterize existence of compatible K s.t.

- $[E^K, A^K, Q^K, C^K]$ is DD
- $[E^K, A^K]$ is autonomous

additional assumption: $\frac{\mathrm{d}}{\mathrm{d}t} E^K z(t) = A^K z(t)$ has unique solutions

i.e., $[E^K, A^K]$ is autonomous $\iff \ker_{\mathbb{R}[s]}(sE^K - A^K) = \{0\}$

Aim: Characterize existence of compatible K s.t.

- $[E^K, A^K, Q^K, C^K]$ is DD
- $[E^K, A^K]$ is autonomous

Let inv. S, T be s.t.

$$S[sE - A, -B]T = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \end{bmatrix}$$

where $\mathrm{rk}(\lambda E_{22} - A_{22}) = n_3$ for all $\lambda \in \mathbb{C}$;

$$SQ = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad [C, 0]T = [C_1, C_2, C_3], \quad KT = [K_1, K_2, K_3]$$

Lemma:

K is compatible for $[E, A, B, C]$ s.t. $[E^K, A^K]$ is autonomous

$\iff \quad \mathrm{im}\, K_1 \subseteq \mathrm{im}\, K_2 \ \wedge \ \mathrm{rk}\, K_2 = n_2$

Let inv. S, T be s.t.

$$S[sE - A, -B]T = \begin{bmatrix} sI_{n_1} - A_{11} & -A_{12} & 0 \\ 0 & 0 & sE_{22} - A_{22} \end{bmatrix}$$

where $\mathrm{rk}(\lambda E_{22} - A_{22}) = n_3$ for all $\lambda \in \mathbb{C}$;

$$SQ = \begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}, \quad [C, 0]T = [C_1, C_2, C_3], \quad KT = [K_1, K_2, K_3]$$

Lemma:

K is compatible for $[E, A, B, C]$ s.t. $[E^K, A^K]$ is autonomous

$\iff$ $\mathrm{im}\, K_1 \subseteq \mathrm{im}\, K_2 \ \wedge\ \mathrm{rk}\, K_2 = n_2$

If $[E^K, A^K, Q^K, C^K]$ is DD, then

$$\forall\, d \in \mathcal{C}^\infty \ \exists\, (x_1, x_2, x_3) \in \mathcal{C}^\infty : \qquad \begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + Q_1 d \\ E_{22}\dot{x}_3 &= \qquad\qquad A_{22}x_3 + Q_2 d \\ 0 &= C_1x_1 + C_2x_2 + C_3x_3 \\ 0 &= K_1x_1 + K_2x_2 + K_3x_3 \end{aligned}$$

By **Lemma**: $\exists$ inv. V s.t.: $\quad V[K_1, K_2, K_3] = \begin{bmatrix} Z_1 & I_{n_2} & Z_2 \\ 0 & 0 & Z_3 \end{bmatrix}$

$$\forall\, d \in \mathcal{C}^\infty \ \exists\, (x_1, x_3) \in \mathcal{C}^\infty : \quad \begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}Z_1)x_1 - A_{12}Z_2x_3 + Q_1 d \\ E_{22}\dot{x}_3 &= \qquad\qquad\qquad A_{22}x_3 + Q_2 d \\ 0 &= (C_1 - C_2Z_1)x_1 + (C_3 - C_2Z_2)x_3 \end{aligned}$$

If $[E^K, A^K, Q^K, C^K]$ is DD, then

$$\forall\, d \in \mathcal{C}^\infty\ \exists\,(x_1,x_2,x_3) \in \mathcal{C}^\infty : \quad \begin{aligned} \dot x_1 &= A_{11}x_1 + A_{12}x_2 + Q_1 d \\ E_{22}\dot x_3 &= \qquad\qquad A_{22}x_3 + Q_2 d \\ 0 &= C_1x_1 + C_2x_2 + C_3x_3 \\ 0 &= K_1x_1 + K_2x_2 + K_3x_3 \end{aligned}$$

By **Lemma**: $\exists$ inv. V s.t.: $V[K_1, K_2, K_3] = \begin{bmatrix} Z_1 & I_{n_2} & Z_2 \\ 0 & 0 & Z_3 \end{bmatrix}$

$$\forall\, d \in \mathcal{C}^\infty\ \exists\,(x_1,x_3) \in \mathcal{C}^\infty : \quad \begin{aligned} \dot x_1 &= (A_{11} - A_{12}Z_1)x_1 - A_{12}Z_2x_3 + Q_1 d \\ E_{22}\dot x_3 &= \qquad\qquad A_{22}x_3 + Q_2 d \\ 0 &= (C_1 - C_2Z_1)x_1 + (C_3 - C_2Z_2)x_3 \end{aligned}$$

If $[E^K, A^K, Q^K, C^K]$ is DD, then

$$\forall\, d \in \mathcal{C}^\infty\ \exists\,(x_1,x_2,x_3) \in \mathcal{C}^\infty: \quad \begin{aligned} \dot{x}_1 &= A_{11}x_1 + A_{12}x_2 + Q_1 d \\ E_{22}\dot{x}_3 &= A_{22}x_3 + Q_2 d \\ 0 &= C_1x_1 + C_2x_2 + C_3x_3 \\ 0 &= K_1x_1 + K_2x_2 + K_3x_3 \end{aligned}$$

By **Lemma**: $\exists$ inv. V s.t.: $V[K_1, K_2, K_3] = \begin{bmatrix} Z_1 & I_{n_2} & Z_2 \\ 0 & 0 & Z_3 \end{bmatrix}$

$$\forall\, d \in \mathcal{C}^\infty\ \exists\,(x_1,x_3) \in \mathcal{C}^\infty: \quad \begin{aligned} \dot{x}_1 &= (A_{11} - A_{12}Z_1)x_1 - A_{12}Z_2x_3 + Q_1 d \\ E_{22}\dot{x}_3 &= A_{22}x_3 + Q_2 d \\ 0 &= (C_1 - C_2Z_1)x_1 + (C_3 - C_2Z_2)x_3 \end{aligned}$$

$\Longrightarrow \quad \exists\ F = [Z_1, Z_2]$:

$$\left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix} - \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix} F, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix}, 0 \right] \text{ is DD}$$

equivalently, by [Banaszuk et al. '90], with

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{Q}] = \left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix}, \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix}, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix} \right],$$

- $\operatorname{im} \tilde{Q} \subseteq \tilde{E} \mathcal{V}^*_{[\tilde{E},\tilde{A},\tilde{B},0]} + \tilde{A} \mathcal{W}^*_{[\tilde{E},\tilde{A},\tilde{B},0]} + \operatorname{im} \tilde{B}$,
- $\dim \left(\tilde{E} \tilde{\mathcal{R}}^* + \operatorname{im} \tilde{Q} \right) \leq \dim \tilde{\mathcal{R}}^*$
  where $\tilde{\mathcal{R}}^* = \mathcal{V}^*_{[\tilde{E},\tilde{A},[\tilde{B},\tilde{Q}],0]} \cap \mathcal{W}^*_{[\tilde{E},\tilde{A},[\tilde{B},\tilde{Q}],0]}$

$\Longrightarrow \quad \exists\, F = [Z_1, Z_2]$:

$$\left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix} - \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix} F, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix}, 0 \right] \text{ is DD}$$

equivalently, by [Banaszuk et al. '90], with

$$[\tilde{E}, \tilde{A}, \tilde{B}, \tilde{Q}] = \left[\begin{bmatrix} I_{n_1} & 0 \\ 0 & E_{22} \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} A_{11} & 0 \\ 0 & A_{22} \\ C_1 & C_3 \end{bmatrix}, \begin{bmatrix} A_{12} \\ 0 \\ C_2 \end{bmatrix}, \begin{bmatrix} Q_1 \\ Q_2 \\ 0 \end{bmatrix} \right],$$

- $\operatorname{im} \tilde{Q} \subseteq \tilde{E}\mathcal{V}^*_{[\tilde{E},\tilde{A},\tilde{B},0]} + \tilde{A}\mathcal{W}^*_{[\tilde{E},\tilde{A},\tilde{B},0]} + \operatorname{im} \tilde{B}$,
- $\dim \left(\tilde{E}\tilde{\mathcal{R}}^* + \operatorname{im} \tilde{Q} \right) \le \dim \tilde{\mathcal{R}}^*$
  where $\tilde{\mathcal{R}}^* = \mathcal{V}^*_{[\tilde{E},\tilde{A},[\tilde{B},\tilde{Q}],0]} \cap \mathcal{W}^*_{[\tilde{E},\tilde{A},[\tilde{B},\tilde{Q}],0]}$