Funnel Observation and Funnel Control

Thomas Berger and Timo Reis

Fachbereich Mathematik, Universität Hamburg

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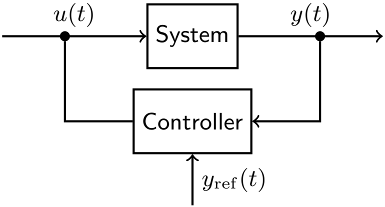
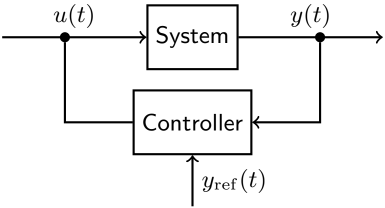
Adaptive Control



- no knowledge of system parameters, only “structure” of mathematical model for the system
- *aim*: design controller such that $y(t)$ “tracks” $y_{\mathrm{ref}}(t)$

Adaptive Control



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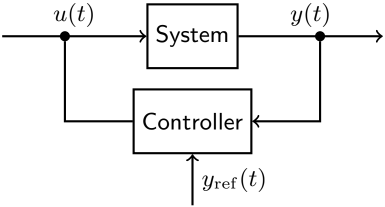

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- *aim*: design controller such that $y(t)$ “tracks” $y_{\mathrm{ref}}(t)$

Relative degree $r \in \mathbb{N}$

$$\dot{x}(t) = Ax(t) + bu(t), \quad y(t) = cx(t)$$

$$\dot{y}(t) = cAx(t) + cbu(t); \quad cb \neq 0 \;\Rightarrow\; r = 1$$

$$cb = 0 \;\Rightarrow\; \ddot{y}(t) = cA^2x(t) + cAbu(t); \quad cAb \neq 0 \;\Rightarrow\; r = 2$$

$$cAb = 0 \;\Rightarrow\; \ldots$$

$$\boxed{cb = cAb = \ldots = cA^{r-2}b = 0 \;\wedge\; cA^{r-1}b \neq 0}$$

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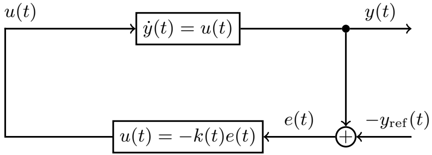
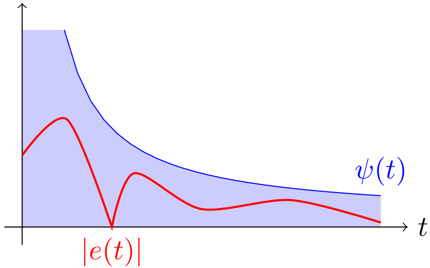
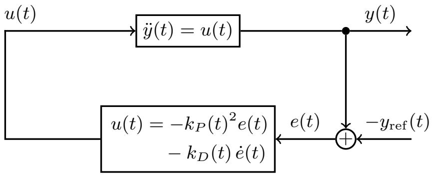
Funnel control

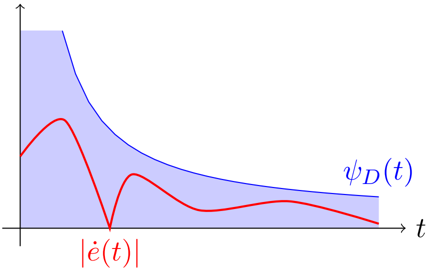




$$k(t) = \frac{\psi(t)}{\psi(t) - |e(t)|}$$

[Ilchmann, Ryan, Sangwin '02]:
This works!





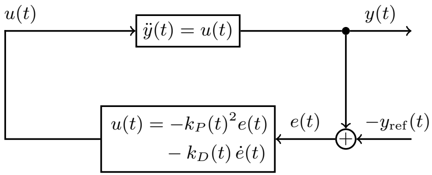
$$k_P(t) = \frac{\psi_P(t)}{\psi_P(t) - |e(t)|},$$

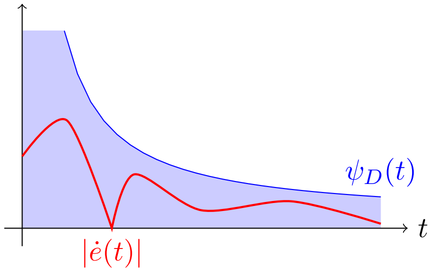
$$k_D(t) = \frac{\psi_D(t)}{\psi_D(t) - |\dot{e}(t)|},$$

$$0 < \delta \leq \frac{\mathrm{d}}{\mathrm{d}t}\psi_P(t) + \psi_D(t)$$

[Hackl, Hopfe, Ilchmann, Müller, Trenn '13]:
This works!

Problem: derivative of y not available!





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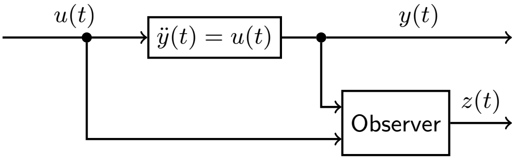
Problem: derivative of y not available!

High-gain observer for $\ddot{y} = u$

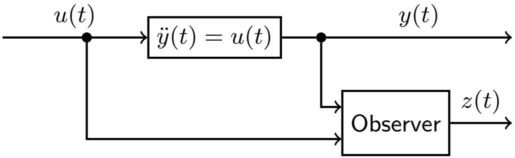


$$
\begin{aligned}
\dot{z}_1 &= z_2 + p_1 k (y - z_1),\\
\dot{z}_2 &= u + p_2 k^2 (y - z_1),\\
& p_1, p_2 > 0
\end{aligned}
$$

[Tornambe '92]: $k > 0$ large enough

$\implies \quad z_1 \approx y \;\wedge\; z_2 \approx \dot{y}$

High-gain observer for $\ddot{y} = u$



$$\dot{z}_1 = z_2 + p_1 k(t)(y - z_1),$$
$$\dot{z}_2 = u + p_2 k(t)^2 (y - z_1),$$

$$\dot{k}(t) = \gamma d^2_{\lambda,\hat{\lambda}}(y(t) - z_1(t)),$$

$$d_{\lambda,\hat{\lambda}}(z) = \begin{cases} \hat{\lambda} - \lambda, & |z| \geq \hat{\lambda} \\ |z| - \lambda, & \lambda \leq |z| \leq \hat{\lambda} \\ 0, & |z| \leq \lambda \end{cases}$$

[Bullinger, Ilchmann, Allgöwer '98]:

$z, k \in L^\infty$ and

$$\operatorname{dist}\big(y - z_1, [-\lambda, \lambda]\big) \to 0$$

Consider the modified observer for $\ddot{y} = u$

$$\dot{z}_1 = z_2 + \big(q_1 + p_1 k(t)\big)(y - z_1), \qquad k(t) = \frac{\psi_O(t)}{\psi_O(t) - |y(t) - z_1(t)|}$$

$$\dot{z}_2 = u + \big(q_2 + p_2 k(t)\big)(y - z_1),$$

$$e_1 := y - z_1,\ \ e_2 := \dot{y} - z_2, \qquad k(t) = \frac{\psi_O(t)}{\psi_O(t) - |e_1(t)|}$$

$$\dot{e}_1 = e_2 - \big(q_1 + p_1 k(t)\big)e_1,$$

$$\dot{e}_2 = -\big(q_2 + p_2 k(t)\big)e_1,$$

$$A := \begin{bmatrix} -q_1 & 1 \\ -q_2 & 0 \end{bmatrix},\ \ P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \text{ s.t. } \ A^\top P + PA = -I$$

$$p_1 = 1, \quad p_2 = -\frac{p_{12}}{p_{22}}; \quad \dot{e} = Ae - k(t)\left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right)e_1$$

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$$\dot{e} = Ae - k(t)\begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1$$

$$\begin{aligned} \tfrac{\mathsf{d}}{\mathsf{d}t} e^\top P e &= e^\top (A^\top P + PA) e - k(t)\left(e_1 \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}^\top P e + e^\top P \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1 \right) \\ &= -\|e\|^2 - k(t)\left(p_{11} - \frac{p_{12}^2}{p_{22}} \right) e_1^2 \le -\|e\|^2 \end{aligned}$$

$\implies \; e_1, e_2 \in L^\infty[0,\omega)$, standard funnel argument : $k \in L^\infty$

$\implies \; \omega = \infty \;\wedge\; e(t) \to 0 \;\wedge\; k(t) \to 1$

$$\dot{e} = Ae - k(t) \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix} \right) e_1$$

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$$\begin{aligned} &\implies e_1, {\color{red} e_2} \in L^\infty[0,\omega), \text{ standard funnel argument : } k \in L^\infty \\ &\implies \omega = \infty \ \wedge \ e(t) \to 0 \ \wedge \ k(t) \to 1 \end{aligned}$$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\text{loc}}$ causal, local Lipschitz, BIBO

Examples:

Extension

$$y^{(r)}(t) = f\big(d(t), T(y,\dot y,\ldots,y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y,\dot y,\ldots,y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\rm loc}$ causal, local Lipschitz, BIBO

Examples:

$\dot x(t) = Ax(t) + Bu(t),\ y(t) = Cx(t)$ with

(A1) $\mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} = n+m$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$;

(A2) $CB = CAB = \ldots = CA^{r-2}B = 0$ and $CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$.

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Is equivalent to $\frac{\mathsf{d}}{\mathsf{d}t}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t)$, $y(t) = \hat{C}\hat{x}(t)$ with

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_r & S \\ P & 0 & \cdots & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} I_m & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y,\dot y,\dots,y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y,\dot y,\dots,y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\mathrm{loc}}$ causal, local Lipschitz, BIBO

Examples:

$\frac{\mathrm{d}}{\mathrm{d}t}\hat x(t) = \hat A \hat x(t) + \hat B u(t)$, $y(t) = \hat C \hat x(t)$ with

$$\hat A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_r & S \\ P & 0 & \cdots & 0 & Q \end{bmatrix}, \quad \hat B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{bmatrix}, \quad \hat C = \begin{bmatrix} I_m & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$y^{(r)}(t) = R_1 y(t) + \dots R_r y^{(r-1)}(t) + S e^{Qt}\eta(0) + \int_0^t S e^{Q(t-\tau)} P y(\tau) d\tau + CA^{r-1}Bu(t).$$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y,\dot y,\ldots,y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y,\dot y,\ldots,y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\text{loc}}$ causal, local Lipschitz, BIBO

Examples:

$$y^{(r)}(t) = T(y,\dot y,\ldots,y^{(r-1)})(t) + \Gamma u(t)$$

with $\Gamma = CA^{r-1}B$ and

$$T(y,\ldots,y^{(r-1)})(t)$$
$$= R_1 y(t) + \ldots R_r y^{(r-1)}(t) + Se^{Qt}\eta(0) + \int_0^t Se^{Q(t-\tau)}Py(\tau)d\tau.$$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\text{loc}}$ causal, local Lipschitz, BIBO

Examples:

$$\begin{aligned}
\dot{x}_1(t) &= x_2(t), \; \ldots, \; \dot{x}_{r-1}(t) = x_r(t), \qquad \hat{x}(t) = (x_1(t)^\top, \ldots, x_r(t)^\top)^\top \\
\dot{x}_r(t) &= g_1(\hat{x}(t), \eta(t)) + g_2(\hat{x}(t), \eta(t))u(t), \\
\dot{\eta}(t) &= g_3(\hat{x}(t), \eta(t)), \\
y(t) &= x_1(t)
\end{aligned}$$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y,\dot y,\dots,y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y,\dot y,\dots,y^{(r-1)})(t)\big)u(t)$$

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Examples:

$$\begin{aligned}
\dot x_1(t) &= x_2(t),\ \dots,\ \dot x_{r-1}(t) = x_r(t), \qquad \hat x(t) = (x_1(t)^\top,\dots,x_r(t)^\top)^\top\\
\dot x_r(t) &= g_1(\hat x(t),\eta(t)) + g_2(\hat x(t),\eta(t))u(t),\\
\dot\eta(t) &= g_3(\hat x(t),\eta(t)),\\
y(t) &= x_1(t)
\end{aligned}$$

$$T(y,\dots,y^{(r-1)})(t) := \big(y(t)^\top,\dots,y^{(r-1)}(t)^\top, \eta(t;\eta^0,y,\dots,y^{(r-1)})^\top\big)^\top$$

$$y^{(r)}(t) = g_1\big(T(y,\dots,y^{(r-1)})(t)\big) + g_2\big(T(y,\dots,y^{(r-1)})(t)\big)\,u(t)$$

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big)u(t)$$

$$\begin{aligned}
\dot{z}_1(t) &= z_2(t) + \big(q_1 + p_1 k(t)\big)\big(y(t) - z_1(t)\big),\\
\dot{z}_2(t) &= z_3(t) + \big(q_2 + p_2 k(t)\big)\big(y(t) - z_1(t)\big),\\
&\vdots\\
\dot{z}_{r-1}(t) &= z_r(t) + \big(q_{r-1} + p_{r-1} k(t)\big)\big(y(t) - z_1(t)\big),\\
\dot{z}_r(t) &= \tilde{\Gamma} u(t) + \big(q_r + p_r k(t)\big)\big(y(t) - z_1(t)\big),\\
k(t) &= \frac{1}{1 - \varphi(t)^2 \|y(t) - z_1(t)\|^2}
\end{aligned}$$

with design parameters $p_i > 0$, $q_i > 0$, $\tilde{\Gamma} \in \mathbb{R}^{m \times m}$ and the funnel function $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\begin{aligned}\dot z_1 &= z_2 + \big(q_1 + p_1 k(t)\big)(y - z_1)\\ \dot z_2 &= z_3 + \big(q_2 + p_2 k(t)\big)(y - z_1)\\ &\vdots\\ \dot z_{r-1} &= z_r + \big(q_{r-1} + p_{r-1} k(t)\big)(y - z_1)\\ \dot z_r &= \tilde\Gamma u + \big(q_r + p_r k(t)\big)(y - z_1)\end{aligned}$$

$$A := \begin{bmatrix} -q_1 & 1 & & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1\\ -q_r & & & 0\end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12}\\ P_{12}^\top & P_{22}\end{bmatrix} \quad P_{11} \in \mathbb{R},\ \ P_{22} \in \mathbb{R}^{(r-1)\times(r-1)}$$

$$\text{s.t.}\quad A^\top P + PA = -I \quad \text{and} \quad \begin{pmatrix} p_1\\ \vdots\\ p_r\end{pmatrix} = \begin{pmatrix} 1\\ -P_{22}^{-1} P_{12}^\top\end{pmatrix}$$

Theorem [B., Reis '16]:

$$y, \ldots, y^{(r-1)} \in L^\infty \implies z_1, \ldots, z_r, k \in L^\infty \ \wedge\ |y(t) - z_1(t)| < \psi_O(t)$$

$$\limsup_{t\to\infty} \|e(t)\| \le M, \quad e(t) = \big(y(t) - z_1(t), \ldots, y^{(r)}(t) - z_r(t)\big)$$

$$\begin{aligned}
\dot z_1 &= z_2 + \big(q_1 + p_1 k(t)\big)(y - z_1)\\
\dot z_2 &= z_3 + \big(q_2 + p_2 k(t)\big)(y - z_1)\\
&\ \vdots\\
\dot z_{r-1} &= z_r + \big(q_{r-1} + p_{r-1} k(t)\big)(y - z_1)\\
\dot z_r &= \tilde\Gamma u + \big(q_r + p_r k(t)\big)(y - z_1)
\end{aligned}$$

$$A := \begin{bmatrix} -q_1 & 1 & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1 \\ -q_r & & & 0 \end{bmatrix}, \quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22} \end{bmatrix} \quad P_{11} \in \mathbb{R},\ \ P_{22} \in \mathbb{R}^{(r-1)\times(r-1)}$$

$$\text{s.t.} \quad A^\top P + PA = -I \quad \text{and} \quad \begin{pmatrix} p_1 \\ \vdots \\ p_r \end{pmatrix} = \begin{pmatrix} 1 \\ -P_{22}^{-1} P_{12}^\top \end{pmatrix}$$

Theorem [B., Reis '16]:

$$y, \ldots, y^{(r-1)} \in L^\infty \implies z_1, \ldots, z_r, k \in L^\infty \ \wedge\ |y(t) - z_1(t)| < \psi_O(t)$$

$$\limsup_{t\to\infty} \|e(t)\| \le M, \quad e(t) = (y(t) - z_1(t), \ldots, y^{(r)}(t) - z_r(t))$$

Simulation - bioreactor

$$\dot{m}(t) = \frac{a_1 m(t) s(t)}{a_2 m(t) + s(t)} - m(t)u(t),$$
$$\dot{s}(t) = -\frac{a_1 a_3 m(t) s(t)}{a_2 m(t) + s(t)} + \big(a_4 - s(t)\big)u(t),$$
$$y(t) = m(t),$$

m - concentration of microorganisms, $m(0) = 0.075$

s - concentration of substrate, $s(0) = 0.03$

u - substrate inflow rate

$$u(t) = \begin{cases} 0.08, & t \in [0,30) \\ 0.02, & t \in [30,50) \\ 0.08, & t \geq 50, \end{cases}$$

$$\tilde{\Gamma} = 0,\ q_1 = 1,\ q_2 = 0.2,$$
$$p_1 = 1,\ p_2 = \tfrac{1}{11}$$
$$\varphi(t) = \tfrac{1}{2} t e^{-t} + \tfrac{100}{\pi} \arctan t.$$

Simulation - bioreactor

$$\begin{aligned} \dot m(t) &= \frac{a_1 m(t) s(t)}{a_2 m(t) + s(t)} - m(t)u(t), \\ \dot s(t) &= -\frac{a_1 a_3 m(t) s(t)}{a_2 m(t) + s(t)} + \big(a_4 - s(t)\big)u(t), \\ y(t) &= m(t), \end{aligned}$$

m - concentration of microorganisms, $m(0) = 0.075$

s - concentration of substrate, $s(0) = 0.03$

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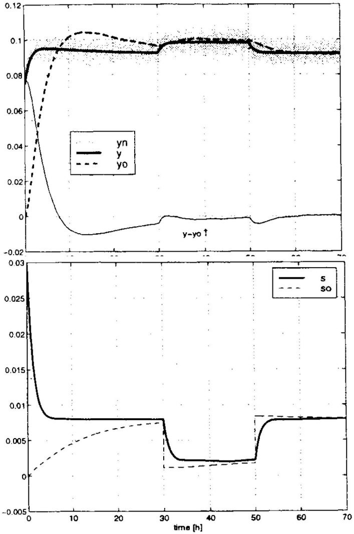
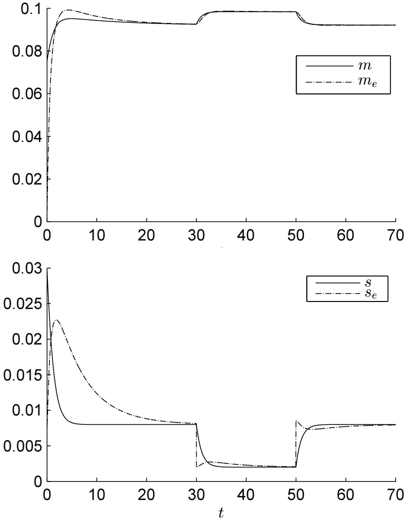
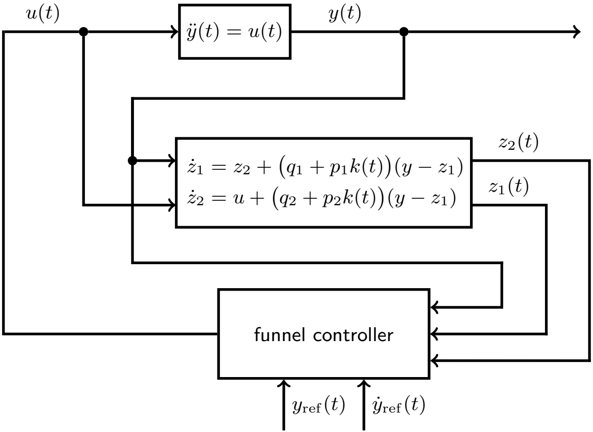
$$\tilde\Gamma = 0,\ q_1 = 1,\ q_2 = 0.2,$$
$$p_1 = 1,\ p_2 = \tfrac{1}{11}$$
$$\varphi(t) = \tfrac{1}{2} t e^{-t} + \tfrac{100}{\pi} \arctan t.$$

λ-strip observer [BIA '98]:



funnel observer:





- given $\psi(t)$, we have $|y(t) - y_{\text{ref}}(t)| < \psi(t)$ for all $t > 0$
- controller only needs y, z_1, z_2

straightforward approach:

$$u(t) = -k_P(t)^2(y(t) - y_{\mathrm{ref}}(t)) - k_D(t)(z_2(t) - \dot{y}_{\mathrm{ref}}(t)),$$
$$z_2(t) = \dot{y}(t) - e_2(t)$$

Problem: performance depends on $\|e_2\|_\infty$; we need $\inf_{t>0} \psi_D(t) > \|e_2\|_\infty$

alternative approach: What if we put $z_1 - y_{\mathrm{ref}}$ into the funnel?

$$|y(t) - y_{\mathrm{ref}}(t)| \leq |z_1(t) - y_{\mathrm{ref}}(t)| + |e_1(t)| < \psi_P(t) + \psi_O(t)$$

Given $\psi(t)$, let $\psi_P(t) = \psi_O(t) = \frac{1}{2}\psi(t)$

Seek a system of relative degree 2 with output $\hat{y} := z_1$!

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Given $\psi(t)$, let $\psi_P(t) = \psi_O(t) = \frac{1}{2}\psi(t)$

Seek a system of relative degree 2 with output $\hat{y} := z_1$!

$$\Sigma = \begin{cases} \dot z_1 &= z_2 + \big(q_1 + p_1 k(t)\big)e_1, \qquad \dot e_1 = e_2 - \big(q_1 + p_1 k(t)\big)e_1, \\ \dot z_2 &= u + \big(q_2 + p_2 k(t)\big)e_1, \qquad \dot e_2 = -\big(q_2 + p_2 k(t)\big)e_1, \\ \hat y &= z_1 \\ & \end{cases}$$

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$$\tfrac{\mathrm{d}}{\mathrm{d}t}\hat y = z_2 + \big(q_1 + p_1 k(t)\big) e_1,$$

$$\left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^2 \hat y = u + \big(q_2 + p_2 k(t)\big) e_1 + p_1 \dot k(t) e_1 + \big(q_1 + p_1 k(t)\big) \dot e_1$$

$$\Sigma = \begin{cases} \dot z_1 &= z_2 + \big(q_1 + p_1 k(t)\big) e_1, \\ \dot z_2 &= u + \big(q_2 + p_2 k(t)\big) e_1, \\ \hat y &= z_1 \end{cases} \qquad \underbrace{\begin{aligned} \dot e_1 &= e_2 - \big(q_1 + p_1 k(t)\big) e_1, \\ \dot e_2 &= -\big(q_2 + p_2 k(t)\big) e_1, \end{aligned}}_{\text{internal dynamics}}$$

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t}\hat y &= z_2 + \big(q_1 + p_1 k(t)\big) e_1, \\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big)^2 \hat y &= u + \underbrace{\big(q_2 + p_2 k(t)\big) e_1 + p_1 \dot k(t) e_1 + \big(q_1 + p_1 k(t)\big) \dot e_1}_{T(\psi_O, \dot\psi_O, e_1, e_2) \in L^\infty} \end{aligned}$$

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[Hackl et al. '13]: PD-funnel controller can be applied to Σ!

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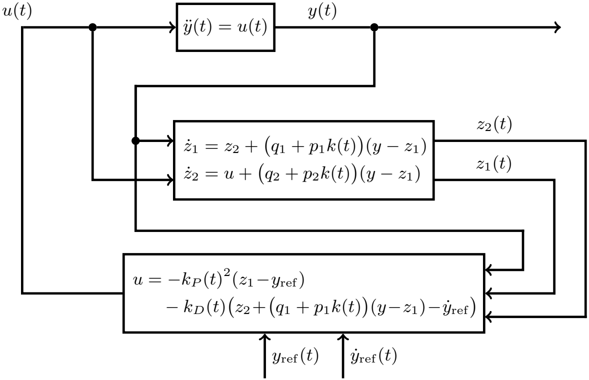
$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \hat y &= z_2 + \big(q_1 + p_1 k(t)\big) e_1, \\ \big(\tfrac{\mathrm{d}}{\mathrm{d}t}\big)^2 \hat y &= u + \underbrace{\big(q_2 + p_2 k(t)\big) e_1 + p_1 \dot k(t) e_1 + \big(q_1 + p_1 k(t)\big) \dot e_1}_{T(\psi_O, \dot\psi_O, e_1, e_2) \in L^\infty} \end{aligned}$$

[Hackl et al. '13]: PD-funnel controller can be applied to Σ!

$$\begin{aligned} u(t) &= -k_P(t)^2 (z_1(t) - y_{\mathrm{ref}}(t)) - k_D(t) (\dot z_1(t) - \dot y_{\mathrm{ref}}(t)) \\ &= -k_P(t)^2 (z_1(t) - y_{\mathrm{ref}}(t)) \\ &\quad - k_D(t) \Big(z_2(t) + \big(q_1 + p_1 k(t)\big) (y(t) - z_1(t)) - \dot y_{\mathrm{ref}}(t) \Big) \end{aligned}$$



$$|y(t) - z_1(t)| < \psi_O(t), \qquad |z_1(t) - y_{\text{ref}}(t)| < \psi_P(t)$$

$$|\dot z_1(t) - \dot y_{\text{ref}}(t)| < \psi_D(t), \qquad |y(t) - y_{\text{ref}}(t)| < \psi_P(t) + \psi_O(t)$$

Extension

$$\left.\begin{aligned} \dot x_1 &= x_2 \\ \dot x_2 &= T(x_1,x_2)+\gamma u \\ y &= x_1 \end{aligned}\right\} + \left\{\begin{aligned} \dot z_1 &= z_2+\big(q_1+p_1k(t)\big)(y-z_1) \\ \dot z_2 &= \tilde\gamma u+\big(q_2+p_2k(t)\big)(y-z_1) \\ u &= -k_P(t)^2(z_1-y_{\rm ref}) \\ &\quad -k_D(t)\Big(z_2+\big(q_1+p_1k(t)\big)(y-z_1)-\dot y_{\rm ref}\Big) \end{aligned}\right.$$

$$\gamma > 0,\ \tilde\gamma > 0$$
$$\forall\, x_1,x_2 \in \mathcal{C}:\ x_1 \in L^\infty \ \Rightarrow\ T(x_1,x_2) \in L^\infty$$

Theorem [B., Reis '16]:

$$|y(t)-z_1(t)| < \psi_O(t), \qquad |z_1(t)-y_{\rm ref}(t)| < \psi_P(t)$$
$$|\dot z_1(t)-\dot y_{\rm ref}(t)| < \psi_D(t), \qquad |y(t)-y_{\rm ref}(t)| < \psi_P(t)+\psi_O(t)$$

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$$\left.\begin{aligned} \dot x_1 &= x_2 \\ \dot x_2 &= T(x_1,x_2)+\gamma u \\ y &= x_1 \end{aligned}\right\} + \left\{\begin{aligned} \dot z_1 &= z_2+\big(q_1+p_1k(t)\big)(y-z_1) \\ \dot z_2 &= \tilde\gamma u+\big(q_2+p_2k(t)\big)(y-z_1) \\ u &= -k_P(t)^2(z_1-y_{\rm ref}) \\ &\quad -k_D(t)\Big(z_2+\big(q_1+p_1k(t)\big)(y-z_1)-\dot y_{\rm ref}\Big) \end{aligned}\right.$$

$$\gamma > 0,\ \tilde\gamma > 0$$

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$$|y(t)-z_1(t)| < \psi_O(t), \qquad |z_1(t)-y_{\rm ref}(t)| < \psi_P(t)$$

$$|\dot z_1(t)-\dot y_{\rm ref}(t)| < \psi_D(t), \qquad |y(t)-y_{\rm ref}(t)| < \psi_P(t)+\psi_O(t)$$

Simulation

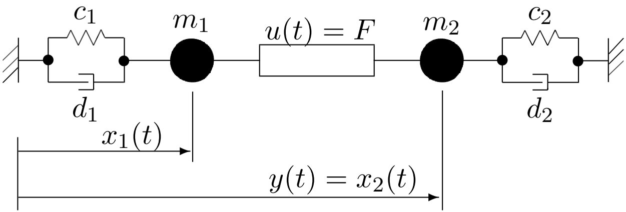
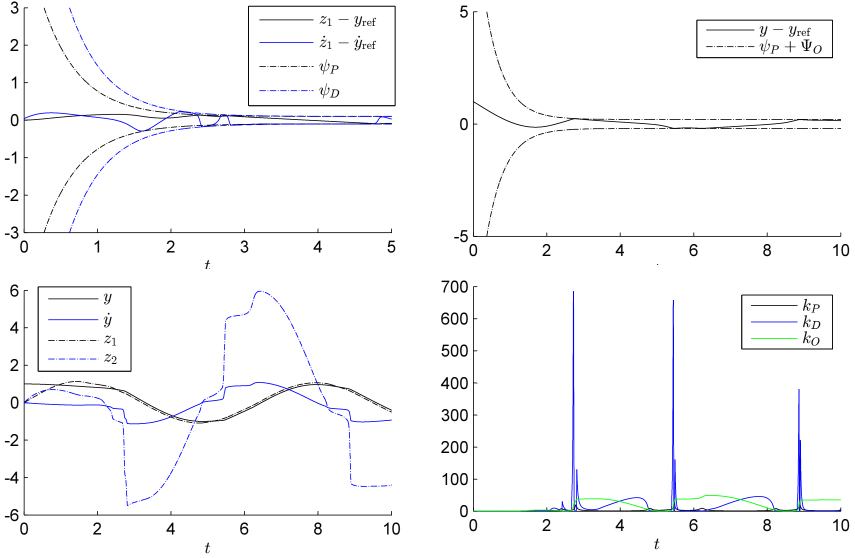
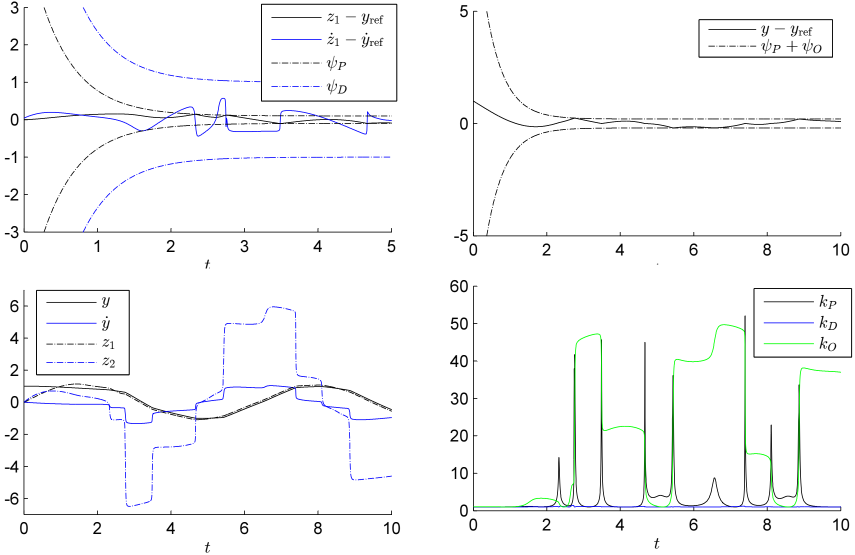


Figure: Mass-spring-damper system

$$y_{\text{ref}}(t) = \sin t$$


$z_1 - y_{\text{ref}}$
$\dot{z}_1 - \dot{y}_{\text{ref}}$
ψ_P
ψ_D
t
$y - y_{\text{ref}}$
$\psi_P + \Psi_O$
y
$\dot{y}$
z_1
z_2
t
k_P
k_D
k_O
t


$z_1 - y_{\rm ref}$
$\dot z_1 - \dot y_{\rm ref}$
ψ_P
ψ_D
t
$y - y_{\rm ref}$
$\psi_P + \psi_O$
y
$\dot y$
z_1
z_2
t
k_P
k_D
k_O
t

Open problems

- choice of design parameters ?
- $\gamma \to g(x_1, x_2) > 0$?
- condition on T: $\forall\, x_1, x_2 \in L^\infty : \; T(x_1, x_2) \in L^\infty$?
- higher relative degree ?