The Mackey bijection for real groups and geometrical realizations

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June 29, 2020

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- Equipped with $\begin{cases} \text{Plancherel measure} \\ \text{Fell topology} \end{cases}$
- Knapp-Zuckerman classification for $\widehat{G}_{\mathrm{temp}}$

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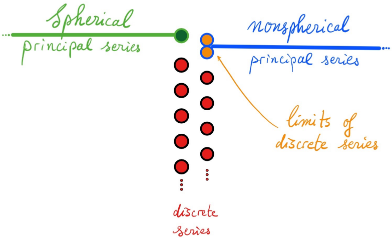
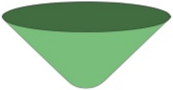
$\widehat{G}_{\mathrm{temp}}$ for $G = \mathrm{SL}(2,\mathbb{R})$



Cartan motion group



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Tangent space
$\mathfrak{p} = T_{1_G K}(G/K)$

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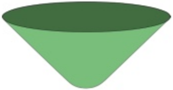

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Tangent space
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Cartan motion group:

$$G_0 = K \ltimes \mathfrak{p}$$

(isometry group of flat $\mathfrak{p}$)

Deforming G to G_0

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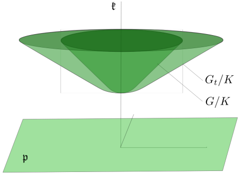


The family $(G_t)_{t>0}$:

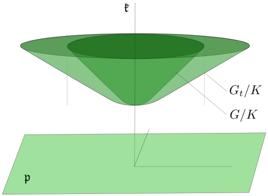
For $t > 0$, use

$$\varphi_t : K \times \mathfrak{p} \to G$$
$$(k, v) \mapsto \exp(tv)k,$$

to define

$$G_t = \text{group} \begin{cases} \text{with underlying set } K \times \mathfrak{p} \\ \text{product law making } \varphi_t \text{ isom.} \end{cases}$$

A puzzle



Group	G	$G_t \simeq G$ $(t > 0)$	G_0

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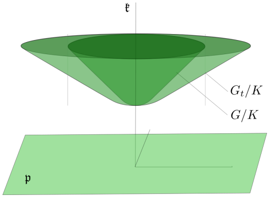

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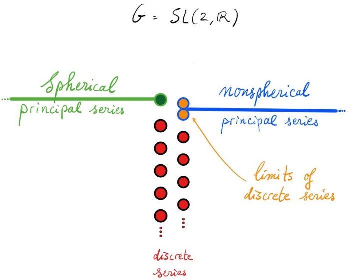

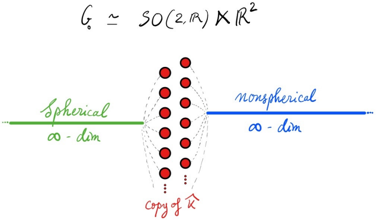

A puzzle, continued

Two problems:

Mackey, 1971-75 : there should exist a natural bijection $\widehat{G}_{\text{temp}} \leftrightarrow \widehat{G_0}$.

Connes & Higson, 1990-94 :

Group	G	$G_t \simeq G$ $(t > 0)$	G_0
Dual space	$\widehat{G}$	$\widehat{G_t}$	$\widehat{G_0}$

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Two problems:

Mackey, 1971-75 : there should exist a natural bijection $\widehat{G}_{\text{temp}} \leftrightarrow \widehat{G_0}$.

Connes & Higson, 1990-94 : the Baum-Connes-Kasparov isomorphism, between $K(C_r^*(G_0))$ and $K(C_r^*(G))$, should be a reflection of its properties.

Group	G	$G_t \simeq G$ $(t > 0)$	G_0
Dual space	$\widehat{G}_{\text{temp}}$	$(\widehat{G_t})_{\text{temp}}$	$\widehat{G_0}$
Reduced C^*-algebra	$C_r^*(G)$	$C_r^*(G_t)$	$C_r^*(G_0)$

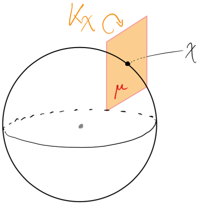
Constructing the (tempered) Mackey bijection

$$\widehat{G_0} \leftrightarrow \widehat{G}_{\mathrm{temp}}$$

Unitary dual of G_0

Main tool: action of K on $\mathfrak{p}^\star$.

Fix $\chi \in \mathfrak{p}^\star$ and $\mu \in \widehat{K_\chi}$.



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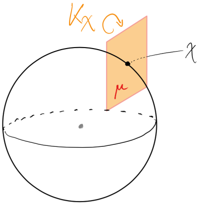

1. Consider the centralizer

$$Z_{G_0}(\chi) = K_\chi \ltimes \mathfrak{p}$$

2. Out of (χ, μ), build a representation of $Z_{G_0}(\chi)$:

$$\mu \otimes e^{i\chi}$$

3. Form the induced representation

$$\mathbf{M}_0(\chi, \mu) = \mathrm{Ind}_{K_\chi \ltimes \mathfrak{p}}^{G_0}(\mu \otimes e^{i\chi}).$$

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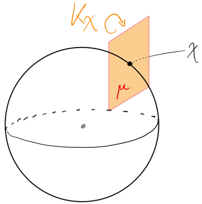

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Theorem (Mackey 1949):

All unitary irreducible representations of G_0 have this form.

$$G = \mathrm{SL}(2,\mathbb{R})$$

action of K on $\mathfrak{p}$ $\longleftrightarrow$ $\mathrm{SO}(2) \circlearrowright \mathbb{R}^2$

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$$Z_{G_0}(\chi) = K_\chi \ltimes \mathfrak{p}$$

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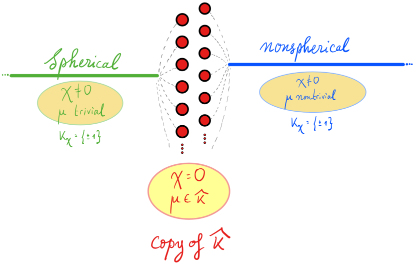
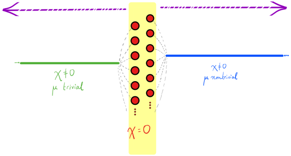


What should a Mackey bijection look like?

Rescaling maps in the duals

$$\text{for } \alpha > 0, \quad \mathbf{R}_{\alpha}^{\widehat{G_0}} : \widehat{G_0} \to \widehat{G_0}$$



- Start with $\pi \in \widehat{G_0}$
- Write it as $\mathbf{M}_0(\chi, \mu)$, where $\begin{cases} \chi \in \mathfrak{p}^\star \\ \mu \in \widehat{K_\chi} \end{cases}$
- Send it to $\mathbf{M}_0(\frac{\chi}{\alpha}, \mu)$

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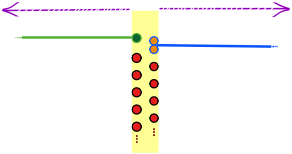

- Start with $\pi \in \widehat{G}_{\mathrm{temp}}$
- Write π as a submodule of $\mathrm{Ind}_{MAN}^{G}(\sigma \otimes e^{i\chi})$
  - LN: cuspidal parabolic subgroup
  - $L = MA$: Langlands decomposition of L
  - σ: discrete series representation of M
  - $\chi \in \mathfrak{a}^\star$
- Send π to the irreducible subrepresentation of

$$\mathrm{Ind}_{MAN}^{G}(\sigma \otimes e^{i\frac{\chi}{\alpha}})$$

that has the same restriction to K as π.

What should a Mackey bijection look like?

Rescaling-invariant representations

- **For** G_0: representations of the form $\mathbf{M}_0(0, \mu)$, with $\mu \in \widehat{K}$.

copy of $\hat{K}$

- **For** G: reps that occur in some $\mathrm{Ind}_{MAN}^{G}(\sigma \otimes e^{i0})$, $\sigma \in \widehat{M}_{\mathrm{DS}}$:

Irreducible tempered reps of G **with real infinitesimal character**

Any bijection that commutes with the rescaling maps must induce a bijection between $\widehat{K}$ and $\widehat{G}_{\mathrm{RIC}}$.

Lowest K-types and a theorem of Vogan

- Vogan introduces a positive-valued function $\|\cdot\|_{\widehat{K}}$ on $\widehat{K}$
- For $R > 0$, there are only a finite number of $\lambda \in \widehat{K}$ such that $\|\lambda\|_{\widehat{K}} \leqslant R$.
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Theorem (Vogan 1981):

1. If π has real infinitesimal character, then π has a unique lowest K-type.
2. Inequivalent π in $\widehat{G}_{\mathrm{RIC}}$ have different lowest K-types.
3. Every K-type occurs as the lowest K-type of a representation in $\widehat{G}_{\mathrm{RIC}}$.

So there exists a unique bijection

$$\begin{aligned} \widehat{K} &\to \widehat{G}_{\mathrm{RIC}} \\ \mu &\mapsto \mathbf{V}_G(\mu) \end{aligned}$$

that is compatible with lowest K-types.

What should a Mackey bijection $\widehat{G_0} \to \widehat{G}_{\mathrm{temp}}$ look like?

If it is compatible with rescaling maps and preserves lowest K-types, then it must coincide with $\mu \mapsto \mathbf{V}_G(\mu)$ on $\widehat{K}$.

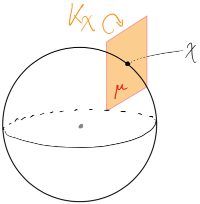
Every representation in $\widehat{G_0}$ reads $\mathbf{M}_0(\chi, \mu)$ for some (χ, μ).

Can we build a representation of G out of a pair (χ, μ) when $\chi \neq 0$?

Building a representation of G by a "Mackey recipe"

Main tool: action of K on $\mathfrak{p}^\star$.

Fix $\chi \in \mathfrak{p}^\star$ and $\mu \in \widehat{K_\chi}$.



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$$L_\chi = \{g \in G\ ,\ \mathrm{Ad}^\star(g)\chi = \chi\}$$
$$= \text{ centralizer of } \chi \text{ in } G.$$

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- Define
$$\sigma = \mathbf{V}_{L_\chi}(\mu) \otimes e^{i\chi},$$
a tempered irreducible representation of L_χ.

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a parabolic subgroup with Levi factor L_χ *(it contracts on $K_\chi \ltimes \mathfrak{p}$)*.

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- Inflate σ to a representation of P_χ, and define
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Theorem ($\sim$ 2016):

- $\mathbf{M}(\chi,\mu)$ is always irreducible ;
- The correspond^{ce} $(\chi,\mu) \mapsto \mathbf{M}(\chi,\mu)$ induces a map $\widehat{G_0} \to \widehat{G}_{\mathrm{temp}}$;
- **The correspondence is a bijection.**

The Mackey-Higson bijection: a few reasons to like it

$$\boxed{\mathrm{Ind}_{K_\chi \ltimes \mathfrak{p}}^{G_0}\left(\mu \otimes e^{i\chi}\right) \quad \longleftrightarrow \quad \mathrm{Ind}_{L_\chi N_\chi}^{G}\left(\mathbf{V}_{L_\chi}(\mu) \otimes e^{i\chi}\right)}$$

- preserves lowest K-types and commutes with the rescaling maps
- is a **piecewise homeomorphism**

  → *leads to a new proof of the Baum-Connes-Kasparov 'conjecture'*
- is **continuous from** $\widehat{G}_{\mathrm{temp}}$ **to** $\widehat{G_0}$ (joint work with A. M. Aubert)

  → *see recent work of Higson & Romàn on C^*-algebra embeddings*
- extends (easily) to a bijection between the **admissible duals**

  → *related to several results of Higson & Subag*

  → *related to work of Bernstein, Higson & Subag on algebraic families*

Deformations of tempered representations

One representation at a time

Pursuing contractions

Starting with (χ, μ),

- View it as a Mackey datum for $G \rightsquigarrow$ construction of $\pi \circlearrowright \mathbf{V}$
- For $t > 0$, view it as a Mackey datum for $G_t \rightsquigarrow$ construction of $\pi_t \circlearrowright \mathbf{V}_t$

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- A way to give this a meaning:
  - Embed all $\mathbf{V}_t$s in a common space $\mathbf{E}$.
  - This will determine **evolution operators $\mathbf{V} \to \mathbf{V}_t$**.

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- A way to give this a meaning:
  - Embed all $\mathbf{V}_t$s in a common space $\mathbf{E}$.
  - This will determine **evolution operators $\mathbf{V} \to \mathbf{V}_t$**.
  - Could there be a topology on $\mathbf{E}$ for which "everything" converges?

The discrete series

G: connected semisimple with $\text{rank}(G) = \text{rank}(K)$. Fix π in $\widehat{G}_{\text{discrete series}}$.

- We want to see **how π "contracts" onto its lowest K-type μ.**
- Parthasarathy-Atiyah-Schmid :

$$\pi \simeq \text{space of } \mathbf{L}^2 \text{ sol}^{ns} \text{ of a } \textbf{Dirac equation} \text{ on } G/K.$$

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For a special finite-dim. K-module $U = V^{\mu^\flat} \otimes S$ that contains μ exactly once,

• the equivariant bundle $\mathfrak{E} = G \times_K U$ over G/K

• and the G-invariant Dirac operator D (acting on smooth sections of $\mathfrak{E}$)

satisfy:

Theorem (Atiyah-Schmid – 1977) :

1. The $\mathbf{L}^2$ kernel of D carries an irreducible rep^n of G with class π.

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1. The $\mathbf{L}^2$ kernel of D carries an irreducible rep^n of G with class π.

2. In fact, sections in the $\mathbf{L}^2$ kernel of D explore the sub-bundle $G \times_K W^\mu$.

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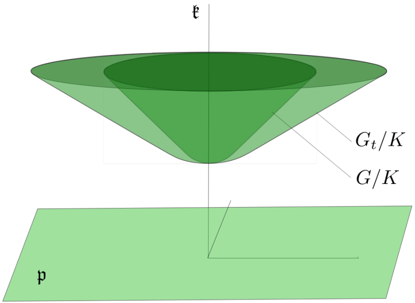
Transfer everything to $\mathfrak{p}$.



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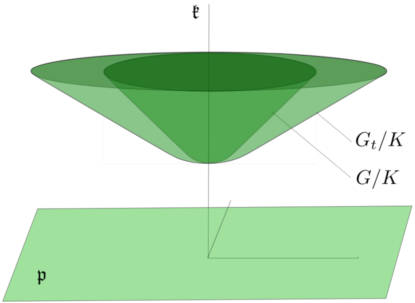

$\rightsquigarrow$ action of G_t and G_t-inv. metric on $\mathfrak{p}$.
$\rightsquigarrow$ **G_t-invariant Dirac operator**

$$D_t \circlearrowright \Gamma(\mathfrak{p}, V^{\mu^\flat} \otimes \mathcal{S}).$$

Trivialize and project $\rightsquigarrow$ diff^l operator

$$\Delta_t \circlearrowright \mathcal{C}^\infty(\mathfrak{p}, W^\mu)$$

$\mathbf{L}^2$ kernel $\mathbf{H}_t$ carries irred. rep. $\simeq \mathbf{V}_{G_t}(\mu)$.

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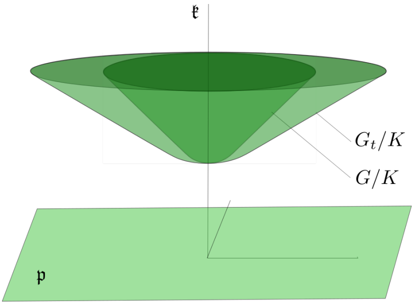


The dilation

$$z_t : \mathfrak{p} \to \mathfrak{p}$$
$$v \mapsto v/t$$

intertwines the G- and G_t- actions on $\mathfrak{p}$.

- Metrics η_t and $z_t^\star \eta_1$ are **proportional**
- "Dirac" operators Δ_t and Δ_1 are **conjugate** (up to $\times$ constant)
- **The zooming-in operator** $f \mapsto f \circ z_t^{-1}$ yields a **contraction $\mathbf{H} \to \mathbf{H}_t$**.

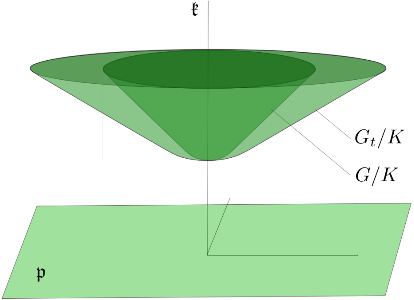
The discrete series

G: connected semisimple with $\mathrm{rank}(G) = \mathrm{rank}(K)$. Fix π in $\widehat{G}_{\text{discrete series}}$.

- We want to see **how π "contracts" onto its lowest K-type μ.**
- Parthasarathy-Atiyah-Schmid :

$$\pi \simeq \text{space of } \mathbf{L}^2 \text{ sol}^{ns} \text{ of a } \textbf{Dirac equation} \text{ on } G/K.$$

Transfer everything to $\mathfrak{p}$.



The zooming-in operator $f \mapsto f \circ z_t^{-1}$ defines a contraction $\mathbf{H} \to \mathbf{H}_t$.

The K-equivariant zooming-in eventually contracts $\mathbf{H}$ onto W^μ, for the top^{gy} of $\mathscr{C}^\infty(\mathfrak{p}, W^\mu)$.

In the K-isotypical subspaces of $\mathbf{H}$ for K-types other than μ, all sections vanish at 1_K.

The spherical principal series

The spherical principal series

Fix an Iwasawa decomposition $G = KAN$. For each χ in $\mathfrak{a}^\star$,

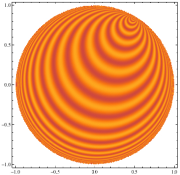
---

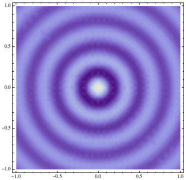
$$G/K \to \mathbb{C}$$
$$ne^H K \mapsto e^{\langle i\chi+\rho, H\rangle}$$

$\langle 2\rho, H\rangle := \mathbf{tr}\ \mathbf{ad}_{|\mathfrak{n}}(H)$

Analogue of a plane wave on G/K :
Horocycle wave $e_{\chi,b}$ *(Helgason)*

(eigenvector for all. G-invariant diff. ops,
constant on the orbits of some N-conjugate.)





Sum of all $e_{\chi,b}$s as b varies:
spherical function φ_χ *(Harish-Chandra)*

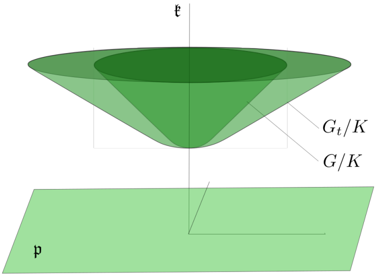
(K-invariant function,
eigenvector for all G-invariant diff.ops.)

$$\varphi_\chi : G/K \to \mathbb{C}$$
$$x \mapsto \int_K e_{\chi,b}(x)\,db$$

The spherical principal series

Start with **regular** χ in $\mathfrak{a}^\star$. Both $\mathbf{M}_0(\chi, 1)$ and $\mathbf{M}(\chi, 1)$ can be realized either

- on $\mathbf{H} = \mathbf{L}^2(K\text{-orbit of } \chi \text{ in } \mathfrak{p}^\star)$, or
- on a space of functions on $\mathfrak{p}$.



Helgason's waves transferred to $\mathfrak{p}$.

$$G_t \circlearrowright \left\{ \int_B \varepsilon^t_{\chi,b} F(b) db \mid F \in \mathbf{L}^2(B) \right\}$$

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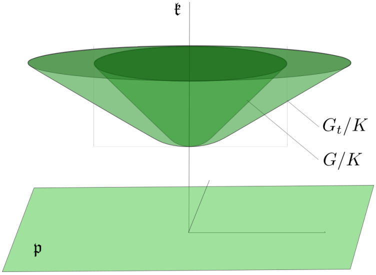

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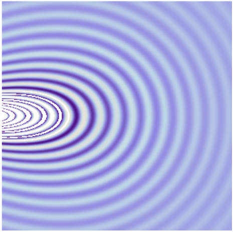

$\varepsilon^1_{\chi,b}$

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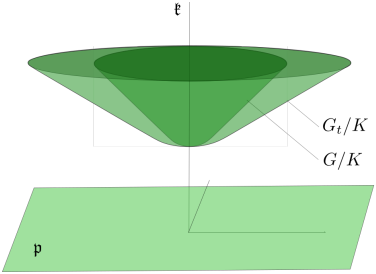

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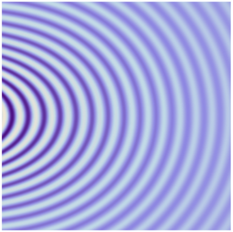

$\varepsilon^{0.5}_{\chi,b}$

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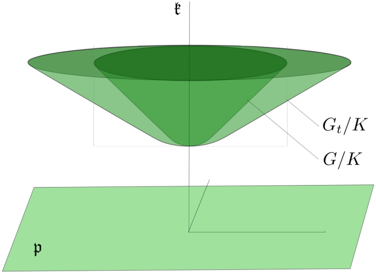

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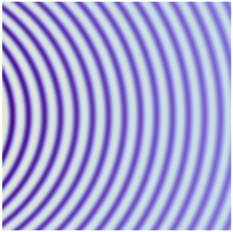

$\varepsilon^{0.25}_{\chi,b}$

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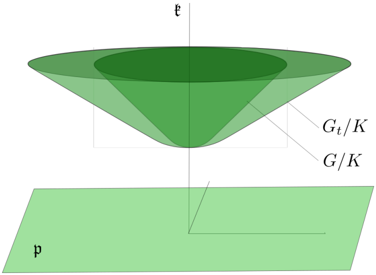

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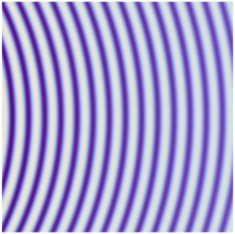

$\varepsilon^{0.125}_{\chi,b}$

The spherical principal series

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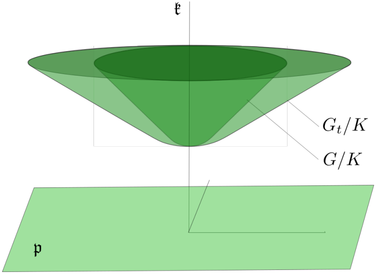

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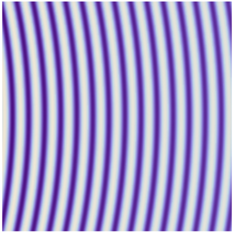

$\varepsilon^{0.0625}_{\chi,b}$

Deforming representations: contractions in Fréchet spaces

Start with $\pi \in \widehat{G}_{\mathrm{temp}}$.

- Look for a Fréchet space $\mathbf{E}$, and for each $t > 0$,
    - a subspace $\mathbf{V}_t \subset \mathbf{E}$
    - a map $\pi_t : K \times \mathfrak{p} \to \mathrm{End}(\mathbf{V}_t)$ defining a rep^n of G_t on $\mathbf{V}_t$,

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- Try to arrange the choice of $(\mathbf{E}, (\mathbf{V_t})_{t>0}, (\pi_t)_{t>0})$ so that as $t \to 0$,
  - $\forall f \in \mathbf{V}_1$, the vector $f_t := \mathbf{C}_t f$ goes to some limit f_0,
  - $\forall (k, v) \in K \times \mathfrak{p}$, $\pi_t(k, v)\,[f_t]$ goes to some limit $\pi_0(k, v) f_0$.

  Then one obtains a representation $(\mathbf{V}_0, \pi_0)$ of G_0.

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Theorem (2018-2020)

For every $\pi \in \widehat{G}_{\mathrm{temp}}$, a choice $(\mathbf{E}, (\mathbf{V_t})_{t>0}, (\pi_t)_{t>0})$ can be made so that the representation $(\mathbf{V}_0, \pi_0)$ of G_0 is irreducible and corresponds to π in the Mackey bijection.

Reduction to real infinitesimal character

- We saw how to contract representations of the form

$$\mathrm{Ind}_{MAN}^{G}(e^{i\chi})$$

where $L = MA$ is **minimal** and $\chi \in \mathfrak{a}^\star$ is regular.

- Every irreducible representation $\pi \in \widehat{G}_{\mathrm{temp}}$ reads

$$\mathrm{Ind}_{MAN}^{G}(\tau \otimes e^{i\chi})$$

where $L = MA$ parabolic and $\tau \in \widehat{M}_{\mathrm{temp}}$ has **real infinitesimal character**.

- To contract arbitrary tempered representations, we can proceed in two steps:

  1. find a contraction for real-infinitesimal-character representations;
  2. use the previous ideas to reduce the general case to that one.

The second step is full of technicalities... I will now focus on the first.

Real infinitesimal character: a strategy

Fix $\pi \in \widehat{G}_{\mathrm{RIC}}$. Work of Vogan-Zuckerman, Knapp-Vogan, Wong:

Realizing π in a Dolbeault cohomology space for an elliptic coadjoint orbit G/L

Real infinitesimal character: a strategy

Fix $\pi \in \widehat{G}_{\mathrm{RIC}}$. Work of Vogan-Zuckerman, Knapp-Vogan, Wong:

Realizing π in a Dolbeault cohomology space for an elliptic coadjoint orbit G/L

There exists

- a *quasi-split* Levi subgroup $L \subset G$
- an irreducible representation (V, σ) with real infinitesimal character,
- and an infinite-rank bundle

$$\mathcal{V}^\sharp \xrightarrow{V^\sharp} G/L \qquad V^\sharp : V \text{ twisted by a character of } L$$

such that

$$\pi \simeq H^{(0,s)}(G/L, \mathcal{V}^\sharp) \qquad \text{where } s = \dim_{\mathbb{C}}(K/(K \cap L)).$$

Real infinitesimal character: a strategy, continued

Realizing π in a Dolbeault cohomology space:

$$\pi \simeq H^{(0,s)}(G/L, \mathcal{V}^\sharp),$$

- L: a quasi-split Levi subgroup
- $s = \dim_{\mathbb{C}}(K/(K \cap L))$
- V: an irreducible tempered representation of L with real infinitesimal character
- $V^\sharp$: a twist of V by a character of L

A theorem of Mostow (1955) on the structure of G/L:

There exists a subspace $\mathfrak{s} \subset \mathfrak{g}$, stable under $\mathrm{Ad}(K \cap L)$, s.t. the Cartan map

$$K \times \mathfrak{s} \to G$$

factors through the quotient $(K \times \mathfrak{s}) \to (K \times \mathfrak{s})/(K \cap L)$, and

$$G/L \simeq (K \times \mathfrak{s})/(K \cap L).$$

Contracting real-infinitesimal-character representations

Start with a real-infinitesimal-character representation π, realize it on

$$\mathcal{H}_{\mathsf{start}} = H^{(0,s)}(G/L, \mathcal{V}^\sharp) \qquad\qquad s = \dim_{\mathbb{C}}(K/(K \cap L)),$$

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We can then associate to every element of $\mathcal{H}_{\text{start}}$ an element of

$$\mathcal{H}_{\text{contracted}} = H^{(0,s)}(K/(K \cap L), \mathcal{W}^\sharp) \qquad \text{où } s = \dim_{\mathbb{C}}(K/(K \cap L))$$

Theorem (2019) :

The K-module $\mathcal{H}_{\text{contracted}}$ yields a realization for the lowest K-type of π.

Quasi-split groups and fine K-types

Last case that remains to be settled:

- G: quasi-split group
- π: real-infinitesimal-character irr. representation with a 'fine' K-type.

In that case

$$\pi \hookrightarrow \mathrm{Ind}_{MAN}(\zeta)$$

where $L = MA$ is minimal, M **is abelian and** ζ **is a character of** M.

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"Theorem" (2018-2020):

Can realize π on a space of sections of the bundle $G \times_K V^\mu$ over G/K

The construction mimics Helgason "waves", in a rather unusual setting.

I will spare you the details...

Three slogans

- There is a simple and natural bijection between the irreducible representations of G and those of $G_0 = K \ltimes \mathfrak{p}$.
- Realizing it as a deformation is possible, but (at the moment) uses the fine details of available constructions.
- Understanding it more conceptually seems to remain challenging.