Funnel control for DAE systems

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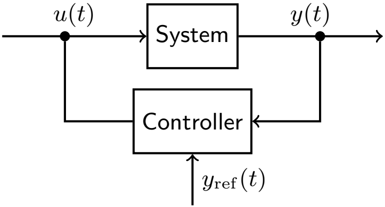
Munich, June 30, 2016

Adaptive Control



- no knowledge of system parameters, only "structure" of mathematical model for the system
- *aim*: design controller such that $y(t)$ "tracks" $y_{\mathrm{ref}}(t)$

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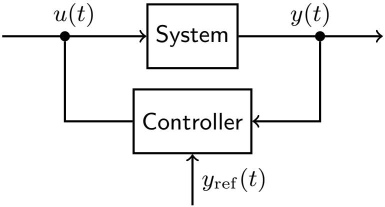

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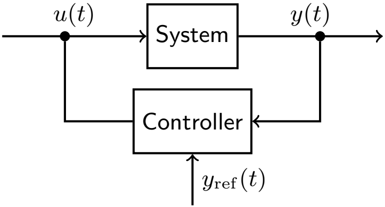

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- *aim*: design controller such that $y(t)$ "tracks" $y_{\mathrm{ref}}(t)$

Relative degree $r \in \mathbb{N}$

$$\dot{x}(t) = Ax(t) + bu(t), \quad y(t) = cx(t)$$

$$\dot{y}(t) = cAx(t) + cbu(t); \quad cb \neq 0 \;\Rightarrow\; r = 1$$

$$cb = 0 \;\Rightarrow\; \ddot{y}(t) = cA^2x(t) + cAbu(t); \quad cAb \neq 0 \;\Rightarrow\; r = 2$$

$$cAb = 0 \;\Rightarrow\; \ldots$$

$$\boxed{cb = cAb = \ldots = cA^{r-2}b = 0 \;\wedge\; cA^{r-1}b \neq 0}$$

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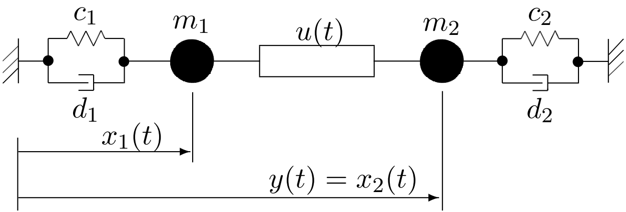

Figure: Mass-spring-damper system

Position control:

$$u(t) = x_2(t) - x_1(t)$$

$$\implies \quad \text{relative degree} = 0$$

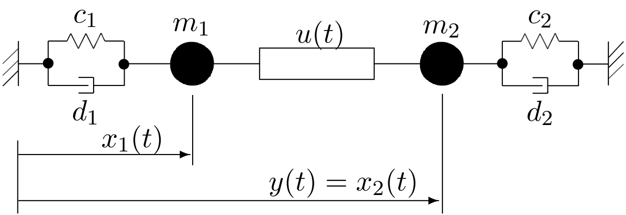

Figure: Mass-spring-damper system

Velocity control:

$$u(t) = \dot{x}_2(t) - \dot{x}_1(t)$$

$$\Longrightarrow \quad \text{relative degree} = 1$$

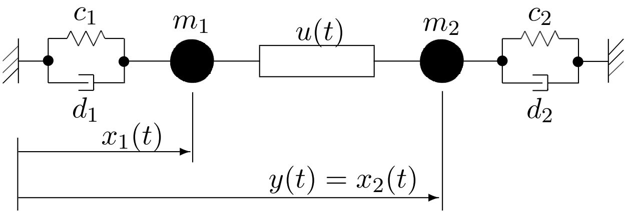

Figure: Mass-spring-damper system

Force control:

$$u(t) = M(\ddot{x}_2(t) - \ddot{x}_1(t))$$

$$\Longrightarrow \quad \text{relative degree} = 2$$

Zero dynamics

$$\dot{x}(t) = Ax(t) + bu(t), \quad y(t) = cx(t)$$

$$\mathcal{ZD} := \{\ (x,u)\ \mid \dot{x}(t) = Ax(t) + bu(t), \ \ 0 = cx(t)\ \}$$

$\mathcal{ZD}$ **stable** $:\Longleftrightarrow \ \forall\, (x,u) \in \mathcal{ZD}: \ \lim_{t\to\infty} (x(t), u(t)) = 0$

Theorem:

$\mathcal{ZD}$ **stable** $\Longleftrightarrow \ \forall\, \lambda \in \overline{\mathbb{C}_+}: \ \det \begin{bmatrix} A - \lambda I_n & b \\ c & 0 \end{bmatrix} \neq 0$

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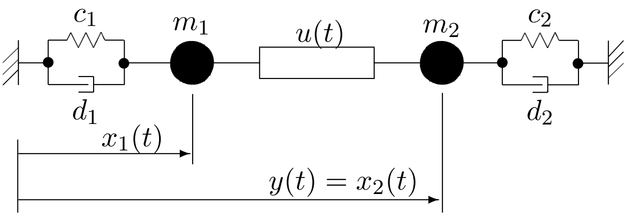

Figure: Mass-spring-damper system

Zero dynamics:

$$y(t) = 0 \quad \Rightarrow \quad x_2(t) = 0 \quad \Rightarrow \quad x_1(t) \to 0,\ u(t) \to 0$$

ODE systems

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (1)$$
$$\dot{y}(t) = f_2(y(t)) + f_3(x(t)) + \Gamma(y(t))u(t), \qquad y(0) = y^0. \quad (2)$$

$x : \mathbb{R} \to \mathbb{R}^n$, $u, y : \mathbb{R} \to \mathbb{R}^m$; f_1, f_2, f_3 differentiable;
$\Gamma : \mathbb{R}^m \to \mathbb{R}^{m \times m}$ differentiable

(1) internal dynamics
(2) input-output behavior

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relative degree:

$$\dot{z}(t) = f(z(t)) + g(z(t))u(t)$$
$$y(t) = h(z(t))$$

hat **strict relative degree** $r \in \mathbb{N}$ $:\Longleftrightarrow$ [Isidori '95]

- $\forall\, \xi \in \mathbb{R}^n \;\forall\, k = 0, \ldots, r-2 : \;\; L_g L_f^k h(\xi) = 0,$
  $[L_f h = (\partial h / \partial z)\, f]$
- $\forall\, \xi \in \mathbb{R}^n : \;\; \det L_g L_f^{r-1} h(\xi) \neq 0.$

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  $[L_f h = (\partial h / \partial z)\, f]$
- $\forall\, \xi \in \mathbb{R}^n : \; \det L_g L_f^{r-1} h(\xi) \neq 0.$

(1), (2) has strict rel. deg. 1 $\Longleftrightarrow \det \Gamma(y) \neq 0 \;\forall\, y \in \mathbb{R}^m$

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (1)$$
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$\mathcal{ZD}$ **stable** $:\Longleftrightarrow \ \forall\,(z,u) \in \mathcal{ZD}: \ \lim_{t\to\infty}(z(t),u(t)) = 0$

$\mathcal{ZD}_{(1),(2)}$ stable, if (1) **input-to-state stable**: [Sontag '89]

$$\forall\,(x^0, y)\ \forall\, t \geq 0: \ \|x(t; x^0, y)\| \leq \alpha(\|x^0\|, t) + \sup_{s\in[0,t]} \beta(\|y(s)\|),$$

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (1)$$
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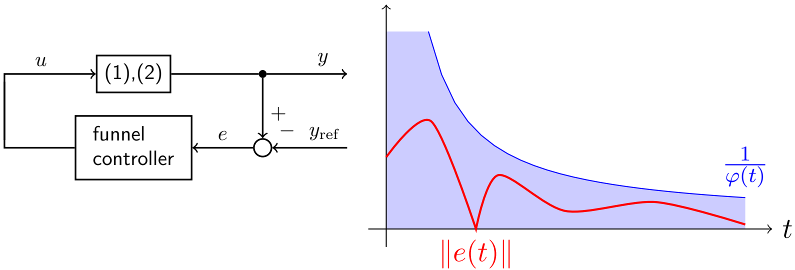

[Ilchmann & Ryan '09]: funnel control works if

- strict relative degree 1 and "high-gain matrix" $\Gamma(y)$ pos. def. for all $y \in \mathbb{R}^m$
- (1) is ISS (zero dynamics are stable)

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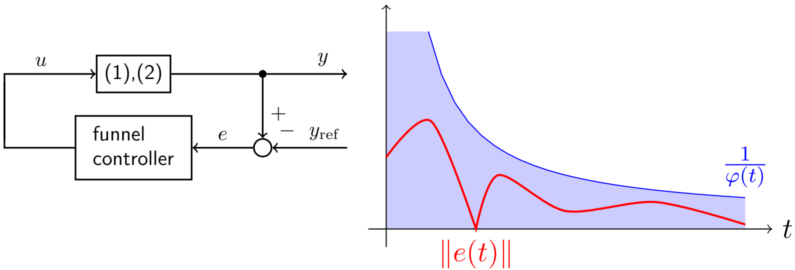

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$\{(1),(2)$ with strict rel. deg. $1\} \overset{\tilde{\Gamma}=\Gamma^{-1}}{\subseteq} \{(3),(4)\}$

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$$\dot x(t) = x(t) + y_1(t)$$

$$\begin{bmatrix} 0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \dot y_1(t)\\ \dot y_2(t)\\ \dot y_3(t) \end{pmatrix} = \begin{pmatrix} 0\\ y_2(t)\\ y_3(t) \end{pmatrix} + \begin{pmatrix} x(t)\\ 0\\ 0 \end{pmatrix} + \begin{pmatrix} u_1(t)\\ u_2(t)\\ u_3(t) \end{pmatrix}$$

$$y_1 = u_1 - \dot u_1 \quad \Longrightarrow \quad \int y_1 = \int u_1 - u_1$$

$$G(s) = \begin{bmatrix} s-1 & 0 & 0\\ 0 & -1 & \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}, \quad \lim_{s\to\infty} \begin{bmatrix} s^{-1} & 0 & 0\\ 0 & s^{0} & 0\\ 0 & 0 & s^{1} \end{bmatrix} G(s) \in \mathbb{R}^{3\times 3} \text{ inv.}$$

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \tag{3}$$

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Theorem [B., Ilchmann, Reis '14]

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (3)$$

$$\tilde{\Gamma}(y(t))\,\dot{y}(t) = f_2(y(t)) + f_3(x(t)) + u(t), \qquad y(0) = y^0. \quad (4)$$

- (3) is ISS
- $\tilde{\Gamma}(y) = RG(y)R^\top$, $G(y) > 0$
- f_2' bounded on $\ker R^\top$
- $\hat{k}$ "large enough"
- y^0 consistent

Then the *funnel controller*

$$u(t) = -k(t)\,e(t), \qquad \text{where} \qquad e(t) = y(t) - y_{\mathrm{ref}}(t)$$

$$k(t) = \hat{k}/\big(1 - \varphi(t)^2\|e(t)\|^2\big)\,,$$

achieves: $(x, y, k) \in L^\infty, \;\wedge\; \exists\, \varepsilon > 0\, \forall\, t > 0 :\; \|e(t)\| \le \varphi(t)^{-1} - \varepsilon$

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (3)$$
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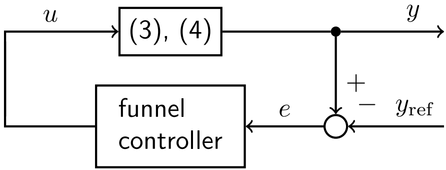

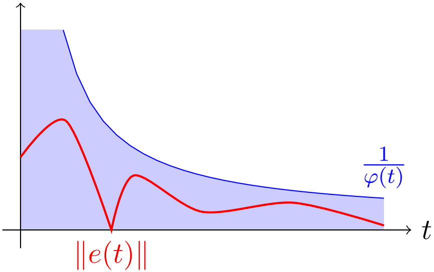

Comparison of ODEs and DAEs

$$\dot{x}(t) = f_1(x(t), y(t)), \qquad x(0) = x^0, \quad (1)$$
$$\dot{y}(t) = f_2(y(t)) + f_3(x(t)) + \Gamma(y(t))u(t), \quad y(0) = y^0. \quad (2)$$

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ODEs	DAEs
(1) is ISS	(3) is ISS
$\Gamma > 0$ (rel. deg. 1)	$\tilde{\Gamma} \geq 0$ (mixed rel. deg.)
$\hat{k} = 1$	$\hat{k}$ "large enough"
$y^0 \in \mathbb{R}^m$	y^0 consistent

Electrical circuits

$$\boxed{\tfrac{\mathrm{d}}{\mathrm{d}t} E x(t) = A x(t) + B u(t), \quad y(t) = C x(t)} \quad u = \begin{pmatrix} i_{\mathcal{I}} \\ v_{\mathcal{V}} \end{pmatrix}, \quad y = \begin{pmatrix} -v_{\mathcal{I}} \\ -i_{\mathcal{V}} \end{pmatrix}$$

$$sE - A = \begin{bmatrix} s\mathrm{A}_{\mathcal{C}} \mathcal{C} \mathrm{A}_{\mathcal{C}}^\top + \mathrm{A}_{\mathcal{R}} \mathcal{G} \mathrm{A}_{\mathcal{R}}^\top & \mathrm{A}_{\mathcal{L}} & \mathrm{A}_{\mathcal{V}} \\ -\mathrm{A}_{\mathcal{L}}^\top & s\mathcal{L} & 0 \\ -\mathrm{A}_{\mathcal{V}}^\top & 0 & 0 \end{bmatrix}, \quad B = C^\top = \begin{bmatrix} -\mathrm{A}_{\mathcal{I}} & 0 \\ 0 & 0 \\ 0 & -I_{n_{\mathcal{V}}} \end{bmatrix}$$

$\mathrm{A}_{\mathcal{C}}$, $\mathrm{A}_{\mathcal{R}}$, $\mathrm{A}_{\mathcal{L}}$, $\mathrm{A}_{\mathcal{V}}$, $\mathrm{A}_{\mathcal{I}}$ – element-related incidence matrices

$\mathcal{C}$, $\mathcal{G}$, $\mathcal{L}$ – constitutive relations of capacitances, resistances and inductances

passivity: $\mathcal{C} = \mathcal{C}^\top > 0, \mathcal{L} = \mathcal{L}^\top > 0, \mathcal{G} + \mathcal{G}^\top > 0$

Theorem [B., Reis '14]

$[E, A, B, C]$ MNA model of a circuit; assume one of the following:

- neither $\mathcal{IL}$-loops, nor $\mathcal{VCL}$-cutsets except for $\mathcal{VL}$-cutsets with at least one inductor
- neither $\mathcal{VC}$-cutsets, nor $\mathcal{ICL}$-loops except for $\mathcal{IC}$-loops with at least one capacitor

$\implies$ $\mathcal{ZD}$ are stable.

Theorem [B., Reis '14]

- $[E, A, B, C]$ MNA model of a circuit with stable $\mathcal{ZD}$

Then the *funnel controller*

$$u(t) = -k(t)\, e(t), \qquad \text{where} \qquad e(t) = y(t) - y_{\rm ref}(t)$$

$$k(t) = \frac{1}{1 - \varphi(t)^2 \|e(t)\|^2},$$

applied to $[E, A, B, C]$ achieves that

$$x \in L^\infty,\ k \in L^\infty \quad \wedge \quad \exists\, \varepsilon > 0\ \forall\, t > 0 :\ \|e(t)\| \le \varphi(t)^{-1} - \varepsilon.$$

$$
\begin{aligned}
E\dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t)
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$$