Decomposition of Haar measures

Let G be a Lie group and let S and T be two closed subgroups of G. Assume that

a. the set $\Omega := \{st : s \in S, t \in T\}$ is open and dense in G

b. $K := S \cap T$ is compact.

The aim of this exercise is to express the left Haar measure of G in terms of the left Haar measures of S and T.

Note that Ω has a natural structure of a manifold induced from the manifold structure of G.

(i) Consider the map $(S \times T) \times \Omega \to \Omega$; $\big((s,t),x\big) \mapsto sxt^{-1}$. Show that it is a smooth action of $S \times T$ on Ω. Prove that the stabilizer of $e \in \Omega$ equals the subgroup

$$\text{diag}(K) := \{(k,k) \in S \times T : k \in K\}.$$

Prove that

$$\Phi : (S \times T)/\text{diag}(K) \to \Omega; \qquad (s,t) \cdot \text{diag}(K) \mapsto st^{-1} \tag{1}$$

is a diffeomorphism.

(ii) Prove that for every Radon measure μ_Ω on Ω there exists a unique Radon measure ν on $(S \times T)/\text{diag}(K)$, such that

$$\int_{(S\times T)/\text{diag}(K)} \phi \circ \Phi(x)\, \nu(x) = \int_\Omega \phi(y)\, \mu_\Omega(y) \qquad \big(\phi \in C_c(\Omega)\big). \tag{2}$$

Let μ_G be a left Haar measure on G and let μ_Ω be the restriction of μ_G to Ω.

(iii) Show that μ_Ω is a Radon measure on Ω.

(iv) Show that for every $\phi \in C_c(G)$

$$\int_G \phi(x)\, \mu_G(x) = \int_\Omega \phi(x)\, \mu_\Omega(x).$$

Let ν be the Radon measure on $(S \times T)/\text{diag}(K)$ such that (2) holds and let μ_K be a left Haar measure on K that is normalized such that the volume of K equals 1. (Recall that K is assumed to be compact.)

(v) For $\psi \in C_c(S \times T)$, we define $\overline{\psi} : (S \times T)/\text{diag}(K) \to \mathbf{C}$ to be the function given by

$$\overline{\psi}\big((s,t) \cdot \text{diag}(K)\big) = \int_K \psi(sk, tk)\, \mu_K(k).$$

Show that $\overline{\psi}$ is a compactly supported continuous function on $(S \times T)/\text{diag}(K)$ for every $\psi \in C_c(S \times T)$.

(vi) Prove that

$$C_c(S \times T) \ni \psi \mapsto \int_{(S\times T)/\mathrm{diag}(K)} \overline{\psi}(x)\,\nu(x)$$

defines a Radon measure on $S \times T$. We denote this measure by η.

(vii) Recall the definition of Φ from (1). Let π be the projection $S\times T \to (S\times T)/\mathrm{diag}(K)$. Prove that

$$\int_G \phi(x)\,\mu_G(x) = \int_{S\times T} \phi \circ \Phi \circ \pi(s,t)\,\eta(s,t)$$

for every $\phi \in C_c(G)$.

We write Δ_G for the modular function of G, i.e., the function $G \to \mathbf{R}_+$ given by

$$\Delta(g) = |\det \mathrm{Ad}(g)|.$$

Similarly, we write Δ_T for the modular function of T.

(viii) Let $\psi \in C_c(S \times T)$. Prove that for every $s_0 \in S$ and $t_0 \in T$

$$\int_{S\times T} \psi(s_0 s, t_0 t)\,\eta(s,t) = \int_{S\times T} \psi(s,t)\Delta_G(t_0)^{-1}\,\eta(s,t).$$

Prove that

$$C_c(S \times T) \ni \psi \mapsto \int_{S\times T} \psi(s,t)\Delta_G(t)\,\eta(s,t)$$

defines a left Haar measure of $S \times T$.

Let μ_S and μ_T be left Haar measure on S and T respectively.

(ix) Prove that

$$C_c(S \times T) \ni \psi \mapsto \int_S \int_T \psi(s,t^{-1})\Delta_T(t)\,\mu_T(t)\,\mu_S(s)$$

defines a left Haar measure of $S \times T$. Show that there exists a constant $c > 0$ such that

$$\int_{S\times T} \psi(s,t)\,\eta(s,t) = c\int_S \int_T \psi(s,t^{-1})\frac{\Delta_T(t)}{\Delta_G(t)}\,\mu_T(t)\,\mu_S(s) \qquad \big(\psi \in C_c(S\times T)\big)$$

(x) Prove that for every $\phi \in C_c(S \times T)$ we have

$$\int_G \phi(x)\,\mu_G(x) = \int_S \int_T \phi(st)\frac{\Delta_T(t)}{\Delta_G(t)}\,\mu_T(t)\,\mu_S(s).$$

In the remainder of this exercise we treat some examples.

(xi) Let $\mathbf{GL}_+(n,\mathbf{R})$ be the group of $n\times n$-matrices with strictly positive determinant. Note that $\mathbf{GL}_+(n,\mathbf{R})$ is an open submanifold of the vector space $\mathrm{Mat}(n,\mathbf{R})$. Let λ be the Lebesgue measure on $\mathrm{Mat}(n,\mathbf{R})$. Show that

$$C_c(\mathbf{GL}_+(n,\mathbf{R})) \ni \phi \mapsto \int_{\mathbf{GL}_+(n,\mathbf{R})} \phi(x)\det(x)^{-n}\lambda(x)$$

defines both a left and right Haar measure of $\mathbf{GL}_+(n,\mathbf{R})$. Prove that $\mathbf{GL}_+(n,\mathbf{R})$ is unimodular.

(xii) Prove that

$$\mathbf{GL}_+(n,\mathbf{R}) \to \mathbf{R}_+ \times \mathbf{SL}(n,\mathbf{R}); \qquad g \mapsto \big(\det g, (\det g)^{-\frac{1}{n}} g\big) \tag{3}$$

is a Lie isomorphism.

(xiii) Let μ be a Haar measure on $\mathbf{SL}(n,\mathbf{R})$. Prove that there exists a positive constant c such that

$$\int_{\mathbf{GL}_+(n,\mathbf{R})} \phi(x)(\det x)^{-n}\,\lambda(x) = c\int_{\mathbf{R}_+}\int_{\mathbf{SL}(n,\mathbf{R})} \phi(ry)\,\mu(y)\,\frac{dr}{r}$$

for every $\phi \in C_c(\mathbf{GL}_+(n,\mathbf{R}))$.

Note that a Lie group is unimodular if and only if its modular function equals the constant 1.

(xiv) Let G_1 and G_2 be two Lie groups. Let $\Delta_{G_1\times G_2}$, Δ_{G_1} and Δ_{G_2} be the modular functions of $G_1\times G_2$, G_1 and G_2 respectively. Prove that $\Delta_{G_1\times G_2}(g_1,g_2) = \Delta_{G_1}(g_1)\Delta_{G_2}(g_2)$ for every $(g_1,g_2)\in G_1\times G_2$.

(xv) Use (3) to show that $\Delta_{\mathbf{SL}(n,\mathbf{R})}$ equals the constant 1, or equivalently, prove that $\mathbf{SL}(n,\mathbf{R})$ is unimodular.

Let $K = \mathbf{SO}(n)$, A the subgroup of $\mathbf{SL}(n,\mathbf{R})$ consisting of diagonal matrices with strictly positive diagonal entries, and N the subgroup of $\mathbf{SL}(n,\mathbf{R})$ consisting of upper triangular matrices with each diagonal entry equal to 1. In a previous exercise we have shown that AN is a closed Lie subgroup of $\mathbf{SL}(n,\mathbf{R})$ (as well as K, A and N) and that the maps

$$A\times N \to AN; \qquad (a,n)\mapsto an$$

$$K\times AN \to \mathbf{SL}(n,\mathbf{R}); \qquad (k,x)\mapsto kx$$

are diffeomorphisms. We write Δ_A, Δ_N and Δ_{AN} for the modular functions of A, N and AN respectively and μ_K, μ_A and μ_N for left Haar measures on K, A, and N respectively.

(xvi) Prove that there exists a constant $c>0$ such that for every $\phi\in C_c(\mathbf{SL}(n,\mathbf{R}))$

$$\int_{\mathbf{SL}(n,\mathbf{R})} \phi(x)\,\mu(x) = c\int_K\int_A\int_N \phi(kan)\frac{\Delta_{AN}(an)\Delta_N(n)}{\Delta_{AN}(n)}\,\mu_N(n)\,\mu_A(a)\,\mu_K(k)$$

A relatively easy computation shows that the exponential mappings of A and N are diffeomorphism. Both groups are unimodular and the Haar measures are given by the Lebesgue measures on the respective Lie algebras. The group AN is not unimodular; in fact

$$\Delta(an) = \prod_{j=1}^{n} a_{jj}^{n+1-2j}.$$

This leads to the integral formula

$$\int_{\mathbf{SL}(n,\mathbf{R})} \phi(x)\,\mu(x) = c \int_K \int_{\mathfrak{a}} \int_{\mathfrak{n}} \phi\big(k \exp(X) \exp(Y)\big) e^{\sum_{j=1}^{n}(n+1-2j)X_{j,j}}\, dY\, dX\, \mu_K(k),$$

where dY is the Lebesgue measure on $\mathfrak{n}$ and dX is the Lebesgue measure on $\mathfrak{a}$.