Funnel control for overhead crane model

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We consider an overhead crane whose control variables are the length of the rope and force/velocity at the gantry. The output consists of the position of the load. The objective is to design a closed-loop tracking controller which also takes into account the transient behavior. First we show that this system has no well-defined relative degree, which unfortunately does not allow to apply established methods for adaptive control to achieve the objective. To circumvent this problem, we design a dynamic state feedback which results in a system with relative degree four. Thereafter, we apply a funnel controller to this feedback system. We show that our approach can be used to move loads from one to another given position in the situation where there are several obstacles which have to be circumnavigated.

1 Introduction

In this paper, we develop a funnel controller for a simplified model of an overhead crane which was discussed in [1]. The model belongs to the class of nonlinear systems

$$\dot{x} = f(x) + \sum_{i=1}^{m} g_i(x)u_i, \quad y = h(x), \quad i = 1,\dots,m, \tag{1.1}$$

where $f, g_i \in \mathscr{C}^\infty(\mathbb{R}^n \to \mathbb{R}^n), h = (h_1,\dots,h_m)^\top \in \mathscr{C}^\infty(\mathbb{R}^n \to \mathbb{R}^m)$. We recall the concept of vector relative degree of (1.1) from [2, Sec. 5.1]: The system (1.1) has *vector relative degree* $(r_1, r_2, \cdots, r_m)$ at a point $x^0 \in \mathbb{R}^n$, if there exists an open neighborhood U of x^0 such that for all $1 \le i \le m$, $1 \le j \le m$ and $0 \le k \le r_i - 2$ we have $L_{g_j} L_f^{k-1} h_i(x) = 0$ for all $x \in U$, and the matrix

$$\Gamma(x^0) := \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x^0) & \cdots & L_{g_m} L_f^{r_1-1} h_1(x^0) \\ \vdots & & \vdots \\ L_{g_1} L_f^{r_m-1} h_m(x^0) & \cdots & L_{g_m} L_f^{r_m-1} h_m(x^0) \end{bmatrix}$$

is invertible. Furthermore, if $r_1 = \cdots = r_m = r$, then the system (1.1) is said to have *strict relative degree* r at x^0.

The control objective is tracking of a reference trajectory $y_{\mathrm{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^m)$ with prescribed performance, i.e., we seek an output error derivative feedback such that in the closed-loop system the tracking error $e(t) = y(t) - y_{\mathrm{ref}}(t)$ evolves within a prescribed performance funnel, that is $\varphi(t)\|e(t)\| < 1$ for all $t \ge 0$, where φ belongs to

$$\Phi_r := \left\{ \varphi \in \mathscr{C}^r(\mathbb{R}_{\ge 0} \to \mathbb{R}) \;\middle|\; \begin{array}{l} \varphi, \dot{\varphi}, \dots, \varphi^{(r)} \text{ are bounded,} \\ \varphi(\tau) > 0 \text{ for all } \tau > 0, \\ \text{and } \liminf_{\tau \to \infty} \varphi(\tau) > 0 \end{array} \right\}.$$

Furthermore, the state x and the input $u = (u_1,\dots,u_m)$ in (1.1) should remain bounded. We follow the framework of *Funnel Control* which was developed in [3]. The funnel controller is an adaptive controller of high-gain type and thus inherently robust. It has been successfully applied e.g. in control of industrial servo-systems [4] and voltage and current control of electrical circuits [5].

In the recent work [6], a funnel controller has been developed which can be applied to a class of systems with arbitrary relative degree. The controller has the form

$$\boxed{\begin{aligned} e_0(t) &= e(t) = y(t) - y_{\mathrm{ref}}(t), \\ e_1(t) &= \dot{e}_0(t) + k_0(t)\, e_0(t), \\ e_2(t) &= \dot{e}_1(t) + k_1(t)\, e_1(t), \\ &\;\;\vdots \\ e_{r-1}(t) &= \dot{e}_{r-2}(t) + k_{r-2}(t)\, e_{r-2}(t), \\ k_i(t) &= \tfrac{1}{1-\varphi_i(t)^2\|e_i(t)\|^2}, \quad i = 0,\dots,r-1, \\ u(t) &= -k_{r-1}(t)\, e_{r-1}(t) \end{aligned}} \tag{1.2}$$

where the reference signal and funnel functions have the following properties:

$$\boxed{\begin{aligned} &y_{\mathrm{ref}} \in \mathscr{W}^{r,\infty}(\mathbb{R}_{\ge 0} \to \mathbb{R}^p), \\ &\varphi_0 \in \Phi_r,\;\; \varphi_1 \in \Phi_{r-1},\; \dots,\;\; \varphi_{r-1} \in \Phi_1. \end{aligned}} \tag{1.3}$$

Now we introduce the model that we want to consider. The overhead crane model, which was introduced in [1], is an overhead gantry crane with a trolley moving along a horizontal axis. A suspended load is attached to the trolley by four ropes, which are assumed to be rigid and massless. Moreover, the winches on the trolley are synchronized which help controlling the length of the ropes so that the attached load does not rotate around itself. Therefore, this model can be represented by a point mass connected to the trolley by a single rope as shown in Fig. 1.

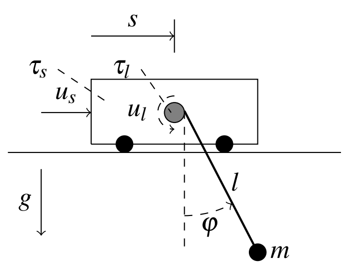

Fig. 1: Two dimensions crane model.

The equations of motion of the overhead crane model are

$$\begin{aligned} \tau_s \ddot{s} + \dot{s} &= u_s \\ \tau_l \ddot{l} + \dot{l} &= u_l \\ \cos(\varphi)\ddot{s} + l\ddot{\varphi} + 2\dot{\varphi}\dot{l} &= -g\sin(\varphi), \end{aligned} \tag{1.4}$$

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where s (in m) is the trolley position, l (in m) is the rope length, and φ (in rad) is the swing angle. The velocity of the trolley u_s (in $\mathrm{m\,s^{-1}}$) and the velocity of the rope u_l (in $\mathrm{m\,s^{-1}}$) serve as the system inputs, and τ_s (in second), τ_l (in second) are time constants of trolley and winch actuator, resp., and $g = 9.81\,\mathrm{m/s^2}$ is the gravitational constant. The constants do not depend on the trolley or load mass. As output of the model we choose the position of the load $(y_1, y_2) = (s + l\sin\varphi, l\cos\varphi)$. Now, we transform the equations of motion (1.4) into the form (1.1). Denote $x_1 := s$, $x_2 := \dot{s}$, $x_3 := l$, $x_4 := \dot{l}$, $x_5 := \varphi$, $x_6 := \dot{\varphi}$, $u_1 := u_s$, $u_2 := u_l$, then we have

$$\begin{aligned} \dot{x} &= f(x) + g_1(x)u_1 + g_2(x)u_2, \\ y &= (x_1 + x_3\cos x_5, x_3\cos x_5)^\top \end{aligned} \tag{1.5}$$

where $x = (x_1, x_2, \ldots, x_6)^\top$, $f(x) = \Big(x_2, -\frac{x_2}{\tau_s}, x_4, -\frac{x_4}{\tau_l}, x_6, \frac{x_2\cos x_5}{x_3\tau_s} - \frac{2x_4x_6 + g\sin x_5}{x_3}\Big)^\top$, $g_1(x) = \Big(0, \frac{1}{\tau_s}, 0, 0, 0, -\frac{\cos x_5}{x_3\tau_s}\Big)^\top$, $g_2(x) = \Big(0, 0, 0, \frac{1}{\tau_l}, 0, 0\Big)^\top$. It is easy to check that

$$\Gamma(x) = \begin{bmatrix} L_{g_1}(L_f y_1) & L_{g_2}(L_f y_1) \\ L_{g_1}(L_f y_2) & L_{g_2}(L_f y_2) \end{bmatrix} = \begin{bmatrix} \frac{\sin^2 x_5}{\tau_s} & \frac{\sin x_5}{\tau_l} \\ \frac{\sin x_5 \cos x_5}{\tau_s} & \frac{\cos x_5}{\tau_l} \end{bmatrix}.$$

Observe that $\operatorname{rank}\Gamma(x) = 1 < 2$ for all $x \in \mathbb{R}^6$, hence system (1.5) does not have a well-defined vector relative degree.

To treat this problem we use the Dynamic Extension Algorithm introduced in [2, Sec.5.4], applied to the system (1.5). The dynamic extension leads to

$$\boxed{\begin{aligned} &\dot{x}_7 = x_8,\ \dot{x}_8 = v_2,\ u_1 = v_1, \\ &u_2 = x_4 + x_3x_6^2\tau_l + (x_2 - v_1)\tau_l\sin x_5/\tau_s \\ &\qquad + (x_7 - g\sin^2 x_5)\tau_l/\cos x_5, \end{aligned}} \tag{1.6}$$

and as a result we obtain the new system

$$\begin{aligned} \dot{\tilde{x}} &= \tilde{f}(\tilde{x}) + \tilde{g}_1(\tilde{x})v_1 + \tilde{g}_2(\tilde{x})v_2, \\ y &= (x_1 + x_3\cos x_5, x_3\cos x_5)^\top, \end{aligned} \tag{1.7}$$

where $\tilde{x} = (x_1, \ldots, x_8)^\top$, $\tilde{f}(\tilde{x}) = \Big(x_2, -\frac{x_2}{\tau_s}, x_3x_6^2 + \frac{x_2\sin x_5}{\tau_s} + \frac{x_7 - g\sin^2 x_5}{\cos x_5}, -\frac{x_4}{\tau_l}, x_6, \frac{x_2\cos x_5}{x_3\tau_s} - \frac{2x_4x_6 + g\sin x_5}{x_3}, x_8, 0\Big)^\top$, $\tilde{g}_1(\tilde{x}) = \Big(0, \frac{1}{\tau_s}, 0, -\frac{\sin x_5}{\tau_s}, 0, -\frac{\cos x_5}{x_3\tau_s}, 0, 0\Big)^\top$, $\tilde{g}_2(\tilde{x}) = (0, \ldots, 0, 1)^\top$. Checking the relative degree of system (1.7) we find that

$$\Gamma(\tilde{x}) = \begin{bmatrix} L_{\tilde{g}_1}(L^3_{\tilde{f}} y_1) & L_{\tilde{g}_2}(L^3_{\tilde{f}} y_1) \\ L_{\tilde{g}_1}(L^3_{\tilde{f}} y_2) & L_{\tilde{g}_2}(L^3_{\tilde{f}} y_2) \end{bmatrix} = \begin{bmatrix} \frac{g - x_7}{x_3\tau_s\cos x_5} & \frac{\sin x_5}{\cos x_5} \\ 0 & 1 \end{bmatrix}.$$

$\Gamma(\tilde{x})$ is invertible at any point of the extended state space where $x_3 \neq 0$, $\cos x_5 \neq 0$ and $x_7 \neq g$. Therefore, system (1.7) has strict relative degree $r = 4$ at each of these points. Furthermore, since the system dimension is 8, the internal dynamics of (1.7) are trivial. As a consequence, two crucial assumptions for feasibility of the funnel controller (1.2) are satisfied, cf. [6]. However, the last crucial assumption, that the symmetric part of $\Gamma(\tilde{x})$ is positive definite everywhere, is not satisfied. A feasibility proof for this case is not available yet, but the simulations look promising. The final feedback controller, consisting of the dynamic extension (1.6) and the funnel controller (1.2) is depicted in Fig. 2.

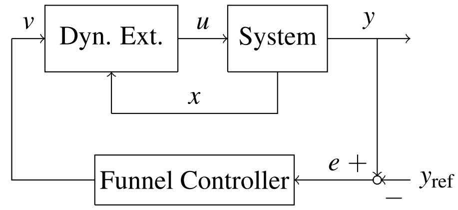

Fig. 2: Combination of funnel controller and dynamic extension.

2 Simulations

We choose the parameter values $\tau_s = 0.03\,\mathrm{s}$, $\tau_l = 0.02\,\mathrm{s}$, the reference signal $y_{\mathrm{ref}}(t) = \big(3(t - \sin t)\mathrm{m}, (9 + 3\cos t)\mathrm{m}\big)^\top$ and the funnel functions $\varphi_0(t) = 3$, $\varphi_1(t) = (2e^{-2t} + 0.05)^{-1}$, $\varphi_2(t) = (4e^{-2t} + 0.1)^{-1}$, $\varphi_3(t) = (20e^{-2t} + 0.5)^{-1}$ resp. The initial values are $\tilde{x}^0 = (0, 0, 12, 0, 0, 0, 0, 0)$. The simulation of the controller (1.2) applied to (1.7) over the time interval $[0, 2\pi]$ has been performed in MATLAB (solver: *ode45*, rel. tol: 10^{-14}, abs. tol: 10^{-10}) and is depicted in Figs. 3 and 4.

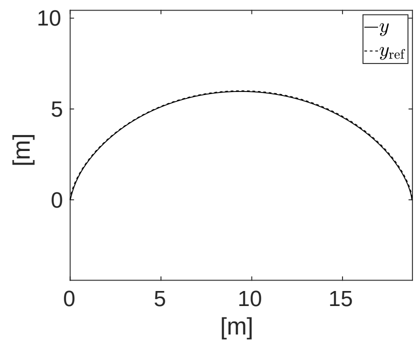

Fig. 3: Load trajectories

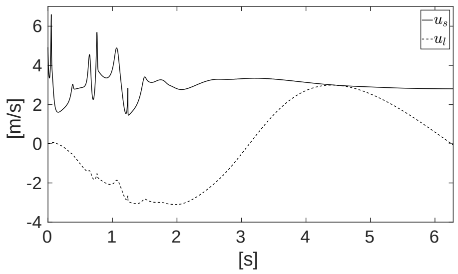

Fig. 4: Input functions

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