Some ideas on the funnel observer

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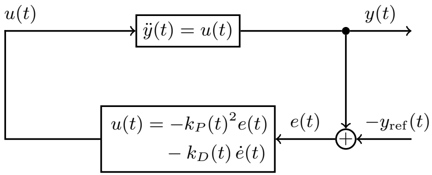
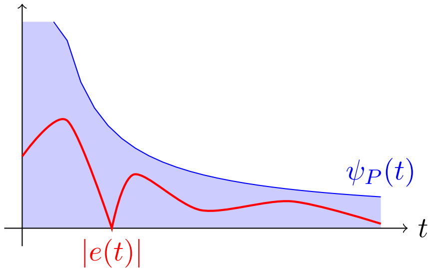
ZD
I.) Find
s.t. funnel control is achieved
II.) Find





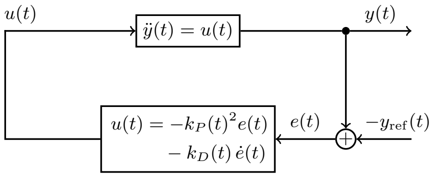
$$k_P(t) = \frac{\psi_P(t)}{\psi_P(t) - |e(t)|},$$

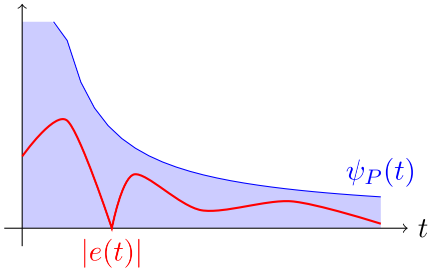
$$k_D(t) = \frac{\psi_D(t)}{\psi_D(t) - |\dot{e}(t)|},$$

$$0 < \delta \leq \tfrac{\mathrm{d}}{\mathrm{d}t}\psi_P(t) + \psi_D(t)$$

[Hackl, Hopfe, Ilchmann, Müller, Trenn '13]:
This works!

Problem: derivative of y not available!





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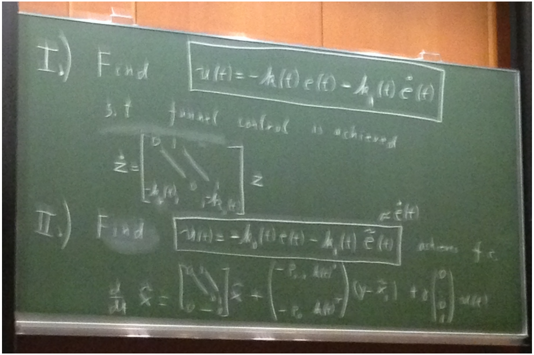
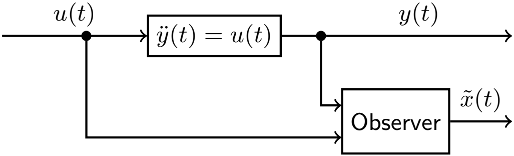
High-gain observer for $\ddot{y} = u$





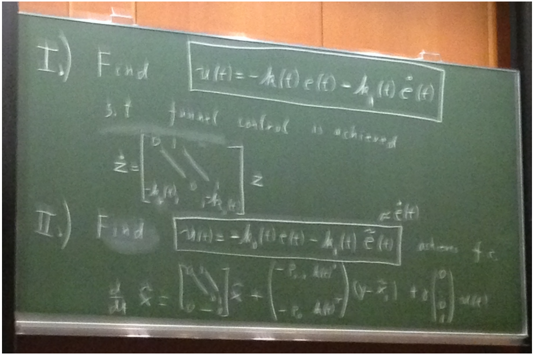
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1 = \tilde{x}_2 + p_1 k (y - \tilde{x}_1),$$
$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_2 = u + p_2 k^2 (y - \tilde{x}_1),$$
$$p_1, p_2 > 0$$

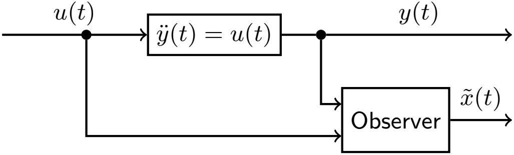
[TORNAMBE '92]:

$k > 0$ large enough

$$\implies \quad \tilde{x}_1 \approx y \ \wedge \ \tilde{x}_2 \approx \dot{y}$$

High-gain observer for $\ddot{y} = u$





$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1 = \tilde{x}_2 + p_1 k(t)(y - \tilde{x}_1),$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_2 = u + p_2 k(t)^2 (y - \tilde{x}_1),$$

$$\dot{k}(t) = \gamma d_{\lambda,\hat{\lambda}}^2 (y(t) - \tilde{x}_1(t)),$$

$$d_{\lambda,\hat{\lambda}}(z) = \begin{cases} \hat{\lambda} - \lambda, & |z| \geq \hat{\lambda} \\ |z| - \lambda, & \lambda \leq |z| \leq \hat{\lambda} \\ 0, & |z| \leq \lambda \end{cases}$$

[Bullinger, Ilchmann, Allgöwer '98]:

$\tilde{x}, k \in L^\infty$ and

$\operatorname{dist}\big(y - \tilde{x}_1, [-\lambda, \lambda]\big) \to 0$

Achim: "It should be possible with $k(t) = \frac{\psi_O(t)}{\psi_O(t) - |y(t) - \tilde{x}_1(t)|}$!"

Consider a modified observer for $\ddot{y} = u$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1 = \tilde{x}_2 + \big(q_1 + p_1 k(t)\big)(y - \tilde{x}_1), \qquad k(t) = \frac{\psi_O(t)}{\psi_O(t) - |y(t) - \tilde{x}_1(t)|}$$

$$\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_2 = u + \big(q_2 + p_2 k(t)\big)(y - \tilde{x}_1),$$

$$e_1 := y - \tilde{x}_1,\ e_2 := \dot{y} - \tilde{x}_2, \qquad k(t) = \frac{\psi_O(t)}{\psi_O(t) - |e_1(t)|}$$

$$\dot{e}_1 = e_2 - \big(q_1 + p_1 k(t)\big)e_1,$$

$$\dot{e}_2 = -\big(q_2 + p_2 k(t)\big)e_1,$$

$$A := \begin{bmatrix} -q_1 & 1 \\ -q_2 & 0 \end{bmatrix},\ P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \text{ s.t. } A^\top P + PA = -I$$

$$p_1, p_2 \text{ s.t. } \binom{p_1}{p_2} = \begin{pmatrix} 1 \\ -\frac{p_{12}}{p_{22}} \end{pmatrix}; \quad \dot{e} = Ae - k(t)\binom{p_1}{p_2} e_1$$

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$$\dot{e} = Ae - k(t) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1$$

$$\begin{aligned} \frac{\mathsf{d}}{\mathsf{d}t} e^\top P e &= e^\top (A^\top P + PA) e - k(t) \left(e_1 \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}^\top P e + e^\top P \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1 \right) \\ &= -\|e\|^2 - k(t) \left(p_{11} - \frac{p_{12}^2}{p_{22}} \right) e_1^2 \leq -\|e\|^2 \end{aligned}$$

$\Longrightarrow \; e_1, e_2 \in L^\infty[0,\omega)$, standard funnel argument : $k \in L^\infty$

$\Longrightarrow \; \omega = \infty \;\wedge\; e(t) \to 0 \;\wedge\; k(t) \to 1$

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$$\implies e_1, {\color{red} e_2} \in L^\infty[0,\omega), \text{ standard funnel argument : } k \in L^\infty$$
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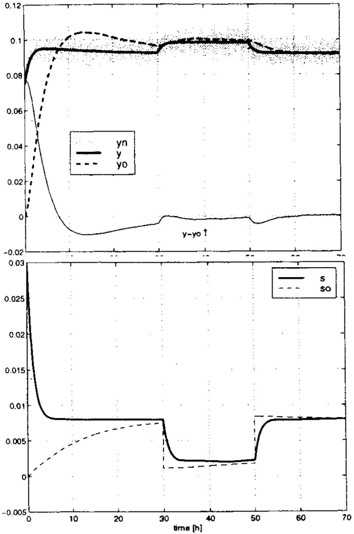
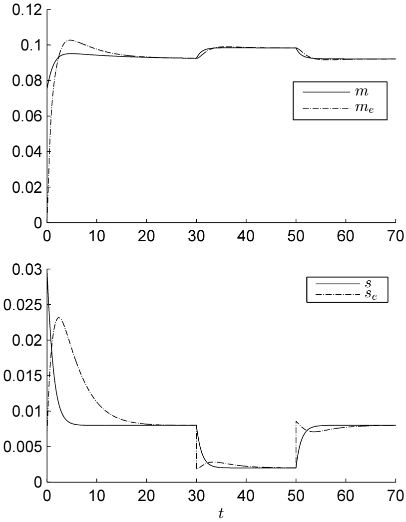
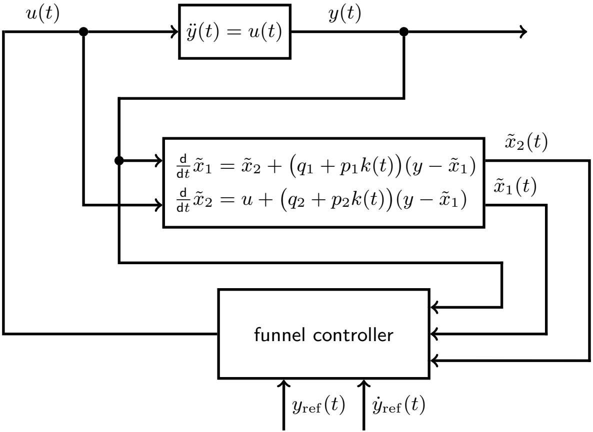
λ-strip observer [BIA '98]:



funnel observer:





- given $\psi(t)$, we have $|y(t) - y_{\rm ref}(t)| < \psi(t)$ for all $t > 0$
- controller only needs $y, \tilde{x}_1, \tilde{x}_2$

straightforward approach:

$$u(t) = -k_P(t)^2(y(t) - y_{\text{ref}}(t)) - k_D(t)(\tilde{x}_2(t) - \dot{y}_{\text{ref}}(t)),$$
$$\tilde{x}_2(t) = \dot{y}(t) - e_2(t)$$

Problem: performance depends on $\|e_2\|_\infty$; we need $\inf_{t>0} \psi_D(t) > \|e_2\|_\infty$

alternative approach: What if we put $\tilde{x}_1 - y_{\text{ref}}$ into the funnel?

$$|y(t) - y_{\text{ref}}(t)| \leq |\tilde{x}_1(t) - y_{\text{ref}}(t)| + |e_1(t)| < \psi_P(t) + \psi_O(t)$$

Given $\psi(t)$, let $\psi_P(t) = \psi_O(t) = \frac{1}{2}\psi(t)$

Seek a system of relative degree 2 with output $\hat{y} := \tilde{x}_1$!

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$$\frac{\mathsf{d}}{\mathsf{d}t}\hat{y} = \tilde{x}_2 + \big(q_1 + p_1 k(t)\big)e_1,$$

$$\left(\frac{\mathsf{d}}{\mathsf{d}t}\right)^2\hat{y} = u + \big(q_2 + p_2 k(t)\big)e_1 + p_1\dot{k}(t)e_1 + \big(q_1 + p_1 k(t)\big)\dot{e}_1$$

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[Hackl et al. '13]: PD-funnel controller can be applied to Σ!

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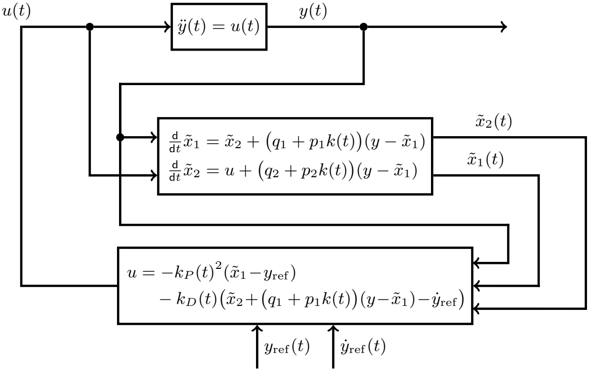
$$\begin{aligned} u(t) &= -k_P(t)^2(\tilde{x}_1(t) - y_{\mathrm{ref}}(t)) - k_D(t)\big(\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1(t) - \dot{y}_{\mathrm{ref}}(t)\big) \\ &= -k_P(t)^2(\tilde{x}_1(t) - y_{\mathrm{ref}}(t)) \\ &\quad - k_D(t)\Big(\tilde{x}_2(t) + \big(q_1 + p_1 k(t)\big)(y(t) - \tilde{x}_1(t)) - \dot{y}_{\mathrm{ref}}(t)\Big) \end{aligned}$$



$$|y(t) - \tilde{x}_1(t)| < \psi_O(t), \qquad |\tilde{x}_1(t) - y_{\mathrm{ref}}(t)| < \psi_P(t)$$

$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde{x}_1(t) - \dot{y}_{\mathrm{ref}}(t)\right| < \psi_D(t), \qquad |y(t) - y_{\mathrm{ref}}(t)| < \psi_P(t) + \psi_O(t)$$

Simulation

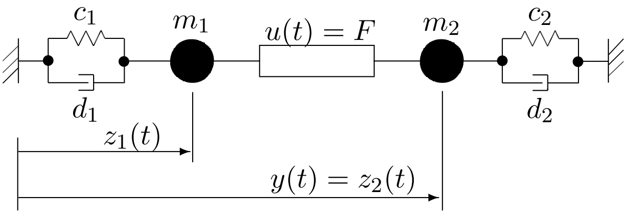
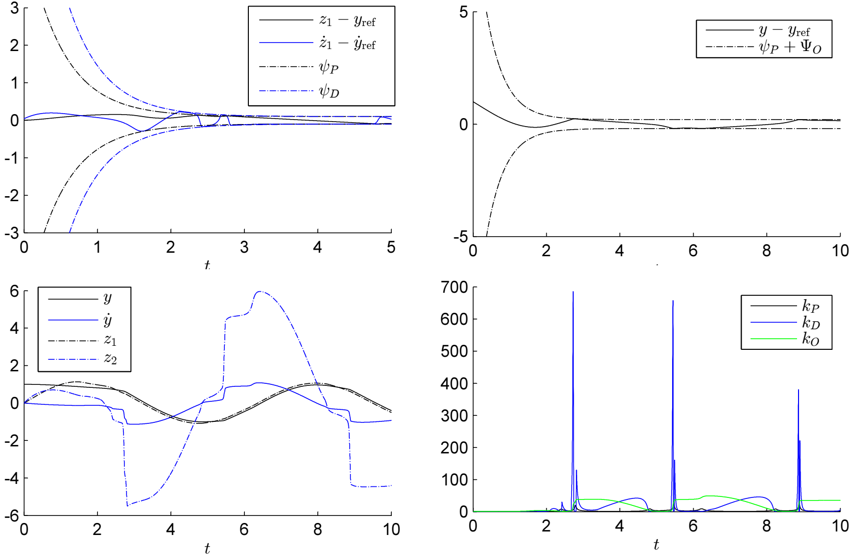
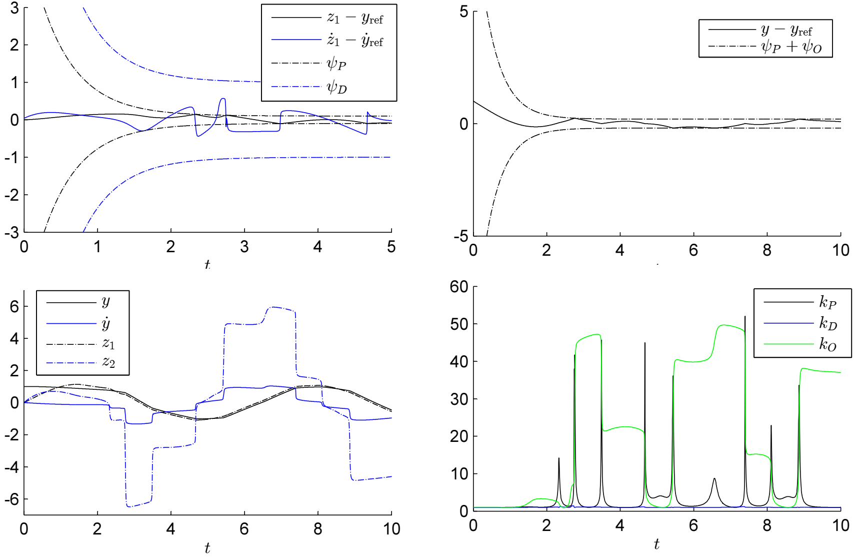


Figure: Mass-spring-damper system

$$y_{\mathrm{ref}}(t) = \sin t$$


$z_1 - y_{\mathrm{ref}}$
$\dot{z}_1 - \dot{y}_{\mathrm{ref}}$
ψ_P
ψ_D
t
$y - y_{\mathrm{ref}}$
$\psi_P + \Psi_O$
y
$\dot{y}$
z_1
z_2
t
k_P
k_D
k_O
t


$z_1 - y_{\mathrm{ref}}$
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ψ_P
ψ_D
t
$y - y_{\mathrm{ref}}$
$\psi_P + \psi_O$
y
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z_1
z_2
t
k_P
k_D
k_O
t

Extensions

$$\left.\begin{aligned} \dot x_1 &= x_2 \\ &\vdots \\ \dot x_{r-1} &= x_r \\ \dot x_r &= T(x_1,\ldots,x_r)+\gamma u \\ y &= x_1 \end{aligned}\right\} \xrightarrow{y,u} \left\{\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_1 &= \tilde x_2+\big(q_1+p_1k(t)\big)(y-\tilde x_1) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_2 &= \tilde x_3+\big(q_2+p_2k(t)\big)(y-\tilde x_1) \\ &\vdots \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_{r-1} &= \tilde x_r+\big(q_{r-1}+p_{r-1}k(t)\big)(y-\tilde x_1) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_r &= \tilde\gamma u+\big(q_r+p_rk(t)\big)(y-\tilde x_1) \end{aligned}\right.$$

$$\forall\, x_1,\ldots,x_r \in L^\infty:\ T(x_1,\ldots,x_r)\in L^\infty,\quad \gamma\neq 0,\ \tilde\gamma\in\mathbb{R}$$

$$A := \begin{bmatrix} -q_1 & 1 & & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1 \\ -q_r & & & 0 \end{bmatrix},\ P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix}\ P_{11}\in\mathbb{R},\ P_{22}\in\mathbb{R}^{(r-1)\times(r-1)}$$

s.t. $A^\top P + PA = -I$ and $\begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} = \begin{pmatrix} 1 \\ -P_{22}^{-1}P_{12}^\top \end{pmatrix}$

Theorem:

$y,\ldots,y^{(r-1)}\in L^\infty \implies \tilde x_1,\ldots,\tilde x_r, k\in L^\infty \wedge |y(t)-\tilde x_1(t)|<\psi_O(t)$

Extensions

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Extensions

$$\left.\begin{aligned}\dot x_1 &= x_2\\ \dot x_2 &= T(x_1,x_2)+\gamma u\\ y &= x_1\end{aligned}\right\} + \left\{\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_1 &= \tilde x_2 + \big(q_1+p_1 k(t)\big)(y-\tilde x_1)\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_2 &= \tilde\gamma u + \big(q_2+p_2 k(t)\big)(y-\tilde x_1)\\ u &= -k_P(t)^2(\tilde x_1 - y_{\mathrm{ref}})\\ &\quad -k_D(t)\big(\tilde x_2 + \big(q_1+p_1 k(t)\big)(y-\tilde x_1) - \dot y_{\mathrm{ref}}\big)\end{aligned}\right.$$

$$\gamma > 0,\ \tilde\gamma > 0$$
$$\forall\, x_1, x_2 \in \mathcal{C}:\ x_1 \in L^\infty \ \Rightarrow\ T(x_1,x_2) \in L^\infty$$

Theorem:

$$|y(t)-\tilde x_1(t)| < \psi_O(t), \qquad |\tilde x_1(t) - y_{\mathrm{ref}}(t)| < \psi_P(t)$$
$$\left|\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_1(t) - \dot y_{\mathrm{ref}}(t)\right| < \psi_D(t), \qquad |y(t) - y_{\mathrm{ref}}(t)| < \psi_P(t) + \psi_O(t)$$

Proof: use saturation of T

Extensions

$$\left.\begin{aligned}\dot x_1 &= x_2\\ \dot x_2 &= T(x_1,x_2)+\gamma u\\ y &= x_1\end{aligned}\right\} + \left\{\begin{aligned}\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_1 &= \tilde x_2+\big(q_1+p_1k(t)\big)(y-\tilde x_1)\\ \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_2 &= \tilde\gamma u+\big(q_2+p_2k(t)\big)(y-\tilde x_1)\\ u &= -k_P(t)^2(\tilde x_1-y_{\mathrm{ref}})\\ &\quad -k_D(t)\Big(\tilde x_2+\big(q_1+p_1k(t)\big)(y-\tilde x_1)-\dot y_{\mathrm{ref}}\Big)\end{aligned}\right.$$

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Proof: use saturation of T

Extensions

$$\left.\begin{aligned} \dot x_1 &= x_2 \\ \dot x_2 &= T(x_1,x_2)+\gamma u \\ y &= x_1 \end{aligned}\right\} + \left\{\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_1 &= \tilde x_2 + \big(q_1+p_1k(t)\big)(y-\tilde x_1) \\ \tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_2 &= \tilde\gamma u + \big(q_2+p_2k(t)\big)(y-\tilde x_1) \\ u &= -k_P(t)^2(\tilde x_1 - y_{\mathrm{ref}}) \\ &\quad -k_D(t)\Big(\tilde x_2 + \big(q_1+p_1k(t)\big)(y-\tilde x_1) - \dot y_{\mathrm{ref}}\Big) \end{aligned}\right.$$

$$\gamma > 0,\ \tilde\gamma > 0$$

$$\forall\, x_1, x_2 \in \mathcal{C}:\ x_1 \in L^\infty \ \Rightarrow\ T(x_1,x_2) \in L^\infty$$

Theorem:

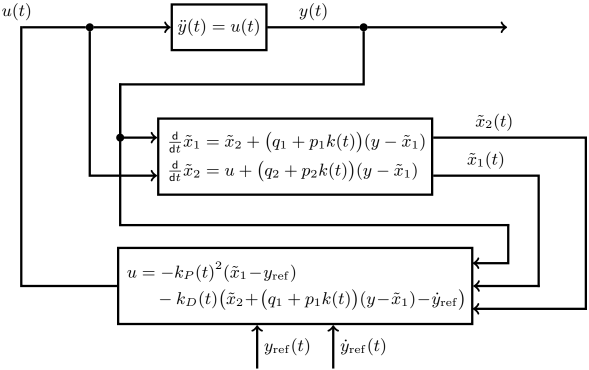
$$|y(t) - \tilde x_1(t)| < \psi_O(t), \qquad |\tilde x_1(t) - y_{\mathrm{ref}}(t)| < \psi_P(t)$$

$$|\tfrac{\mathrm{d}}{\mathrm{d}t}\tilde x_1(t) - \dot y_{\mathrm{ref}}(t)| < \psi_D(t), \qquad |y(t) - y_{\mathrm{ref}}(t)| < \psi_P(t) + \psi_O(t)$$

Proof: use saturation of T

Open problems

- choice of design parameters ?
- $\gamma \to g(x_1, x_2) > 0$?
- condition on T: $\forall\, x_1, x_2 \in L^\infty : \; T(x_1, x_2) \in L^\infty$?
- higher relative degree ?



Happy Birthday, Achim!