Funnel Observation and Funnel Control

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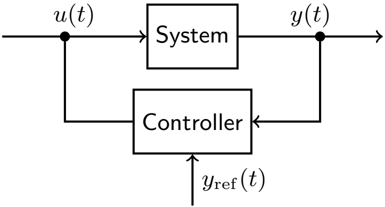
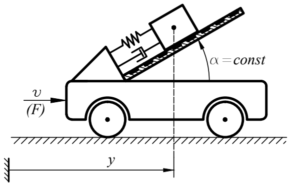
Control systems



$$
\begin{aligned}
\dot{x}(t) &= f\big(x(t), u(t)\big), \quad x(0) = x^0 \\
y(t) &= h\big(x(t)\big)
\end{aligned}
$$

- **Aim:** design controller such that $y(t)$ "tracks" $y_{\mathrm{ref}}(t)$
- no knowledge of system parameters, only "structural assumptions" on the model of the system



[Seifried, Blajer '13]

Angle: $0^\circ < \alpha \leq 90^\circ$

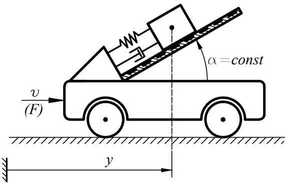
spring, damper with nonlinear characteristics: $K(z)$, $D(\dot{z})$

$$u(t) = F$$
$$y(t) = q(t) + z(t)\cos\alpha$$

$$\begin{bmatrix} m_1 + m_2 & m_2 \cos\alpha \\ m_2 \cos\alpha & m_2 \end{bmatrix} \begin{pmatrix} \ddot{q} \\ \ddot{z} \end{pmatrix} = \begin{pmatrix} u \\ -K(z) - D(\dot{z}) + m_2 g \sin\alpha \end{pmatrix}$$

$$\ddot{y} = -c_1 K\left(\frac{\eta - y\cos\alpha}{\gamma}\right) - c_1 D\left(\frac{\dot{\eta} - \dot{y}\cos\alpha}{\gamma}\right) + c_2 g \sin\alpha + c_3 u$$
$$\ddot{\eta} = -c_4 K\left(\frac{\eta - y\cos\alpha}{\gamma}\right) - c_4 D\left(\frac{\dot{\eta} - \dot{y}\cos\alpha}{\gamma}\right) + c_5 g \sin\alpha$$

relative degree $= 2$



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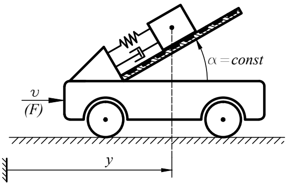

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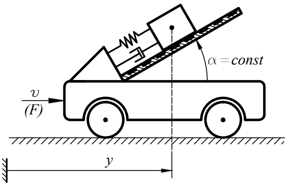

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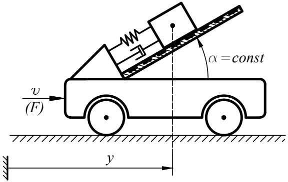
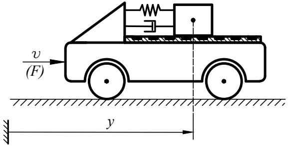


Angle: $0^\circ < \alpha \leq 90^\circ$

Input: velocity

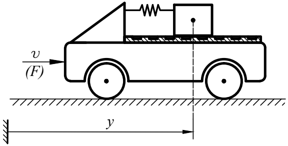
relative degree $= 1$



Angle: $\alpha = 0^\circ$

Input: force

relative degree $= 3$

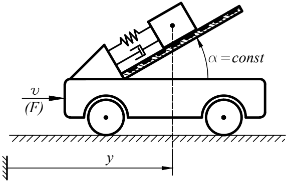


Angle: $\alpha = 0^\circ$

Input: force

no damping: $D(\dot{z}) = 0$

relative degree $= 4$



Internal dynamics: remaining dynamics when output is fixed

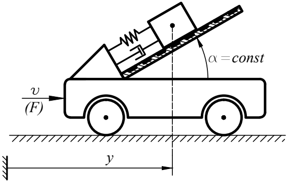
$$\ddot{\eta} = -c_4 K\left(\frac{\eta - y\cos\alpha}{\gamma}\right) - c_4 D\left(\frac{\dot{\eta} - \dot{y}\cos\alpha}{\gamma}\right) + c_5 g \sin\alpha$$

Lyapunov function: kinetic + potential energy

$$V = \frac{1}{2}(\dot{\eta} - \dot{y}\cos\alpha)^2 + \gamma V_K\left(\frac{\eta - y\cos\alpha}{\gamma}\right), \qquad \frac{\mathrm{d}}{\mathrm{d}z}V_K(z) = K(z)$$

$$\dot{V} \le \underbrace{-c_4 D\left(\frac{\dot{\eta} - \dot{y}\cos\alpha}{\gamma}\right)(\dot{\eta} - \dot{y}\cos\alpha)}_{\ge \beta\|\dot{\eta} - \dot{y}\cos\alpha\|^2} + (c_6 + \|\ddot{y}\|)\|\dot{\eta} - \dot{y}\cos\alpha\|$$

$y, \dot{y}, \ddot{y} \in L^\infty \implies \eta, \dot{\eta}, \ddot{\eta} \in L^\infty$ **(stable internal dynamics)**



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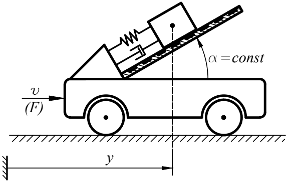

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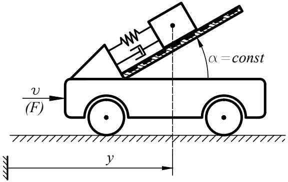

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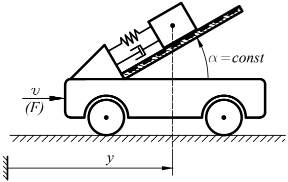
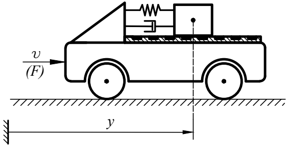


Angle: $0^\circ < \alpha \leq 90^\circ$

Input: velocity

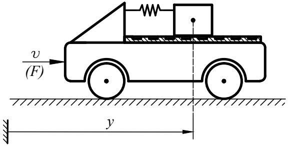
stable internal dynamics



Angle: $\alpha = 0^\circ$

Input: force

stable internal dynamics

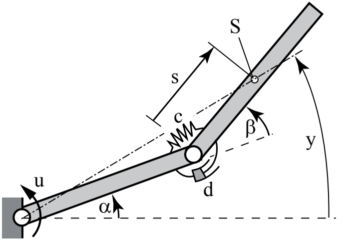


Angle: $\alpha = 0^\circ$

Input: force

no damping: $D(\dot{z}) = 0$

no internal dynamics



[Seifried, Blajer '13]

Rotational Manipulator Arm

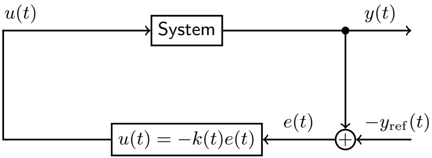
Input: angular velocity of first link

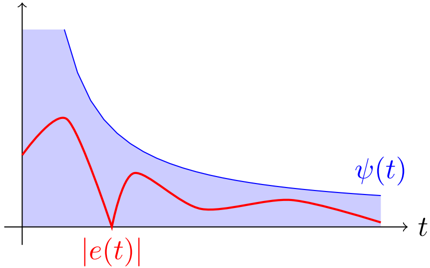
Output: position of S described by angle y

relative degree $= 1$

unstable internal dynamics

Funnel Control





$$k(t) = \frac{\psi(t)}{\psi(t) - |e(t)|}$$

[Ilchmann, Ryan, Sangwin '02]:
Works, if

- relative degree $= 1$
- stable internal dynamics

Problem: higher relative degree

Relative degree 1:

$$\dot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \textbf{as. stable for } k \gg 0$$

Relative degree 2:

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -ky(t) \implies \textbf{not as. stable}$$

$$\ddot{y}(t) + cy(t) = u(t) \stackrel{!}{=} -k_1 y(t) - k_2 \dot{y}(t) \implies \textbf{as. stable for } k_1, k_2 \gg 0$$

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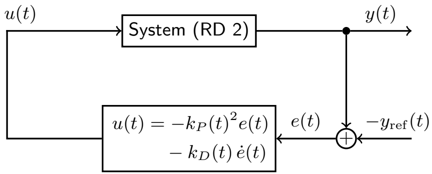
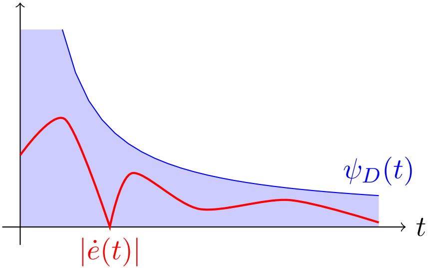




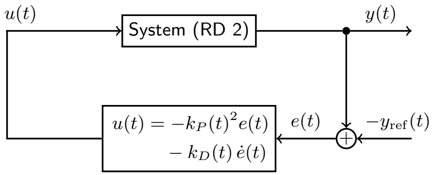
$$k_P(t) = \frac{\psi_P(t)}{\psi_P(t) - |e(t)|},$$

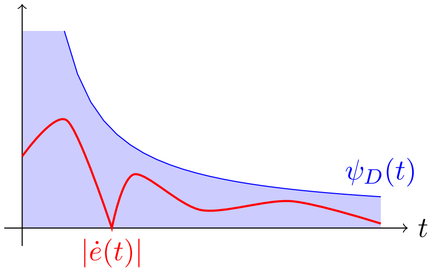
$$k_D(t) = \frac{\psi_D(t)}{\psi_D(t) - |\dot e(t)|},$$

$$0 < \delta \le \tfrac{\mathrm{d}}{\mathrm{d}t}\psi_P(t) + \psi_D(t)$$

[Hackl, Hopfe, Ilchmann, Müller, Trenn '13]: Works...

But: derivative of y not available!





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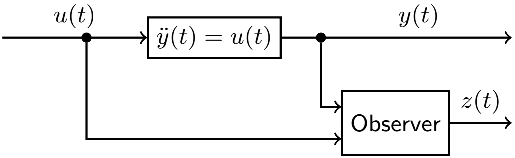
High-gain observer for $\ddot{y} = u$

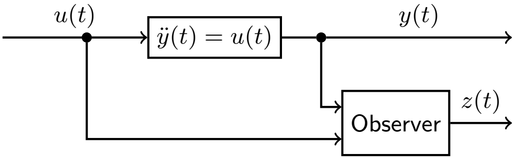


$$\begin{aligned} \dot{z}_1 &= z_2 + p_1 k (y - z_1), \\ \dot{z}_2 &= u + p_2 k^2 (y - z_1), \\ & p_1, p_2 > 0 \end{aligned}$$

[Tornambe '92]: $k > 0$ large enough

$\implies \quad z_1 \approx y \;\wedge\; z_2 \approx \dot{y}$

High-gain observer for $\ddot{y} = u$



$$\dot{z}_1 = z_2 + p_1 k(t)(y - z_1),$$
$$\dot{z}_2 = u + p_2 k(t)^2 (y - z_1),$$

$$\dot{k}(t) = \gamma d^2_{\lambda,\hat{\lambda}}(y(t) - z_1(t)),$$

$$d_{\lambda,\hat{\lambda}}(z) = \begin{cases} \hat{\lambda} - \lambda, & |z| \geq \hat{\lambda} \\ |z| - \lambda, & \lambda \leq |z| \leq \hat{\lambda} \\ 0, & |z| \leq \lambda \end{cases}$$

[Bullinger, Ilchmann, Allgöwer '98]: $z, k \in L^\infty$ and $\operatorname{dist}\big(y - z_1, [-\lambda, \lambda]\big) \to 0$

Funnel Observer for $\ddot{y} = u$

$$\dot{z}_1 = z_2 + \big(q_1 + p_1 k_O(t)\big)(y - z_1), \qquad k_O(t) = \frac{\psi_O(t)}{\psi_O(t) - |y(t) - z_1(t)|}$$

$$\dot{z}_2 = u + \big(q_2 + p_2 k_O(t)\big)(y - z_1)$$

$$e_1 := y - z_1,\ e_2 := \dot{y} - z_2, \qquad k_O(t) = \frac{\psi_O(t)}{\psi_O(t) - |e_1(t)|}$$

$$\dot{e}_1 = e_2 - \big(q_1 + p_1 k_O(t)\big) e_1,$$

$$\dot{e}_2 = -\big(q_2 + p_2 k_O(t)\big) e_1$$

$$A := \begin{bmatrix} -q_1 & 1 \\ -q_2 & 0 \end{bmatrix}, \ P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \text{ s.t. } \boxed{\boldsymbol{A^\top P + PA = -I}}$$

$$p_1 = 1, \quad p_2 = -\frac{p_{12}}{p_{22}}; \qquad \dot{e} = Ae - k_O(t) \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) e_1$$

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$$p_1 = 1, \quad p_2 = -\frac{p_{12}}{p_{22}}; \qquad \dot{e} = Ae - k_O(t) \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix} \right) e_1$$

$$\dot{e} = Ae - k_O(t) \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1$$

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} e^\top P e &= e^\top (A^\top P + PA) e - k_O(t) \left(e_1 \begin{pmatrix} p_1 \\ p_2 \end{pmatrix}^\top P e + e^\top P \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} e_1 \right) \\ &= -\|e\|^2 - k_O(t) \left(p_{11} - \frac{p_{12}^2}{p_{22}} \right) e_1^2 \leq -\|e\|^2 \end{aligned}$$

$\implies \; e_1, e_2 \in L^\infty[0,\omega)$, standard funnel argument : $k_O \in L^\infty$

$\implies \; \omega = \infty \;\wedge\; e(t) \to 0 \;\wedge\; k_O(t) \to 1$

$$\dot{e} = Ae - k_O(t) \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) e_1$$

$$\begin{aligned} \tfrac{\mathsf{d}}{\mathsf{d}t} e^\top P e &= e^\top (A^\top P + PA) e - k_O(t) \left(e_1 \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right)^\top P e + e^\top P \left(\begin{smallmatrix} p_1 \\ p_2 \end{smallmatrix}\right) e_1 \right) \\ &= -\|e\|^2 - k_O(t) \left(p_{11} - \frac{p_{12}^2}{p_{22}} \right) e_1^2 \le -\|e\|^2 \end{aligned}$$

$\implies$ $e_1, e_2 \in L^\infty[0, \omega)$, standard funnel argument : $k_O \in L^\infty$
$\implies$ $\omega = \infty \;\wedge\; e(t) \to 0 \;\wedge\; k_O(t) \to 1$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\mathrm{loc}}$ causal, local Lipschitz, BIBO

Examples:

Extension

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\mathrm{loc}}$ causal, local Lipschitz, BIBO

Examples:

$\dot{x}(t) = Ax(t) + Bu(t),\ y(t) = Cx(t)$ with

(A1) $\mathrm{rk}_{\mathbb{C}} \begin{bmatrix} \lambda I - A & B \\ C & 0 \end{bmatrix} = n + m$ for all $\lambda \in \mathbb{C}$ with $\mathrm{Re}\,\lambda \geq 0$;

(A2) $CB = CAB = \ldots = CA^{r-2}B = 0$ and $CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$.

Extension

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big)u(t)$$

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(A2) $CB = CAB = \ldots = CA^{r-2}B = 0$ and $CA^{r-1}B \in \mathbf{Gl}_m(\mathbb{R})$.

Is equivalent to $\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t)$, $y(t) = \hat{C}\hat{x}(t)$ with

$$\hat{A} = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_r & S \\ P & 0 & \cdots & 0 & Q \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} I_m & 0 & \cdots & 0 & 0 \end{bmatrix}$$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y, \dot y, \dots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot y, \dots, y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\text{loc}}$ causal, local Lipschitz, BIBO

Examples:

$\frac{\mathrm{d}}{\mathrm{d}t}\hat x(t) = \hat A \hat x(t) + \hat B u(t)$, $y(t) = \hat C \hat x(t)$ with

$$\hat A = \begin{bmatrix} 0 & I_m & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & I_m & 0 \\ R_1 & R_2 & \cdots & R_r & S \\ P & 0 & \cdots & 0 & Q \end{bmatrix}, \quad \hat B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ CA^{r-1}B \\ 0 \end{bmatrix}, \quad \hat C = \begin{bmatrix} I_m & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$y^{(r)}(t) = R_1 y(t) + \dots R_r y^{(r-1)}(t) + S e^{Qt}\eta(0) + \int_0^t S e^{Q(t-\tau)} P y(\tau) d\tau + CA^{r-1}Bu(t).$$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \ldots, y^{(r-1)})(t)\big)u(t)$$

$T : \mathcal{C} \to \mathcal{L}^{\infty}_{\mathrm{loc}}$ causal, local Lipschitz, BIBO

Examples:

$$y^{(r)}(t) = T(y, \dot{y}, \ldots, y^{(r-1)})(t) + \Gamma u(t)$$

with $\Gamma = CA^{r-1}B$ and

$$\begin{aligned} &T(y, \ldots, y^{(r-1)})(t) \\ &= R_1 y(t) + \ldots R_r y^{(r-1)}(t) + Se^{Qt}\eta(0) + \int_0^t Se^{Q(t-\tau)} P y(\tau) d\tau. \end{aligned}$$

Extension

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$T : \mathcal{C} \to \mathcal{L}^\infty_{\text{loc}}$ causal, local Lipschitz, BIBO

Examples:

$$\dot{x}_1(t) = x_2(t),\ \ldots,\ \dot{x}_{r-1}(t) = x_r(t), \qquad \hat{x}(t) = (x_1(t)^\top, \ldots, x_r(t)^\top)^\top$$

$$\dot{x}_r(t) = g_1(\hat{x}(t), \eta(t)) + g_2(\hat{x}(t), \eta(t))u(t),$$

$$\dot{\eta}(t) = g_3(\hat{x}(t), \eta(t)),$$

$$y(t) = x_1(t)$$

Extension

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)\big) u(t)$$

$T : \mathcal{C} \to \mathcal{L}^\infty_{\mathrm{loc}}$ causal, local Lipschitz, BIBO

Examples:

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \ \dots, \ \dot{x}_{r-1}(t) = x_r(t), \qquad \hat{x}(t) = (x_1(t)^\top, \dots, x_r(t)^\top)^\top \\ \dot{x}_r(t) &= g_1(\hat{x}(t), \eta(t)) + g_2(\hat{x}(t), \eta(t)) u(t), \\ \dot{\eta}(t) &= g_3(\hat{x}(t), \eta(t)), \\ y(t) &= x_1(t) \end{aligned}$$

$$T(y, \dots, y^{(r-1)})(t) := \big(y(t)^\top, \dots, y^{(r-1)}(t)^\top, \eta(t; \eta^0, y, \dots, y^{(r-1)})^\top\big)^\top$$

$$y^{(r)}(t) = g_1\big(T(y, \dots, y^{(r-1)})(t)\big) + g_2\big(T(y, \dots, y^{(r-1)})(t)\big)\, u(t)$$

$$y^{(r)}(t) = f\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)\big) + \Gamma\big(d(t), T(y, \dot{y}, \dots, y^{(r-1)})(t)\big)u(t)$$

$$\boxed{\begin{aligned}
\dot{z}_1(t) &= z_2(t) + \big(q_1 + p_1 k_O(t)\big)(y(t) - z_1(t)),\\
\dot{z}_2(t) &= z_3(t) + \big(q_2 + p_2 k_O(t)\big)(y(t) - z_1(t)),\\
&\;\vdots\\
\dot{z}_{r-1}(t) &= z_r(t) + \big(q_{r-1} + p_{r-1} k_O(t)\big)(y(t) - z_1(t)),\\
\dot{z}_r(t) &= \tilde{\Gamma} u(t) + \big(q_r + p_r k_O(t)\big)(y(t) - z_1(t)),\\
k_O(t) &= \frac{\psi_O(t)}{\psi_O(t) - |y(t) - z_1(t)|}
\end{aligned}}$$

with design parameters $p_i > 0$, $q_i > 0$, $\tilde{\Gamma} \in \mathbb{R}^{m \times m}$ and the funnel function $\psi_O : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$

$$\begin{aligned}\dot z_1 &= z_2 + \big(q_1 + p_1 k_O(t)\big)(y - z_1)\\ &\vdots\\ \dot z_{r-1} &= z_r + \big(q_{r-1} + p_{r-1} k_O(t)\big)(y - z_1)\\ \dot z_r &= \tilde\Gamma u + \big(q_r + p_r k_O(t)\big)(y - z_1)\end{aligned}$$

$$A := \begin{bmatrix} -q_1 & 1 & \\ \vdots & & \ddots & \\ -q_{r-1} & & & 1 \\ -q_r & & & 0\end{bmatrix},\quad P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^\top & P_{22}\end{bmatrix}\quad P_{11} \in \mathbb{R},\ P_{22} \in \mathbb{R}^{(r-1)\times(r-1)}$$

s.t. $A^\top P + PA = -I$ and $\begin{pmatrix} p_1 \\ \vdots \\ p_r\end{pmatrix} = \begin{pmatrix} 1 \\ -P_{22}^{-1} P_{12}^\top\end{pmatrix}$

Theorem [B., Reis '16]

$$y, \dots, y^{(r-1)} \in L^\infty \;\Rightarrow\; z_1, \dots, z_r, k_O \in L^\infty \wedge |y(t) - z_1(t)| < \psi_O(t)$$

$$\limsup_{t\to\infty} \|e(t)\| \le M, \quad e(t) = \big(y(t) - z_1(t), \dots, y^{(r)}(t) - z_r(t)\big)$$

$$\begin{aligned}\dot z_1 &= z_2 + \big(q_1 + p_1 k_O(t)\big)(y - z_1)\\ &\ \vdots\\ \dot z_{r-1} &= z_r + \big(q_{r-1} + p_{r-1} k_O(t)\big)(y - z_1)\\ \dot z_r &= \tilde\Gamma u + \big(q_r + p_r k_O(t)\big)(y - z_1)\end{aligned}$$

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$$\limsup_{t\to\infty}\|e(t)\| \le M,\quad e(t) = \big(y(t)-z_1(t),\dots,y^{(r)}(t)-z_r(t)\big)$$

Simulation - bioreactor

$$\begin{aligned} \dot m(t) &= \frac{a_1 m(t) s(t)}{a_2 m(t) + s(t)} - m(t)u(t), \\ \dot s(t) &= -\frac{a_1 a_3 m(t) s(t)}{a_2 m(t) + s(t)} + \big(a_4 - s(t)\big)u(t), \\ y(t) &= m(t), \end{aligned}$$

m - concentration of microorganisms, $m(0) = 0.075$

s - concentration of substrate, $s(0) = 0.03$

u - substrate inflow rate

$$u(t) = \begin{cases} 0.08, & t \in [0,30) \\ 0.02, & t \in [30,50) \\ 0.08, & t \ge 50, \end{cases}$$

$$\tilde\Gamma = 0,\ q_1 = 1,\ q_2 = 0.2,$$
$$p_1 = 1,\ p_2 = \tfrac{1}{11}$$
$$\psi_O(t)^{-1} = \tfrac{1}{2} t e^{-t} + \tfrac{100}{\pi} \arctan t.$$

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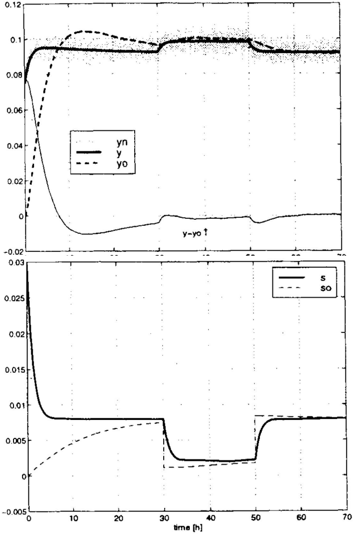
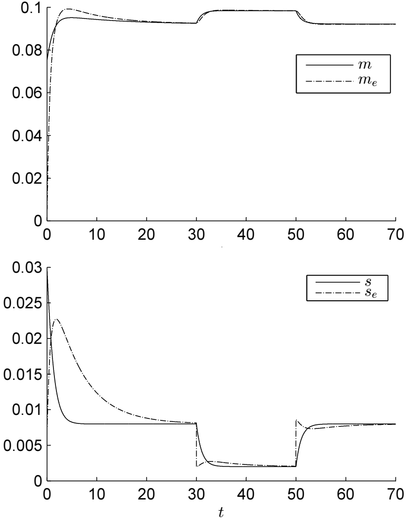
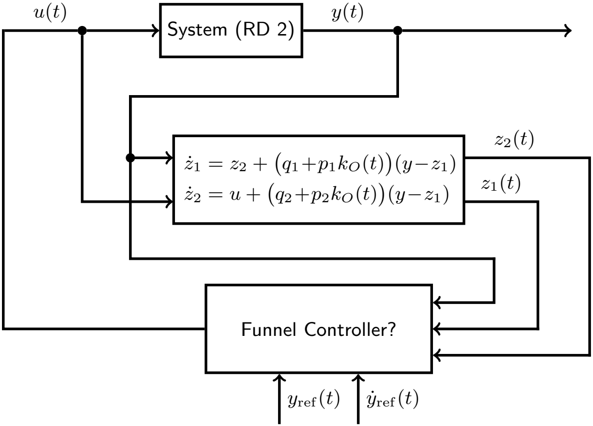
λ-strip observer [BIA '98]:



Funnel Observer:





- given $\psi(t)$, we have $|y(t) - y_{\text{ref}}(t)| < \psi(t)$ for all $t > 0$
- controller only needs y, z_1, z_2

straightforward approach:

$$u(t) = -k_P(t)^2(y(t) - y_{\text{ref}}(t)) - k_D(t)(z_2(t) - \dot{y}_{\text{ref}}(t)),$$
$$z_2(t) = \dot{y}(t) - e_2(t)$$

Problem: performance depends on $\|e_2\|_\infty$; we need $\inf_{t>0} \psi_D(t) > \|e_2\|_\infty$

alternative approach: What if we put $z_1 - y_{\text{ref}}$ into the funnel?

- $\dot{z}_1 = z_2 + \big(q_1 + p_1 k_O(t)\big)(y - z_1)$
- $|y(t) - y_{\text{ref}}(t)| \le |z_1(t) - y_{\text{ref}}(t)| + |y(t) - z_1(t)| < \psi_P(t) + \psi_O(t)$

  Given $\psi(t)$, let $\psi_P(t) = \psi_O(t) = \frac{1}{2}\psi(t)$

Seek a system of relative degree 2 with output $\hat{y} := z_1$!

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- $|y(t)-y_{\text{ref}}(t)| \leq |z_1(t)-y_{\text{ref}}(t)|+|y(t)-z_1(t)| < \psi_P(t)+\psi_O(t)$

  Given $\psi(t)$, let $\psi_P(t) = \psi_O(t) = \frac{1}{2}\psi(t)$

Seek a system of relative degree 2 with output $\hat{y} := z_1$!

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$$z_2(t) = \dot y(t) - e_2(t)$$

Problem: performance depends on $\|e_2\|_\infty$; we need $\inf_{t>0} \psi_D(t) > \|e_2\|_\infty$

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Seek a system of relative degree 2 with output $\hat y := z_1$!

$$\Sigma = \begin{cases} \dot{z}_1 &= z_2 + \big(q_1 + p_1 k_O(t)\big)e_1, \qquad \dot{e}_1 = e_2 - \big(q_1 + p_1 k_O(t)\big)e_1, \\ \dot{z}_2 &= u + \big(q_2 + p_2 k_O(t)\big)e_1, \qquad \dot{e}_2 = -\big(q_2 + p_2 k_O(t)\big)e_1, \\ \hat{y} &= z_1 \\ & \end{cases}$$

$$\Sigma = \begin{cases} \dot{z}_1 &= z_2 + \big(q_1 + p_1 k_O(t)\big) e_1, \qquad \dot{e}_1 = e_2 - \big(q_1 + p_1 k_O(t)\big) e_1, \\ \dot{z}_2 &= u + \big(q_2 + p_2 k_O(t)\big) e_1, \qquad \dot{e}_2 = -\big(q_2 + p_2 k_O(t)\big) e_1, \\ \hat{y} &= z_1 \end{cases}$$

$$\tfrac{\mathsf{d}}{\mathsf{d}t}\hat{y} = z_2 + \big(q_1 + p_1 k_O(t)\big) e_1,$$

$$(\tfrac{\mathsf{d}}{\mathsf{d}t})^2 \hat{y} = u + \big(q_2 + p_2 k_O(t)\big) e_1 + p_1 \dot{k}_O(t) e_1 + \big(q_1 + p_1 k_O(t)\big) \dot{e}_1$$

$$\Sigma = \begin{cases} \dot z_1 &= z_2 + \big(q_1 + p_1 k_O(t)\big) e_1, \\ \dot z_2 &= u + \big(q_2 + p_2 k_O(t)\big) e_1, \\ \hat y &= z_1 \end{cases} \qquad \underbrace{\begin{aligned} \dot e_1 &= e_2 - \big(q_1 + p_1 k_O(t)\big) e_1, \\ \dot e_2 &= -\big(q_2 + p_2 k_O(t)\big) e_1, \end{aligned}}_{\text{internal dynamics}}$$

$$\tfrac{\mathrm d}{\mathrm dt}\hat y = z_2 + \big(q_1 + p_1 k_O(t)\big) e_1,$$
$$(\tfrac{\mathrm d}{\mathrm dt})^2 \hat y = u + \underbrace{\big(q_2 + p_2 k_O(t)\big) e_1 + p_1 \dot k_O(t) e_1 + \big(q_1 + p_1 k_O(t)\big)\dot e_1}_{T(\psi_O, \dot\psi_O, e_1, e_2) \in L^\infty}$$

$$\Sigma = \begin{cases} \dot z_1 &= z_2 + \big(q_1 + p_1 k_O(t)\big) e_1, \\ \dot z_2 &= u + \big(q_2 + p_2 k_O(t)\big) e_1, \\ \hat y &= z_1 \end{cases} \qquad \underbrace{\begin{aligned} \dot e_1 &= e_2 - \big(q_1 + p_1 k_O(t)\big) e_1, \\ \dot e_2 &= -\big(q_2 + p_2 k_O(t)\big) e_1, \end{aligned}}_{\text{internal dynamics}}$$

$$\begin{aligned} \tfrac{\mathrm{d}}{\mathrm{d}t} \hat y &= z_2 + \big(q_1 + p_1 k_O(t)\big) e_1, \\ \left(\tfrac{\mathrm{d}}{\mathrm{d}t}\right)^2 \hat y &= u + \underbrace{\big(q_2 + p_2 k_O(t)\big) e_1 + p_1 \dot k_O(t) e_1 + \big(q_1 + p_1 k_O(t)\big) \dot e_1}_{T(\psi_O, \dot\psi_O, e_1, e_2) \in L^\infty} \end{aligned}$$

[Hackl et al. '13]: PD-funnel controller can be applied to Σ!

$$\Sigma = \begin{cases} \dot z_1 &= z_2 + \big(q_1 + p_1 k_O(t)\big) e_1, \\ \dot z_2 &= u + \big(q_2 + p_2 k_O(t)\big) e_1, \\ \hat y &= z_1 \end{cases} \qquad \underbrace{\begin{aligned} \dot e_1 &= e_2 - \big(q_1 + p_1 k_O(t)\big) e_1, \\ \dot e_2 &= -\big(q_2 + p_2 k_O(t)\big) e_1, \end{aligned}}_{\text{internal dynamics}}$$

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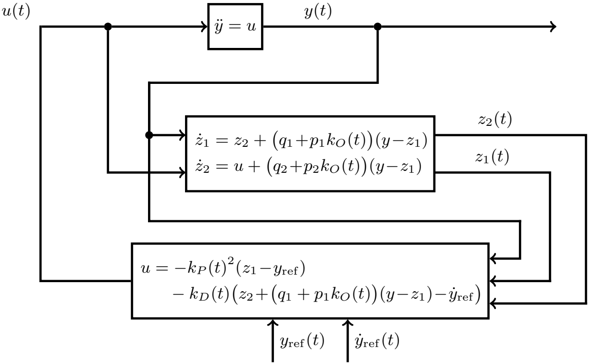
[Hackl et al. '13]: PD-funnel controller can be applied to Σ!

$$\begin{aligned} u(t) &= -k_P(t)^2 (z_1(t) - y_{\mathrm{ref}}(t)) - k_D(t)(\dot z_1(t) - \dot y_{\mathrm{ref}}(t)) \\ &= -k_P(t)^2 (z_1(t) - y_{\mathrm{ref}}(t)) \\ &\quad - k_D(t)\Big(z_2(t) + \big(q_1 + p_1 k_O(t)\big)(y(t) - z_1(t)) - \dot y_{\mathrm{ref}}(t)\Big) \end{aligned}$$



$$|y(t) - z_1(t)| < \psi_O(t), \qquad |z_1(t) - y_{\rm ref}(t)| < \psi_P(t),$$
$$|\dot{z}_1(t) - \dot{y}_{\rm ref}(t)| < \psi_D(t), \qquad |y(t) - y_{\rm ref}(t)| < \psi_P(t) + \psi_O(t)$$

Extension

$$\left.\begin{aligned} \dot x_1 &= x_2 \\ \dot x_2 &= T(x_1,x_2)+\gamma u \\ y &= x_1 \end{aligned}\right\} + \left\{\begin{aligned} \dot z_1 &= z_2+\big(q_1+p_1 k(t)\big)(y-z_1) \\ \dot z_2 &= \tilde\gamma u+\big(q_2+p_2 k_O(t)\big)(y-z_1) \\ u &= -k_P(t)^2(z_1-y_{\rm ref}) \\ &\quad -k_D(t)\Big(z_2+\big(q_1+p_1 k_O(t)\big)(y-z_1)-\dot y_{\rm ref}\Big) \end{aligned}\right.$$

$$\gamma > 0,\ \tilde\gamma > 0$$
$$\forall\, x_1, x_2 \in \mathcal{C}:\ x_1 \in L^\infty \ \Rightarrow\ T(x_1,x_2) \in L^\infty$$

Theorem [B., Reis '16]

$$|y(t)-z_1(t)| < \psi_O(t), \qquad |z_1(t)-y_{\rm ref}(t)| < \psi_P(t),$$
$$|\dot z_1(t)-\dot y_{\rm ref}(t)| < \psi_D(t), \qquad |y(t)-y_{\rm ref}(t)| < \psi_P(t)+\psi_O(t)$$

Extension

$$\left.\begin{aligned} \dot x_1 &= x_2 \\ \dot x_2 &= T(x_1,x_2)+\gamma u \\ y &= x_1 \end{aligned}\right\} + \left\{\begin{aligned} \dot z_1 &= z_2 + \big(q_1+p_1 k(t)\big)(y-z_1) \\ \dot z_2 &= \tilde\gamma u + \big(q_2+p_2 k_O(t)\big)(y-z_1) \\ u &= -k_P(t)^2(z_1-y_{\rm ref}) \\ &\quad -k_D(t)\Big(z_2+\big(q_1+p_1 k_O(t)\big)(y-z_1)-\dot y_{\rm ref}\Big) \end{aligned}\right.$$

$$\gamma > 0,\ \tilde\gamma > 0$$
$$\forall\, x_1, x_2 \in \mathcal{C}:\ x_1 \in L^\infty \ \Rightarrow\ T(x_1,x_2) \in L^\infty$$

Theorem [B., Reis '16]

$$|y(t)-z_1(t)| < \psi_O(t), \qquad |z_1(t)-y_{\rm ref}(t)| < \psi_P(t),$$
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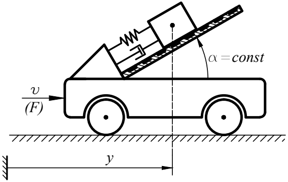
Simulation



Angle: $\alpha = 45°$

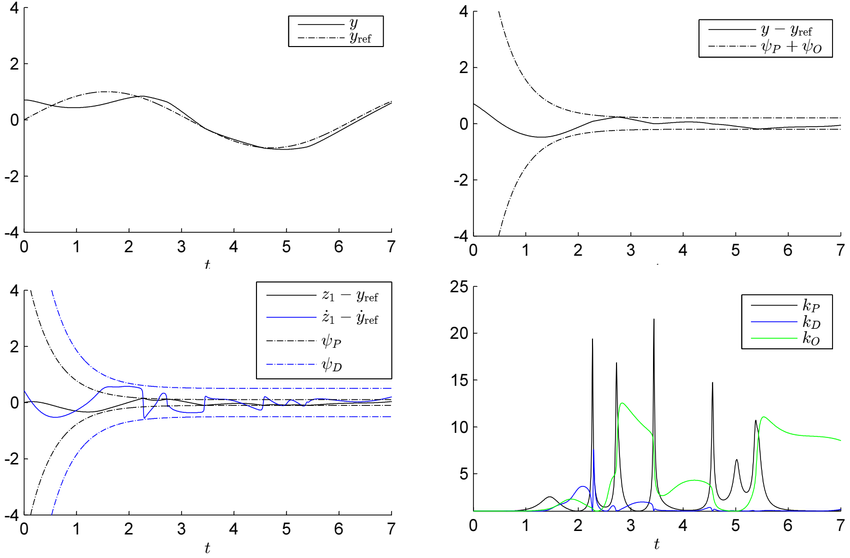
$K(z) = 2z^3 + 5z$

$D(\dot z) = 3\dot z$

$$y_{\mathrm{ref}}(t) = \sin t$$

$$q_1 = 1,\ q_2 = 0.2,\ p_1 = 1,\ p_2 = \frac{1}{11}$$

$$\psi_P(t) = \psi_O(t) = 5e^{-2t} + 0.1, \qquad \psi_D(t) = 10e^{-2t} + 0.5$$


4
2
0
-2
-4
0 1 2 3 4 5 6 7
t
y
y_{ref}
$y - y_{\text{ref}}$
$\psi_P + \psi_O$
$z_1 - y_{\text{ref}}$
$\dot{z}_1 - \dot{y}_{\text{ref}}$
ψ_P
ψ_D
25
20
15
10
5
k_P
k_D
k_O

Outlook

- relative degree higher than 2 $\rightarrow$ cascade of observers
- unstable internal dynamics $\rightarrow$ servo control
- optimal choice of design parameters
- experiments