UNIVERSITÄT PADERBORN

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FUNNEL CONTROL FOR LINEAR NON-MINIMUM PHASE SYSTEMS

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Linear systems with relative degree $r \in \mathbb{N}$

$$\dot{x}(t) = Ax(t) + Bu(t), \quad y(t) = Cx(t), \quad x(0) = x^0 \quad (\Sigma)$$

- $A \in \mathbb{R}^{n\times n}, B, C^\top \in \mathbb{R}^{n\times m}, x^0 \in \mathbb{R}^n$
- $CB = CAB = \ldots = CA^{r-2}B = 0, \quad CA^{r-1}B \in \mathbf{Gl}_n(\mathbb{R})$
- minimum phase $\iff$

$$\forall\, \lambda \in \mathbb{C}_- : \quad \mathrm{rk} \begin{bmatrix} A - \lambda I_n & B \\ C & 0 \end{bmatrix} = n + m$$

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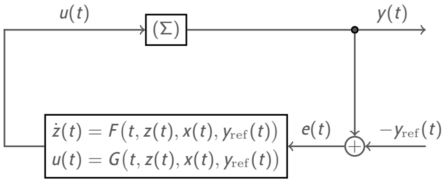
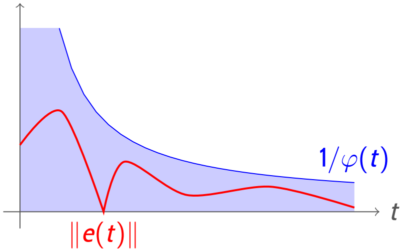
$$\forall\, \lambda \in \mathbb{C}_- : \quad \mathrm{rk} \begin{bmatrix} A - \lambda I_n & B \\ C & 0 \end{bmatrix} = n + m$$ **NOT required!**

Control objective





$$\Phi_r = \left\{ \varphi \in \mathcal{C}^r(\mathbb{R}_{\geq 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \ldots, \varphi^{(r)} \text{ bounded}, \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau \to \infty} \varphi(\tau) > 0 \end{array} \right\}$$

Basic idea

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   - the new system has a higher relative degree $r + \ell$
   - the unstable part of the internal dynamics is eliminated

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2. Define a new reference signal s.t. it belongs to $\mathcal{W}^{r+\ell,\infty}$
3. Apply the available funnel controller to the system with new output and new reference signal
4. Show that the original tracking error satisfies the control objective

Definition of new output

Byrnes-Isidori form: $\exists\, U \in \mathbf{Gl}_n(\mathbb{R})$ s.t.
$Ux(t) = \left(y(t)^\top, \dot{y}(t)^\top, \ldots, y^{(r-1)}(t)^\top, \eta(t)^\top\right)^\top$ transforms (Σ) into

$$y^{(r)}(t) = \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S\eta(t) + \underbrace{\Gamma}_{=CA^{r-1}B} u(t),$$

$$\dot{\eta}(t) = Py(t) + Q\eta(t)$$

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Assumption: $TQT^{-1} = \begin{bmatrix} \hat{Q}_1 & \hat{Q}_2 \\ 0 & \tilde{Q} \end{bmatrix}, \quad TP = \begin{bmatrix} \hat{P} \\ \tilde{P} \end{bmatrix}$

$\tilde{Q} \in \mathbb{R}^{\ell m \times \ell m}$ and $\hat{Q}_1 \in \mathbb{R}^{k \times k}$ with $\sigma(\hat{Q}_1) \subseteq \mathbb{C}_-$, $k = n - rm - \ell m \geq 0$

Definition of new output

$$
\begin{aligned}
y^{(r)}(t) &= \sum_{i=1}^{r} R_i y^{(i-1)}(t) + S_1\eta_1(t) + S_2\eta_2(t) + \Gamma u(t),\\
\dot\eta_1(t) &= \hat Q_1\eta_1(t) + \hat Q_2\eta_2(t) + \hat P y(t),\\
\dot\eta_2(t) &= \qquad\qquad\quad\;\; \tilde Q\eta_2(t) + \tilde P y(t)
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$$K := [0, \ldots, 0, \Gamma^{-1}] \underbrace{[\tilde P, \tilde Q \tilde P, \ldots, \tilde Q^{\ell-1} \tilde P]^{-1}}_{\textit{Assumption}}$$

$$y_{\mathrm{new}}(t) := K \eta_2(t)$$

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$$
\begin{aligned}
\eta_2(t) &= \sum_{i=1}^{\ell} F_i\, y_{\text{new}}^{(i-1)}(t),\\
y(t) &= \Gamma y_{\text{new}}^{(\ell)}(t) + \sum_{i=1}^{\ell} \Gamma K \tilde Q^{\ell} F_i\, y_{\text{new}}^{(i-1)}(t)
\end{aligned}
$$

Definition of new output

$\rightarrow$ y_{new} is flat output for $\dot{\eta}_2(t) = \tilde{Q}\eta_2(t) + \tilde{P}y(t)$

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$$y_{\text{new}}^{(r+\ell)}(t) = \sum_{i=1}^{r+\ell} \hat R_i \, y_{\text{new}}^{(i-1)}(t) + S_1 \eta_1(t) + u(t),$$
$$\dot\eta_1(t) = \sum_{i=1}^{\ell+1} \hat P_i \, y_{\text{new}}^{(i-1)}(t) + \hat Q_1 \eta_1(t)$$

Definition of new reference signal

$$\dot{\eta}_{2,\mathrm{ref}}(t) = \tilde{Q}\eta_{2,\mathrm{ref}}(t) + \tilde{P}y_{\mathrm{ref}}(t), \quad \eta_{2,\mathrm{ref}}(0) = \eta^0_{2,\mathrm{ref}}$$
$$\hat{y}_{\mathrm{ref}}(t) = K\eta_{2,\mathrm{ref}}(t)$$

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$$\hat y_{\mathrm{ref}}(t) = K \eta_{2,\mathrm{ref}}(t)$$

Lemma: We have $\hat y_{\mathrm{ref}} \in \mathcal{W}^{r+\ell,\infty}$, if $y_{\mathrm{ref}} \in \mathcal{W}^{r-1,\infty}$ and

- $W\tilde Q W^{-1} = \begin{bmatrix} Q_1 & 0 & 0 \\ 0 & Q_2 & 0 \\ 0 & 0 & Q_3 \end{bmatrix}$ with $\sigma(Q_1) \subseteq \mathbb{C}_-$, $\sigma(Q_2) \subseteq \mathbb{C}_+$, $\sigma(Q_3) \subseteq \mathrm{i}\mathbb{R}$, $\quad W\tilde P = [P_1^\top, P_2^\top, P_3^\top]^\top$
- $\eta_{2,\mathrm{ref}}^0 = W^{-1} \begin{bmatrix} 0_{k_1 \times k_2} \\ -I_{k_2} \\ 0_{k_3 \times k_2} \end{bmatrix} \int_0^\infty e^{-Q_2 s} P_2 y_{\mathrm{ref}}(s) \, \mathrm{d}s$
- $\dot z(t) = Q_3 z(t) + P_3 y_{\mathrm{ref}}(t),\ z(0) = 0$ has a bounded solution $z(\cdot)$

Definition of new reference signal

exosystem:
$$\dot{w}(t) = A_e w(t), \quad y_{\rm ref}(t) = C_e w(t), \quad w(0) = w^0,$$
$$\dot{\eta}_{2,\rm ref}(t) = \tilde{Q}\eta_{2,\rm ref}(t) + \tilde{P} y_{\rm ref}(t), \quad \eta_{2,\rm ref}(0) = \eta^0_{2,\rm ref}$$
$$\hat{y}_{\rm ref}(t) = K\eta_{2,\rm ref}(t)$$

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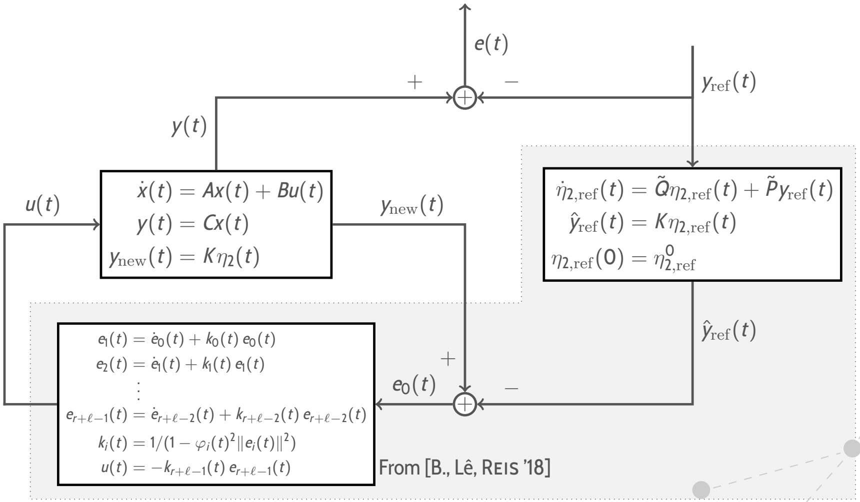
Lemma: $\exists$ unique $X : \; Q_2 X - X A_e = P_2 C_e$ and

$$\eta^0_{2,\mathrm{ref}} = W^{-1} \begin{bmatrix} 0_{k_1 \times k_2} \\ -I_{k_2} \\ 0_{k_3 \times k_2} \end{bmatrix} X w^0$$

Controller structure



From [B., Lê, Reis '18]

Theorem

$y_{\rm ref} \in \mathcal{W}^{r-1,\infty}$, $\varphi_i \in \Phi_{r+\ell-i}$, $\varphi_i(0)\|e_i(0)\| < 1$ for $i = 0, \ldots, r+\ell-1$

$\Longrightarrow \quad x, \eta_{2,{\rm ref}}, u, k_0, \ldots, k_{r+\ell-1} \in L^\infty$ and

$$\|e_i(t)\| \le \varphi_i(t)^{-1} - \varepsilon_i$$

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Original tracking error satisfies: $\|e(t)\| \le \Psi(t)$

- Ψ is *known a priori* and depends on φ_i and $e_i(0)$, $i = 0, \ldots, r+\ell-1$
- $\varphi \in \Phi_r$ given $\to$ choose $\varphi_0, \ldots, \varphi_{r+\ell-1}$ s.t. $\Psi(t) < \varphi(t)^{-1}$

Simulation

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 0 & -2 & 0 \\ 0 & 0 & 3 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix}, \quad C = [1, 0, -3, 0], \quad x^0 = 0, \quad r = 2$$

$$y_{\mathrm{ref}}(t) = \begin{cases} (1 - \cos t), & t \in [0, 2\pi], \\ 0, & t > 2\pi. \end{cases}$$

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$$\begin{aligned} \ddot{y}(t) &= -18y(t) - 7\dot{y}(t) + \eta_1(t) - 8\eta_2(t) + 2u(t) \\ \dot{\eta}_1(t) &= -\eta_1(t) + \eta_2(t) \\ \dot{\eta}_2(t) &= \eta_2(t) + 3y(t) \end{aligned}$$

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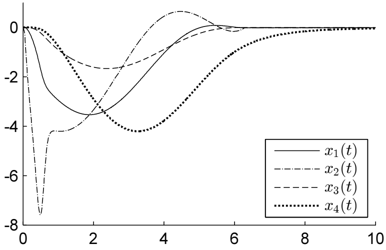
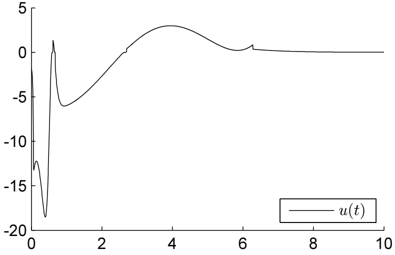
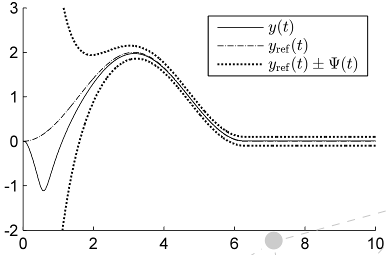
$$\ell = 1, \tilde{Q} = 1, \tilde{P} = 3, K = \tfrac{1}{6} \quad \to \quad y_{\mathrm{new}}(t) = \tfrac{1}{6}\eta_2(t) = \tfrac{1}{2}x_3(t)$$

Simulation

$$\eta_{2,\mathrm{ref}}^0 = -\frac{1069}{714},$$
$$\varphi_0(t) = \left(e^{-2t} + 0.01\right)^{-1},$$
$$\varphi_1(t) = \left(2e^{-2t} + 0.01\right)^{-1},$$
$$\varphi_2(t) = \left(2e^{-10t} + 0.01\right)^{-1}$$







Outlook

1. the controller is robust w.r.t. disturbances which do not affect the unstable part of the internal dynamics $\rightarrow$ extension?
2. a feature of funnel control is lost: not model-free
3. additional measurements required: $y_{\mathrm{new}}, \dot{y}_{\mathrm{new}}, \ldots, y_{\mathrm{new}}^{r+\ell-1}$
4. construction method for φ_i so that $\Psi(t) < \varphi(t)^{-1}$ not available yet