Funnel control für DAEs

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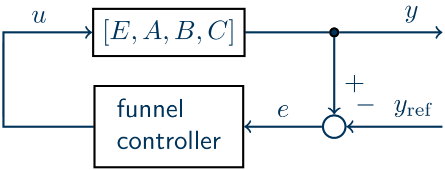

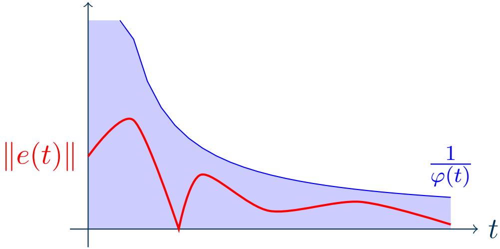

Nulldynamik:

$$\mathcal{ZD} := \left\{ (x,u,y) \;\middle|\; \begin{array}{rrcl} & E\dot{x} & = & Ax + Bu \\ 0 = & y & = & Cx \end{array} \right\}$$

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Lem.: $\mathcal{ZD}$ stabil $\implies$ $\mathcal{ZD}$ autonom

$\mathcal{V} \subseteq \mathbb{R}^n$ heißt **(E, A, B)-invariant** $:\Longleftrightarrow \; A\mathcal{V} \subseteq E\mathcal{V} + \operatorname{im} B$

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Wähle $V \in \mathbb{R}^{n \times k}$ mit $\operatorname{rk} V = k$, so dass

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Proposition:

$$\mathcal{ZD} \text{ autonom} \quad \Longleftrightarrow \quad \begin{cases} \textbf{(A1)} & \operatorname{rk} B = m \\ \textbf{(A2)} & \ker E \cap \operatorname{im} V = \{0\} \\ \textbf{(A3)} & \operatorname{im} B \cap \operatorname{im} EV = \{0\} \end{cases}$$

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Theorem („Normalform“)

- $[E,A,B,C] \in \Sigma_{l,n,m}$ mit $\mathcal{ZD}$ autonom
- $\operatorname{im} V = \sup\{\, \mathcal{V} \mid \mathcal{V}$ ist (E,A,B)-invariant $\wedge\; \mathcal{V} \subseteq \ker C \,\}$

$$\implies \exists\, S \in \mathbf{Gl}_l(\mathbb{R}), W \in \mathbb{R}^{n \times (n-k)}: \; [V,W] \in \mathbf{Gl}_n(\mathbb{R}) \wedge$$

$$SE[V,W] = \begin{bmatrix} I_k & E_2 \\ 0 & E_4 \\ 0 & E_6 \end{bmatrix}, \; SA[V,W] = \begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \\ 0 & A_6 \end{bmatrix}, \; SB = \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix},$$

$$C[V,W] = [0, C_2]$$

Theorem („Normalform“ DAE)

$[E, A, B, C] \in \Sigma_{l,n,m}$ mit

- $\operatorname{rk} C = m$
- $\mathcal{ZD}$ autonom
- $\Gamma = \lim_{s\to\infty} s^{-1}[0, I_m]L(s)[0, I_m]^\top \in \mathbb{R}^{m\times m}$ existiert, wobei $L(s)$ Linksinverse von $\begin{bmatrix} sE-A & -B \\ -C & 0 \end{bmatrix}$ über $\mathbb{R}(s)$ ist

$\Longrightarrow$ $[E, A, B, C]$ hat die Form

$$\begin{aligned} \dot{x}_1 &= Q\,x_1 + A_{12}\,y - E_{13}\,\dot{x}_3 \\ \Gamma\,\dot{y} &= \tilde{A}_{22}\,y + \Psi(x_1(0), y) + u \\ x_3 &= \sum_{k=0}^{\nu-1} N^k E_{32}\,y^{(k+1)} \\ 0 &= A_{42}\,y - E_{42}\,\dot{y} - E_{43}\,\dot{x}_3 \end{aligned}$$

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Lem.: $\mathcal{ZD}$ stabil $\iff$ $\sigma(Q) \subseteq \mathbb{C}_-$

$[\boldsymbol{E}, \boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}]$ rechts-invertierbar $:\Longleftrightarrow$

$$\exists\, k \in \mathbb{N}\ \forall\, y \in \mathcal{C}^k(\mathbb{R};\mathbb{R}^m)\ \exists\, (x,u) \in \mathcal{C}(\mathbb{R};\mathbb{R}^n) \times \mathcal{C}(\mathbb{R};\mathbb{R}^m) :$$

$$(x,u,y) \text{ löst } [E,A,B,C]$$

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Proposition

- $\mathcal{ZD}$ autonom
- $\Gamma = \lim_{s\to\infty} s^{-1}[0, I_m]L(s)[0, I_m]^\top$ existiert

$[E,A,B,C]$ rechts-invertierbar $\iff$ $\operatorname{rk} C = m\ \wedge$

$$\begin{aligned}
\dot x_1 &= Q\,x_1 + A_{12}\,y - E_{13}\,\dot x_3\\
\Gamma\,\dot y &= \tilde A_{22}\,y + \Psi(x_1(0), y) + u\\
x_3 &= \textstyle\sum_{k=0}^{\nu-1} N^k E_{32}\,y^{(k+1)}\\
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\end{aligned}$$

Theorem (funnel control)

$[E, A, B, C] \in \Sigma_{l,n,m}$ mit

- $\mathcal{ZD}$ stabil
- $[E, A, B, C]$ rechts-invertierbar
- $\Gamma = \lim_{s\to\infty} s^{-1}[0, I_m]L(s)[0, I_m]^\top$ existiert und $\Gamma = \Gamma^\top \geq 0$

Dann erreicht der *funnel controller*

$$u(t) = -k(t)\, e(t), \qquad \text{wobei} \qquad e(t) = y(t) - y_{\text{ref}}(t)$$

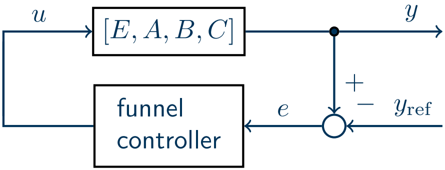
$$k(t) = \frac{\hat{k}}{1 - \varphi(t)^2 \|e(t)\|^2},$$

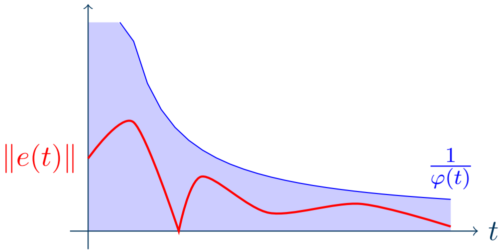
angewendet auf $[E, A, B, C]$, dass

$$x \in L^\infty,\ k \in L^\infty \quad \wedge \quad \exists\, \varepsilon > 0\ \forall\, t > 0 : \ \|e(t)\| \leq \varphi(t)^{-1} - \varepsilon$$

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

$$E, A \in \mathbb{R}^{l \times n}, \qquad B \in \mathbb{R}^{l \times m}, \quad C \in \mathbb{R}^{m \times n}$$





Beweisskizze:

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$$\text{(A3)} \;\; \operatorname{im} B \cap \operatorname{im} EV = \{0\} \quad \Rightarrow \quad [EV, B] \text{ full rank}$$

$$\Rightarrow \quad S\,[EV, B] = \begin{bmatrix} I_k & 0 \\ 0 & I_m \\ 0 & 0 \end{bmatrix}$$

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$$(E,A,B)\text{-Invarianz} \;\Leftrightarrow\; AV = EVN + BM$$

$$\Rightarrow \; S^{-1} \begin{bmatrix} A_1 \\ A_3 \\ A_5 \end{bmatrix} = AV = EVN + BM = S^{-1} \begin{bmatrix} I_k \\ 0 \\ 0 \end{bmatrix} N + S^{-1} \begin{bmatrix} 0 \\ I_m \\ 0 \end{bmatrix} M$$