Locally Compact Contraction Groups

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joint work with

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Overview

Let G be a topological group, with neutral element e.
Let $\alpha \in \mathrm{Aut}(G)$.

Definition

The **contraction subgroup** $\mathrm{con}(\alpha)$ of G is the set of all $g \in G$ such that $\alpha^n(g) \to e$ as $n \to \infty$.

The talk has two parts:

Part 1: Contraction subgroups of totally disconnected, locally compact groups

Part 2: Structure of closed contraction groups in lcp groups

Contraction subgroups $\mathrm{con}(\alpha) \subseteq G$

Let G be a topological group, with neutral element e.
Let $\alpha \in \mathrm{Aut}(G)$.

Contraction subgroups and Levi subgroups

- The **contraction subgroup** $\mathrm{con}(\alpha)$ of G is the set of all $g \in G$ such that $\alpha^n(g) \to e$ as $n \to \infty$.
- The **Levi subgroup** $\mathrm{lev}(\alpha)$ is the set of all $g \in G$ the two-sided orbit $\{\alpha^n(g) \colon n \in \mathbb{Z}\}$ of which is relatively compact.

Using basic facts concerning "tidy subgroups" from the structure theory of totally disconnected, locally compact groups (tdlc-groups) initiated by G. Willis in 1994, one finds:

Theorem (Baumgartner-Willis 2004)

If G is a tdlc-group and $\alpha \in \mathrm{Aut}(G)$, then $\mathrm{lev}(\alpha)$ is closed in G.

Contraction subgroups $\mathrm{con}(\alpha) \subseteq G$

Let G be a tdlc group and $\alpha \in \mathrm{Aut}(G)$. Dynamics on the **big cell**

$$\Omega := \mathrm{con}(\alpha)\,\mathrm{lev}(\alpha)\,\mathrm{con}(\alpha^{-1}) \subseteq G$$

Theorem (JSP Wang, 1984)

If G is a p-adic Lie group, then $\mathrm{con}(\alpha)$ is closed and we have $(*)$: Ω is an α-stable open e-neighbourhood in G, and the product map

$$\pi\colon\ \mathrm{con}(\alpha) \times \mathrm{lev}(\alpha) \times \mathrm{con}(\alpha^{-1}) \to \Omega, \quad (x, y, z) \mapsto xyz$$

is a homeomorphism.

- If $\mathrm{con}(\alpha)$ is closed, then $(*)$ holds for any metrizable tdlc group G (G'05, using Baumgartner-Willis '04)
- G Lie$/\mathbb{K}$, α analytic $\Rightarrow$ π analytic diffeo (cf. G'08)
- Metrizability is irrelevant and same holds for endomorphisms (Bywaters-G-Tornier 2018); Lie case for endomorphisms: G'18

Contraction subgroups $\mathrm{con}(\alpha) \subseteq G$

If $\alpha \in \mathrm{End}(G)$, following Willis (2015) replace $\mathrm{con}(\alpha^{-1})$ with subgroup $\mathrm{con}^{-}(\alpha)$ of all $g_0 \in G$ for which there exist

$$\cdots \mapsto g_{-3} \mapsto g_{-2} \mapsto g_{-1} \mapsto g_0$$

such that $g_{-n} \to e$ as $n \to \infty$.

To see that $\mathrm{con}(\alpha)$, $\mathrm{lev}(\alpha)$ and $\mathrm{con}(\alpha^{-1})$ are submanifolds of a Lie group $G \,/\, \mathbb{K}$, need to adapt concepts and proofs from theory of smooth dynamical systems over $\mathbb{R}$:

$\mathrm{con}(\alpha) = W^s(\alpha, e)$	**stable manifold** around e
$\mathrm{lev}(\alpha) = W^c(\alpha, e)$	**centre manifold**
$\mathrm{con}(\alpha^{-1}) = W^u(\alpha, e)$	**unstable manifold**

Analytic manifold structures on such can be constructed for analytic dynamical systems over $\mathbb{K}$ (G'13, G'16)

Contraction subgroups $\mathrm{con}(\alpha) \subseteq G$

Let $\alpha \in \mathrm{End}(G)$ and $H \subseteq G$ be an α-invariant closed subgroup.

Definition

The **basin of attraction** $\mathrm{con}(\alpha, H)$ of H in G is the set of all $g \in G$ such that for each open e-neighbourhood $U \subseteq G$, there is n_0 with

$$\alpha^n(g) \in UH \quad \text{for all } n \geq n_0.$$

Theorem (Bywaters-G-Tornier '18)

Let G be a tdlc group, $\alpha \in \mathrm{End}(G)$ and $H \subseteq G$ be a closed subgroup. If $\alpha(H) = H$ or $\alpha(H) \subseteq H$ and H is compact, then

$$\mathrm{con}(\alpha, H) = \mathrm{con}(\alpha)H.$$

Special cases for $\alpha \in \mathrm{Aut}(G)$:

Dani-Shah '91 (p-adic Lie groups)
Baumgartner-Willis '04 (G metrizable), Jaworski '09

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If G is a locally compact group, $\alpha \in \mathrm{Aut}(G)$ and $\mathrm{con}(\alpha)$ is closed in G, then $\mathrm{con}(\alpha)$ is a locally compact group and $\alpha|_{\mathrm{con}(\alpha)}$ an automorphism of $\mathrm{con}(\alpha)$ which is *contractive*:

Definition

- An automorphism $\alpha\colon G \to G$ is called **contractive** if $\alpha^n(g) \to e$ as $n \to \infty$, for all $g \in G$.
- A **locally compact contraction group** is a locally compact group G, together with a contractive automorphism $\alpha \in \mathrm{Aut}(G)$.

Locally compact contraction groups

For G locally compact, $\alpha \in \mathrm{Aut}(G)$ contractive
i.e., $\lim_{n\to\infty} \alpha^n(g) = e$ for all $g \in G$

Theorem (Siebert '86)

$G = G_e \times G_{td}$, where $G_e \subseteq G$ is the identity component and G_{td} an α-stable, totally disconnected, normal subgroup

(G_{td} unique by G/Willis '18)

Siebert also characterized the **connected** locally compact contraction groups

(simply connected nilpotent real Lie groups admitting an $(]0, \infty[, +)$-grading on the Lie algebra).

Now: Structure of **totally disconnected**, locally compact contraction groups

Locally compact contraction groups

Structure Theorem (G./Willis '10).

If (G, α) is a tdlc contraction group, then the set $\mathrm{tor}(G)$ *of torsion elements is a closed subgroup of G. Moreover,* $\mathrm{tor}(G)$ *is a torsion group of finite exponent and*

$$G \;=\; \mathrm{tor}(G) \times G_{p_1} \times \cdots \times G_{p_n}$$

internally as a topological group, for certain α-stable closed subgroups G_p which are p-adic Lie groups.

- By J.S.P. Wang (1984), each G_p is nilpotent (a group of $\mathbb{Q}_p$-rational points of a unipotent linear algebraic group over $\mathbb{Q}_p$).
- Any p-adic contraction group (G, α) is determined by its Lie algebra $L(G)$ and the contractive Lie algebra automorphism $L(\alpha)$.
- $\mathfrak{g}$ admits a contractive Lie algebra automorphism iff $\mathfrak{g}$ admits an $(\mathbb{N}, +)$-grading (G'08).

Locally compact contraction groups

Theorem (G/Willis '18)

The G_p in $G = \mathrm{tor}(G) \times G_{p_1} \times \cdots \times G_{p_n}$ are unique.

Structure of $\mathrm{tor}(G)$?

Example (right shift)

F	finite group
$G := F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$	$F^{(-\mathbb{N})} := \bigoplus_{n \in -\mathbb{N}} F$ discrete, $F^{\mathbb{N}_0}$ compact
$\alpha\colon G \to G$	right shift, $\alpha((x_n)_{n\in\mathbb{Z}}) := (x_{n-1})_{n\in\mathbb{Z}}$.

Then α is a contractive automorphism and hence (G, α) a contraction group.

Classification of the simple contraction groups

Structure Theorem (G/Willis '10)

Let (G, α) be a tdlc contraction group such that G is a torsion group. Then (G, α) admits a composition series

$$\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

and each of the simple factors G_j/G_{j-1}, with the induced contractive automorphism, is isomorphic to $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with the right shift for a finite simple group F.

There are only countably possibilities for the simple factors.

Does this imply there are only countably many torsion contraction groups? No!

Locally compact contraction groups

Proposition (G/Willis '18).

If (G,α) is a locally compact contraction group and $N \subseteq G$ an α-stable closed normal subgroup, then

$$q\colon G \to G/N, \quad x \mapsto xN$$

admits an equivariant continuous global section $\sigma\colon G/N \to G$.

Thus $q \circ \sigma = \mathrm{id}_{G/N}$ and $\alpha \circ \sigma = \sigma \circ \alpha'$.

Remark As a consequence, central extensions

$$\mathbf{1} \to A \to \widehat{G} \to G \to \mathbf{1}$$

of locally compact contraction groups can be described by means of equivariant continuous 2-cocycles $\omega\colon G \times G \to A$

(their cohomology classes)

Locally compact contraction groups

Take $A := G := C_p^{(-\mathbb{N})} \times C_p^{\mathbb{N}_0}$ with right shift, $C_p = \mathbb{Z}/p\mathbb{Z}$

Theorem (G/Willis '18)

Suitable choices of $\omega\colon G \times G \to A$ yield an uncountable family of torsion contraction groups $A \times_\omega G$ which are pairwise non-isomorphic as contraction groups.

[Notably, simply take ω bi-additive]

However, only countably many **abelian** torsion contraction groups.

Theorem (G/Willis '18)

Every abelian torsion contraction group is isomorphic to $F^{(-\mathbb{N})} \times F^{\mathbb{N}_0}$ with the right shift for some finite abelian group F (which is uniquely determined).

Locally compact contraction groups

Theorem (G/Willis '20)

If (G, α) is a tdlc contraction group and G has an open subgroup which is a pro-p group, then G is nilpotent.

As a corollary, we answer a question of Caprace '18:

If a contraction group G is an extension of $C_p^{(-\mathbb{N})} \times C_p^{\mathbb{N}_0}$ by itself, then G is nilpotent.

Since every locally pro-nilpotent contraction group is a direct product of finitely many groups which are locally pro-p for some prime p, we also deduce:

Corollary (G/Willis '20)

If (G, α) is a tdlc contraction group and G is locally pro-nilpotent, then G is nilpotent.

Locally compact contraction groups

In the case of Lie contraction groups, a central series of Lie subgroups can be chosen.

Theorem (G '08)

Let G be an analytic Lie group over a totally disconnected local field and $\alpha\colon G \to G$ be a contractive, analytic automorphism. Then G admits a central series

$$\mathbf{1} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that each G_j is a Lie subgroup.

In fact, use $G_j = W^s_{a_j}(\alpha, e)$ for suitable $0 < a_0 < \cdots < a_n < 1$, the a_j-stable manifold of all $g \in G$ such that $\|\phi(\alpha^n(g))\| = o((a_j)^n)$ as $n \to \infty$, using a chart ϕ of G around e such that $\phi(e) = 0$.

Non-closed contraction groups

If G is a Lie group over a td local field and $\alpha \in G$ an analytic automorphism, then con(α) need not be closed. Yet, the stable manifold structure makes con(α) a Lie group which is a locally compact contraction group.

If G is a tdlc group an α an automorphism which is **expansive** in the sense that

$$\bigcap_{n\in\mathbb{Z}} \alpha^n(U) = \{e\}$$

for some identity neighbourhood $U \subseteq G$, then con(α) and con(α^{-1}) can be given finer topologies making them locally compact contraction groups for α and α^{-1}, respectively (Siebert '89). This was used to show:

Theorem (G/Raja '17)

If G is a tdlc group, $\alpha \in \mathrm{Aut}(G)$ an expansive automorphism and N an α-stable, closed, normal subgroup, then the automorphism induced by α on G/N is expansive.

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