PERTURBATION THEOREMS FOR α-TIMES INTEGRATED SEMIGROUPS

CORNELIA KAISER AND LUTZ WEIS

ABSTRACT. We prove perturbation results for α-times integrated semigroups assuming relative "smallness" conditions for the perturbation B on a halfplane. If A is a semigroup generator on a uniformly convex Banach space, then these conditions on B already imply that $A + B$ generates a once integrated semigroup. As an illustration we consider Schrödinger operators and higher order differential operators.

1. INTRODUCTION

Perturbation theory for operator semigroups is an important tool in applications to differential equations and therefore it is a richly developed field. Most of these perturbation theorems assume relative boundedness of the perturbation B, and moreover a "relative smallness" condition that amounts to an estimate

$$\|B(\lambda - A)^{-1}\| \leq M < 1 \tag{1}$$

or

$$\|(\lambda - A)^{-1}Bx\| \leq M\|x\| \tag{2}$$

on a certain subset of the complex plane. In all these results one needs further assumptions either on the generator A or on the perturbation B (e.g., analyticity or contractivity conditions). Such additional conditions are indeed necessary, since in general (1) or (2) by themselves do not guarantee that $A + B$ is a semigroup generator (see Example 7.1). But a somewhat weaker result is true. In this paper we show that if the relative boundedness condition (1) or (2) holds for λ in a halfplane, then $A + B$ generates an α-times integrated semigroup where the rate of integration α depends on the geometry of the underlying Banach space X. E.g., if X is uniformly convex, then $A+B$ generates a once integrated semigroup. These results are consequences of a more general perturbation theorem for α-times integrated semigroups which is of some interest in itself. Aside from some special results in [9, Section I.5] and [15] it seems to be the first genuine perturbation theorem for α-times integrated semigroups.

Integrated semigroups where introduced by Arendt [2, 3] to study resolvent positive operators. In [2] there is a perturbation theorem for resolvent positive operators that is closely related to our results. Hieber [9] refined the theory by introducing α-times integrated semigroups for positive real numbers α.

Integrated semigroups are a natural extension of semigroup theory to deal with operators that have polynomially bounded resolvents in a halfplane and for which the Cauchy problem is solvable for $x \in D(A^\alpha)$, $\alpha > 1$. One important example

2000 *Mathematics Subject Classification.* Primary 47A55, 47D62. Secondary 47D06, 47E05.
Key words and phrases. α-times integrated semigroup, perturbation.
The research is supported in part by the *Landesforschungsschwerpunkt Evolutionsgleichungen des Landes Baden-Württenberg.*

is the Schrödinger operator $i\Delta$ on L^p-spaces. Hörmander [12] proved in 1960 that $i\Delta$ generates a C_0-semigroup on $L^p(\mathbb{R}^n)$ if and only if $p = 2$. But Hieber [9, 10] showed that the Schrödinger operator generates an α-times integrated semigroup on $L^p(\mathbb{R}^n)$ for $\alpha > n|\frac{1}{2} - \frac{1}{p}|$. Other examples are second order Cauchy problems [4, 17] and delay equations [1].

We apply our perturbation theorems to the Schrödinger operator in one dimension: If one adds a potential $V \in L^p + L^\infty$, the sum $i\frac{d^2}{dx^2} + V$ generates a β-times integrated semigroup. Similar results hold also for higher order differential operators (see Section 8). For an application to delay equations see [13].

2. α-times integrated semigroups

Let X be a Banach space. By $\mathcal{L}(X)$ we denote the space of all bounded linear operators from X to X. We recall the definition of an α-times integrated semigroup.

Definition 2.1. Let $\alpha \geq 0$ and $(A, D(A))$ be a linear operator on X. A is called *generator of an α-times integrated semigroup* if there are nonnegative numbers ω, M and a mapping $S: [0, \infty) \to \mathcal{L}(X)$ such that

- $(S(t))_{t\geq 0}$ is strongly continuous and $\| \int_0^t S(s)\, ds\| \leq Me^{\omega t}$ for all $t \geq 0$,
- (ω, ∞) is contained in the resolvent set $\rho(A)$ of A, and
- $R(\lambda, A) := (\lambda - A)^{-1} = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)\, dt$ for $\lambda > \omega$.

In this case, the family $(S(t))_{t\geq 0}$ is the *α-times integrated semigroup* generated by A.

Remarks (1) If $(A, D(A))$ generates an α-times integrated semigroup $(S(t))_{t\geq 0}$, then the halfplane $\{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > \omega\}$ is contained in $\rho(A)$ and $R(\lambda, A) = \lambda^\alpha \int_0^\infty e^{-\lambda t} S(t)dt$ for all $\operatorname{Re}\lambda > \omega$.
(2) By uniqueness of the Laplace transform, $(S(t))_{t\geq 0}$ is uniquely determined.
(3) If $\alpha = 0$, the definition above is consistent with the definition of a C_0-semigroup (see [4, Theorem 3.1.7]). In this case the generator A is densely defined and $(S(t))_{t\geq 0}$ is exponentially bounded. For $\alpha > 0$ this may not be true in general.
(4) If A generates an α-times integrated semigroup $(S_\alpha(t))_{t\geq 0}$, then A also generates a β-times integrated semigroup $(S_\beta(t))_{t\geq 0}$ for each $\beta > \alpha$.
(5) If A generates an α-times integrated semigroup $(S(t))_{t\geq 0}$, then the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t \in [0, \tau], \\ u(0) = x, \end{cases} \tag{3}$$

has a unique classical solution for each $x \in D(A^{n+1})$ where $n \in \mathbb{N}_0$ such that $n-1 < \alpha \leq n$ ([9]). By a classical solution of (3) we mean a function $u \in C^1([0, \infty), X)$ such that $u(t) \in D(A)$ for all $t \geq 0$ and (3) is satisfied.

3. Main Results

Let $(A, D(A))$ be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ on X and let

$$\omega(S) := \inf\{\omega \in \mathbb{R} : \exists K \geq 0 \text{ such that } \|S(t)\| \leq Ke^{\omega t}\}$$

be the *growth bound* of S if $(S(t))_{t\geq 0}$ is exponentially bounded. If not let

$$\omega(S) := \inf\Big\{\omega \in \mathbb{R} : \exists K \geq 0 \text{ such that } \Big\|\int_0^t S(s)ds\Big\| \leq Ke^{\omega t}\Big\}.$$

We consider a linear operator $(B, D(B))$ in X that satisfies one of the following conditions:

(C1) $D(B) \supseteq D(A)$ and there are constants $\lambda_0 > \max\{0, \omega(S)\}$ and $M < 1$ such that
$$\|BR(\lambda, A)\| \leq M$$
for all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda = \lambda_0$.

(C2) B is densely defined and there are constants $\lambda_0 > \max\{0, \omega(S)\}$ and $M < 1$ such that
$$\|R(\lambda, A)Bx\| \leq M\|x\|$$
for all $x \in D(B)$ and all $\lambda \in \mathbb{C}$ with $\operatorname{Re} \lambda = \lambda_0$.

Our first result is the following perturbation theorem for α-times integrated semigroups.

Theorem 3.1. *Let $(A, D(A))$ be the generator of an α-times integrated semigroup $(S(t))_{t\geq 0}$ on X and let $(B, D(B))$ be a linear operator in X. Choose $\beta > \alpha + 1$ if $(S(t))_{t\geq 0}$ is exponentially bounded and $\beta > \alpha + 2$ in the general case.*

(a) *If* (C1) *holds, then $(A+B, D(A))$ generates a β-times integrated semigroup.*
(b) *If we assume* (C2)*, then a closed extension $(C, D(C))$ of $(A + B, D(A) \cap D(B))$ generates a β-times integrated semigroup. If A and its adjoint A^* are densely defined, then C is the part of $(A^* + B^*)^*$ in X, i.e., $Cx = (A^* + B^*)^*x$ for $x \in D(C) = \{x \in D((A^* + B^*)^*) \cap X : (A^* + B^*)^* \in X\}$.*

Under certain assumptions on the geometry of the Banach space X one can improve the bound for β. For this we need the following definition:

Definition 3.2. A Banach space X has *Fourier type* $p \in [1, 2]$ if the Fourier transform extends to a bounded linear operator from $L^p(\mathbb{R}, X)$ to $L^{p'}(\mathbb{R}, X)$ where $\frac{1}{p} + \frac{1}{p'} = 1$.

Each Banach space has Fourier type 1. A Banach space has Fourier type 2 if and only if it is isomorphic to a Hilbert space ([16]). If X has Fourier type p, then it has Fourier type r for each $r \in [1, p]$. Each closed subspace, each quotient space and the dual space X^* of a Banach space X has the same Fourier type as X. The space $L^r(\Omega, \mu)$ has Fourier type $\min\{r, \frac{r}{r-1}\}$ ([19]). Each B-convex Banach space has Fourier type $p > 1$ ([5, 6]).

If we take the Fourier type of X into consideration, we obtain the following refined version of our perturbation result with optimal lower bound for β (cf. Section 7).

Theorem 3.3. *Let X be a Banach space of Fourier type $p \in [1, 2]$. Let $(A, D(A))$ be the generator of an exponentially bounded α-times integrated semigroup $(S(t))_{t\geq 0}$ on X and let $(B, D(B))$ be a linear operator in X. Choose $\beta > \alpha + \frac{1}{p}$.*

(a) *If A is densely defined and* (C1) *holds, then $(A + B, D(A))$ generates a β-times integrated semigroup.*
(b) *If we assume* (C2)*, then a closed extension $(C, D(C))$ of $(A + B, D(A) \cap D(B))$ generates a β-times integrated semigroup. If A and A^* are densely defined, then C is the part of $(A^* + B^*)^*$ in X.*

As a corollary we obtain the following perturbation result for C_0-semigroups on B-convex Banach spaces.

Corollary 3.4. *Let $(A, D(A))$ be the generator of a C_0-semigroup on a B-convex Banach space X and let $(B, D(B))$ be a linear operator in X.*

1. *If* (C1) *holds then $(A + B, D(A))$ generates a once integrated semigroup.*
2. *If we assume* (C2) *then a closed extension $(C, D(C))$ of $(A + B, D(A) \cap D(B))$ generates a once integrated semigroup. If A and A^* are densely defined then C is the part of $(A^* + B^*)^*$ in X.*

4. Existence and representation of the resolvent of $A + B$

In this section we collect some results on the existence and representation of the resolvent of the sum of two linear operators A and B. We assume that the resolvent set of A is nonempty. Our first lemma can be used if condition (C1) from Section 3 is satisfied.

Lemma 4.1. *Let $(A, D(A))$ and $(B, D(B))$ be linear operators in X such that $D(A) \subseteq D(B)$. If there is $\lambda \in \rho(A)$ such that $\|BR(\lambda, A)\| < 1$, then $\lambda \in \rho(A + B)$ and*

$$R(\lambda, A + B) = R(\lambda, A)[I - BR(\lambda, A)]^{-1} = R(\lambda, A) \sum_{k=0}^{\infty} [BR(\lambda, A)]^k.$$

Proof. Our assumptions yield that $I - BR(\lambda, A)$ is invertible in $\mathcal{L}(X)$ and that

$$[I - BR(\lambda, A)]^{-1} = \sum_{k=0}^{\infty} [BR(\lambda, A)]^k.$$

Now it is easy to show that $\lambda \in \rho(A+B)$ and $R(\lambda, A+B) = R(\lambda, A)[I-BR(\lambda, A)]^{-1}$. □

The next lemma is related to condition (C2).

Lemma 4.2. *Let $(A, D(A))$ and $(B, D(B))$ be linear operators in X. We assume that there are a nonempty subset G of $\rho(A)$, a subset D of $D(B)$ that is dense in X and a constant $M < 1$ such that $\|R(\lambda, A)Bx\| \le M\|x\|$ for all $x \in D$ and all $\lambda \in G$. Then the following assertions hold:*

(a) *There is a closed extension $(C, D(C))$ of $(A + B, D(A) \cap D(B))$ such that $G \subseteq \rho(C)$ and*

$$R(\lambda, C) = [I - R(\lambda, A)B]^{-1} R(\lambda, A) = \sum_{k=0}^{\infty} [R(\lambda, A)B]^k R(\lambda, A)$$

for all $\lambda \in G$.

(b) *If A and B are densely defined, then $D(A^*) \subseteq D(B^*)$ and $\|B^* R(\lambda, A^*)\| \le M$ for all $\lambda \in G$.*

(c) *If moreover $\overline{D(A^*)} = X^*$, then the operator C from (a) is the part of $(A^* + B^*)^*$ in X.*

Proof. (a) For $\lambda \in G$ we can extend $R(\lambda, A)B$ to a bounded operator on X with norm $\le M$. We denote this (unique) extension also by $R(\lambda, A)B$. Then $I - R(\lambda, A)B$ is invertible in $\mathcal{L}(X)$ and

$$R_\lambda := [I - R(\lambda, A)B]^{-1} R(\lambda, A) = \sum_{k=0}^{\infty} [R(\lambda, A)B]^k R(\lambda, A).$$

We fix $\lambda \in G$ and define

$$\begin{aligned} D(C) &= \operatorname{Ran} R_\lambda, \\ C &= \lambda I - R_\lambda^{-1}. \end{aligned}$$

Using the theory on pseudo resolvents ([18, Section 1.9]), one can show that $(C, D(C))$ does not depend on $\lambda \in G$. Moreover, $R_\mu = R(\mu, C)$ for all $\mu \in G$ and $(C, D(C))$ is a closed extension of $(A + B, D(A) \cap D(B))$.

(b) Since A and B are densely defined, the adjoint operators A^* and B^* are well-defined. Let $y^* \in D(A^*)$ and $\lambda \in G$. Then there is $x^* \in X^*$ with $y^* = R(\lambda, A^*)x^*$ and for all $x \in D$ we obtain

$$\langle y^*, Bx\rangle = \langle R(\lambda, A^*)x^*, Bx\rangle = \langle R(\lambda, A)^*x^*, Bx\rangle = \langle x^*, R(\lambda, A)Bx\rangle.$$

Therefore $y^* \in D(B^*)$ and $\|B^*y^*\| \leq M\|x^*\|$.

(c) From (b) and Lemma 4.1 we obtain that $(A^* + B^*, D(A^*))$ is closed, $G \subseteq \rho(A^* + B^*)$ and $R(\lambda, A^* + B^*) = R(\lambda, A^*)[I - B^*R(\lambda, A^*)]^{-1}$ for each $\lambda \in G$. Moreover it is easy to show that $R(\lambda, A^* + B^*) = R(\lambda, C)^*$.
If $D(A^*)$ is dense in X^*, then the adjoint $(A^* + B^*)^*$ of $(A^* + B^*, D(A^*))$ is well-defined and

$$\begin{aligned} D(C) &= R(\lambda, C)(X) \;=\; R(\lambda, (A^* + B^*)^*)(X) \\ &= \{x \in X \cap D((A^* + B^*)^*) : \ (A^* + B^*)^*x \in X\}. \end{aligned}$$

This means that C is the part of $(A^* + B^*)^*$ in X. □

5. Proof of Theorem 3.1

In the proof of Theorem 3.1 we use the following result from ([9, Theorem 5.1]).

Proposition 5.1. *Let X be a Banach space and $(A, D(A))$ a linear operator in X. If there are numbers $\omega, L \geq 0$ and $\tau \geq -1$ such that*

- *$\{\lambda \in \mathbb{C} : \ \operatorname{Re}\lambda > \omega\} \subseteq \rho(A)$ and*
- *$\|R(\lambda, A)\| \leq L|\lambda|^\tau$ for $\operatorname{Re}\lambda > \omega$,*

then A generates an α-times integrated semigroup for each $\alpha > \tau + 1$.

Proof of Theorem 3.1. (a) We first consider the case that $(S(t))_{t\geq 0}$ is exponentially bounded. Since $(A, D(A))$ generates an α-times integrated semigroup we obtain the estimate

$$\|R(\lambda, A)\| \leq |\lambda|^\alpha \int_0^\infty e^{-\operatorname{Re}\lambda t}\|S(t)\|dt \leq K|\lambda|^\alpha(\operatorname{Re}\lambda - \omega)^{-1} \tag{4}$$

for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq \lambda_0$. Here $\omega \in (\omega(S), \lambda_0)$ and $K \geq 0$ are chosen such that $\|S(t)\| \leq Ke^{\omega t}$.

For $\mu \in \mathbb{C}$ with $\operatorname{Re}\mu > \lambda_0$ we put $\lambda := \lambda_0 + i\operatorname{Im}\mu$. The resolvent equation yields $R(\mu, A) = R(\lambda, A)[I + (\lambda - \mu)R(\mu, A)]$. Then

$$\begin{aligned} \|BR(\mu, A)\| &\leq \|BR(\lambda, A)\|\|I + (\lambda - \mu)R(\mu, A)\| \\ &\leq M\big[1 + |\lambda - \mu| \cdot K|\mu|^\alpha(\operatorname{Re}\mu - \omega)^{-1}\big] \\ &\leq M(1 + K|\mu|^\alpha) \end{aligned}$$

and $BR(\mu, A)$ satisfies the assumptions of the Phragmen-Lindelöf theorem (see e.g. [7]), which then yields that $\|BR(\lambda, A)\| \leq M$ for all $\lambda \in H_{\lambda_0} = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq \lambda_0\}$. By Lemma 4.1, H_{λ_0} is contained in $\rho(A + B)$ and $R(\lambda, A + B) = R(\lambda, A)[I -$

$BR(\lambda, A)]^{-1}$ for all $\lambda \in H_{\lambda_0}$. Now by (4) there is a constant $L \geq 0$ such that for all $\lambda \in H_{\lambda_0}$ the estimate

$$\|R(\lambda, C)\| \leq \|R(\lambda, A)\| \|[I - BR(\lambda, A)]^{-1}\| \leq L|\lambda|^{\alpha}$$

is satisfied. Our claim now follows from Proposition 5.1.

In the general case (where $(S(t))_{t\geq 0}$ is not exponentially bounded) we use the estimate

$$\|R(\lambda, A)\| = \left\|\lambda^{\alpha+1} \int_0^\infty e^{-\lambda t} \int_0^t S(s) ds\, dt\right\| \leq K|\lambda|^{\alpha+1}(\operatorname{Re}\lambda - \omega)^{-1}$$

instead of (4) where $\omega \in (\omega(S), \lambda_0)$ and $K \geq 0$ are chosen such that $\|\int_0^t S(s)ds\| \leq Ke^{\omega t}$. Then we can proceed in the same way as above.

(b) Since $D(B)$ is dense in X, we can extend $R(\lambda, A)B$ for each $\lambda \in \lambda_0 + i\mathbb{R}$ to a bounded linear operator on X with norm $\leq M$. We denote this operator again by $R(\lambda, A)B$. Now the assertion can be proved in the same way as (a) using Lemma 4.2 instead of Lemma 4.1. □

6. Proof of Theorem 3.3

The case $p = 1$ we have already proved above. Let $p \in (1, 2]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Observe that for $x \in X$, $r \geq \lambda_0$ and $s \in \mathbb{R}$ we have

$$\int_0^\infty (e^{-rt}\|S(t)x\|)^p dt \leq c_1\|x\|^p \tag{5}$$

and

$$(r - is)^{-\alpha}R(r - is, A)x = \int_0^\infty e^{ist}(e^{-rt}S(t)x)dt. \tag{6}$$

We first prove (b). Since X has Fourier type p, we obtain that

$$\int_{-\infty}^\infty \|(r + is)^{-\alpha}R(r + is, A)x\|^q ds \leq c_2\|x\|^q$$

for all $r \geq \lambda_0$ and all $x \in X$. As in the proof of Theorem 3.1 we use the Phragmen-Lindelöf theorem and Lemma 4.2 to show that there exists a closed extension $(C, D(C))$ of $(A + B, D(A) \cap D(B))$ such that for $\operatorname{Re}\lambda \geq \lambda_0$ the resolvent can be written as $R(\lambda, C) = [I - R(\lambda, A)B]^{-1}R(\lambda, A)$. This yields

$$\int_{-\infty}^\infty \|(r + is)^{-\alpha}R(r + is, C)x\|^q ds \leq c_3\|x\|^q.$$

Moreover, $\lambda^{-\alpha}R(\lambda, C)$ is holomorphic for $\operatorname{Re}\lambda \geq \lambda_0$.

Let $\gamma > \frac{1}{p}$. For $t \geq 0$ and $x \in X$ we define

$$U(t)x := \frac{1}{2\pi i}\int_{\operatorname{Re}\lambda = \lambda_0} e^{\lambda t}\lambda^{-\gamma}[\lambda^{-\alpha}R(\lambda, C)x]d\lambda.$$

By Hölder's inequality, $U(t) \in \mathcal{L}(X)$. Using the Riemann-Lebesgue-Lemma and [11, Theorem 6.6.1], we obtain that $(U(t))_{t\geq 0}$ is strongly continuous and

$$\lambda^{-\alpha}R(\lambda, C) = \lambda^{\gamma}\int_0^\infty e^{-\lambda t}U(t)\ dt$$

for each $\operatorname{Re}\lambda > \lambda_0$. The claim now follows with Definition 2.1.

To prove (a), we first observe that (5) and (6) also hold if we replace $S(t)$ by its adjoint $S(t)^*$ and x by $x^* \in X^*$. Recall that X^* has Fourier type p since X has. So we obtain in the same way as above that

$$\int_{-\infty}^{\infty} \|(r+is)^{-\alpha} R(r+is, A+B)^* x^*\|^q ds \leq c\|x^*\|^q$$

for all $r \geq \lambda_0$ and all $x^* \in X^*$.

Again let $\gamma > \frac{1}{p}$. For $t \geq 0$ and $x^* \in X^*$ define

$$U^*(t)x^* := \frac{1}{2\pi i} \int_{\operatorname{Re} \lambda = \lambda_0} e^{\lambda t} \lambda^{-\gamma} [\lambda^{-\alpha} R(\lambda, A+B)^* x^*] d\lambda.$$

Then the family $(U^*(t))_{t \geq 0} \subseteq \mathcal{L}(X^*)$ is strongly continuous and

$$\lambda^{-\alpha} R(\lambda, A+B)^* = \lambda^{\gamma} \int_0^{\infty} e^{-\lambda t} U^*(t)\, dt$$

for $\operatorname{Re} \lambda > \lambda_0$. For $x \in D(A)$ and $t \in [0, \infty)$, the integral in

$$U(t)x := \frac{1}{2\pi i} \int_{\operatorname{Re} \lambda = \lambda_0} e^{\lambda t} \lambda^{-\gamma} [\lambda^{-\alpha} R(\lambda, A+B)x] d\lambda$$

converges absolutely. Therefore $t \mapsto U(t)x$ is continuous in $[0, \infty)$ if $x \in D(A)$ and

$$R(\lambda, A+B)x = \lambda^{\gamma+\alpha} \int_0^{\infty} e^{-\lambda t} U(t)x\, dt.$$

Now the uniqueness theorem for the Laplace transform and the fact that $t \mapsto (U^*(t))^* x$ is weakly continuous yields that $U(t)x = (U^*(t))^* x$ for all $t \geq 0$ and all $x \in D(A)$. Since $((U^*(t))^*)_{t \geq 0}$ is exponentially bounded and $D(A)$ is dense in X, the family $((U^*(t))^*)_{t \geq 0}$ is strongly continuous and the claim follows with Definition 2.1. □

7. An example

The following example shows that the bound for β in Theorem 3.3 is optimal.

Example 7.1. Let $X = L^p(0, \infty)$, $p \in (1, \infty)$ and $\gamma \in \mathbb{C}$. We define the operators A and B_γ by

$$(Af)(x) := \frac{d}{dx} f(x), \qquad (B_\gamma f)(x) := \frac{\gamma}{x} f(x),$$

with maximal domains in X. The closure of $(A + B_\gamma, D(A) \cap D(B_\gamma))$ in X we denote by C_γ. Then:

a) $\|R(\lambda, A) B_\gamma x\|_p \leq p|\gamma| \|x\|_p$ for all $x \in D(B)$ and all $\operatorname{Re} \lambda > 0$, i.e. if $|\gamma| < \frac{1}{p}$ and $\alpha > \max\{\frac{1}{p}, 1 - \frac{1}{p}\}$, then C_γ generates an α-times integrated semigroup.

b) If $0 < \alpha < \gamma < \frac{1}{p}$, then C_γ does not generate an α-times integrated semigroup.

c) If $\gamma \geq \frac{1}{p}$, then there is no $\alpha > 0$ such that C_γ generates an α-times integrated semigroup.

Proof. a) Let $1 < p < \infty$, $|\gamma| < \frac{1}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$, $\operatorname{Re} \lambda > 0$, $f \in D(B_\gamma)$ and $g \in L^q(0, \infty)$. It is well known that the operator $(A, D(A))$ generates the C_0-semigroup

$(T(t))_{t\geq 0}$ given by $T(t)f(x) = f(x+t)$. Using this we obtain

$$\begin{aligned}
|\langle g, R(\lambda,A)B_\gamma f\rangle| &= \left|\int_0^\infty g(x)\int_0^\infty e^{-\lambda t}T(t)B_\gamma f(x)\ dt\ dx\right| \\
&= \left|\int_0^\infty g(x)\int_0^\infty e^{-\lambda t}\frac{\gamma}{x+t}f(x+t)\ dt\ dx\right| \\
&= |\gamma|\left|\int_0^\infty g(x)\int_x^\infty e^{-\lambda(t-x)}\frac{f(t)}{t}\ dt\ dx\right| \\
&= |\gamma|\left|\int_0^\infty \frac{f(t)}{t}\int_0^t e^{-\lambda(t-x)}g(x)\ dx\ dt\right| \\
&\leq |\gamma|\int_0^\infty \frac{|f(t)|}{t}\int_0^t e^{-\operatorname{Re}\lambda(t-x)}|g(x)|\ dx\ dt \\
&\leq |\gamma|\int_0^\infty |f(t)|\ \frac{1}{t}\int_0^t |g(x)|\ dx\ dt.
\end{aligned}$$

Let $G(t) := \frac{1}{t}\int_0^t |g(x)|\ dx$. Then by Hardy's inequality ([8, VI.10.11]) $\|G\|_q \leq p\|g\|_q$ and by Hölder's inequality

$$|\langle g, R(\lambda,A)B_\gamma f\rangle| \leq |\gamma|\int_0^\infty |f(t)|\ G(t)\ dt \leq |\gamma|\ \|f\|_p\ \|G\|_q \leq p|\gamma|\ \|f\|_p\ \|g\|_q.$$

Therefore $\|R(\lambda,A)B_\gamma\|_p \leq p|\gamma|\ \|f\|_p$. Since $(C_\gamma, D(C_\gamma))$ is closed and X is reflexive we have $(C_\gamma^*)^* = C_\gamma$. Theorem 3.3 now yields that $(C_\gamma, D(C_\gamma))$ generates an α-times integrated semigroup if $\alpha > \max\{\frac{1}{p}, 1-\frac{1}{p}\}$.

b) Let $0 < \alpha < \gamma < \frac{1}{p}$. For a test function $f \in C_c^\infty(0,\infty)$ and $t > 0$ we define $S_t f$ by

$$S_t f(x) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}\left(\frac{x+s}{x}\right)^\gamma f(x+s)\ ds.$$

Part a) and Lemma 4.2 yields that $\{\lambda \in \mathbb{C} :\ \operatorname{Re}\lambda > 0\} \subseteq \rho(C_\gamma)$. Moreover, for $f \in C_c^\infty(0,\infty)$ and $\operatorname{Re}\lambda > 0$

$$R(\lambda, C_\gamma)f = \lambda^\alpha\int_0^\infty e^{-\lambda t}S_t f\ dt.$$

If $(C_\gamma, D(C_\gamma))$ generates an α-times integrated semigroup, then by uniqueness of the Laplace transform the α-times integrated semigroup is given by $S_t f$ for $f \in C_c^\infty(0,\infty)$. But S_t can not be extended to a bounded linear operator on X.

c) For $f \in C_c^\infty(0,\infty)$ and $\lambda \in \mathbb{R}$ we define $R_\lambda f$ by

$$R_\lambda f(x) := x^{-\gamma}e^{\lambda x}\int_x^\infty e^{-\lambda t}t^\gamma f(t)\ dt.$$

Then $R_\lambda(\lambda - C_\gamma)f = f = (\lambda - C_\gamma)R_\lambda f$. But if $\gamma \geq \frac{1}{p}$, then R_λ can not be extended to a bounded operator on $L^p(0,\infty)$. So $\mathbb{R} \subseteq \sigma(C_\gamma)$. Hence there can be no $\alpha > 0$ such that $(C_\gamma, D(C_\gamma))$ generates an α-times integrated semigroup. □

8. Application

Let $X = L^p(\mathbb{R})$ where $1 < p < \infty$ and let $m \geq 2$ be an integer. We define the operator $(A_m, D(A_m))$ by

$$A_m f := if^{(m)} \qquad \text{if } m \text{ is even,}$$

and by

$$A_m f := f^{(m)} \qquad \text{if } m \text{ is odd,}$$

with domain $D(A) := W^{m,p}(\mathbb{R})$ in $L^p(\mathbb{R})$.

Then $(A_m, D(A_m))$ generates a C_0-semigroup on X if and only if $p = 2$ ([9]). For $m = 2$ this was proved first by Hörmander [12] in 1960. If $p \neq 2$, $(A_m, D(A_m))$ generates an α-times integrated semigroup on X for $\alpha > \left|\frac{1}{2} - \frac{1}{p}\right|$ ([9]).

We consider the Cauchy problem

$$\begin{cases} u'(t) &= (A_m + B)u(t), \qquad t \geq 0, \\ u(0) &= x, \end{cases}$$

where $(B, D(B))$ is defined by

$$Bf := V \cdot f^{(l)}$$

with maximal domain

$$D(B) := \{f \in L^p(\mathbb{R}) : V \cdot f^{(l)} \in L^p(\mathbb{R})\}$$

in $L^p(\mathbb{R})$. Here, V is a potential and $l \in \mathbb{N} \cup \{0\}$. We will use Theorem 3.3 to show the following proposition.

Proposition 8.1. *Let $X = L^p(\mathbb{R})$ where $1 < p < \infty$. The operators $(A_m, D(A_m))$ and $(B, D(B))$ are defined as above. If one of the conditions*

(i) $l \leq \frac{1}{p}(m-1)$ und $V \in L^p(\mathbb{R})$

or

(ii) $l = 0$ und $V \in L^p(\mathbb{R}) + L^\infty(\mathbb{R})$

are satisfied, then $D(B) \supseteq D(A)$ and $(A_m + B, D(A_m))$ generates a β-times integrated semigroup for each $\beta > \sigma_p$. Here

$$\sigma_p = \begin{cases} \frac{2}{p} - \frac{1}{2} & p \in (1, 2] \\ \frac{3}{2} - \frac{2}{p} & p \in (2, \infty). \end{cases}$$

Proof. We only give the proof for the case that m is even, i.e, $m = 2k$ for some $k \in \mathbb{N}$. If m is odd, the proposition can be shown in a similar way.

One can compute that $\mathbb{C} \setminus (i\mathbb{R}) \subseteq \rho(A_{2k})$ and that for $\lambda \in \mathbb{C} \setminus (i\mathbb{R})$ the resolvent of A_{2k} is given by

$$R(\lambda, A_{2k})f(x) = \frac{i}{2k} \int_{-\infty}^{\infty} \sum_{j=1}^{k} \frac{e^{-\mu_j |x-s|}}{(-\mu_j)^{2k-1}} f(s)\, ds, \qquad x \in \mathbb{R},$$

where f is a function in $L^p(\mathbb{R})$ and μ_j $(j = 1, \ldots, k)$ are the k solutions of the equation $\lambda - i\mu^{2k} = 0$ with $\operatorname{Re} \mu_j > 0$. Moreover, using Young's inequality, we obtain the resolvent estimate

$$\|R(\lambda, A_{2k})f\|_p \leq \frac{\|f\|_p}{|\lambda|^{1-1/(2k)} \min\{\operatorname{Re} \mu_j : j = 1, \ldots, k\}}.$$

Let $\lambda = re^{i\varphi}$ where $r > 0$ and $\varphi \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then a careful computation yields

$$\min\{\operatorname{Re} \mu_j : j = 1, \ldots, k\} = |\lambda|^{1/(2k)} \cos \psi_k$$

where

$$\psi_k = \begin{cases} \frac{\varphi}{2k} - \frac{\pi}{4k} + \frac{\pi}{2}, & \text{if } k \text{ even}, \\ \frac{\varphi}{2k} + \frac{\pi}{4k} - \frac{\pi}{2}, & \text{if } k \text{ odd}. \end{cases}$$

Since $|\lambda| = \frac{\operatorname{Re} \lambda}{\cos \varphi}$, we have

$$|\lambda|^{1-1/(2k)} \min\{\operatorname{Re} \mu_j : j = 1, \ldots, k\} = \operatorname{Re} \lambda \, \frac{\cos \psi_k}{\cos \varphi}.$$

But $\frac{\cos\varphi}{\cos\psi_k}$ is bounded by a positive constant c_k that depends only on k and not on $\varphi\in(-\frac{\pi}{2},\frac{\pi}{2})$. This shows the estimate

$$\|R(\lambda,A_{2k})\|\le\frac{c_k}{\operatorname{Re}\lambda}. \tag{7}$$

We look at $BR(\lambda,A_{2k})$. Take $f\in C_c^\infty(\mathbb{R})$. For $\lambda\in\mathbb{C}\setminus(i\mathbb{R})$ we compute

$$BR(\lambda,A_{2k})f=V(x)\frac{i}{2k}\sum_{j=1}^{k}\bigg(\int_{-\infty}^{x}\frac{e^{-\mu_j(x-s)}}{(-\mu_j)^{2k-l-1}}f(s)\,ds-\int_{x}^{\infty}\frac{e^{\mu_j(x-s)}}{\mu_j^{2k-l-1}}f(s)\,ds\bigg).$$

Then, if $g\in C_c^\infty(\mathbb{R})$ and $\frac{1}{p}+\frac{1}{q}=1$, we find

$$\begin{aligned}
&|\langle g,BR(\lambda,A_{2k})f\rangle|\\
&\quad\le\frac{1}{2k|\lambda|^{1-(l+1)/(2k)}}\sum_{j=1}^{k}\int_{-\infty}^{\infty}|g(x)|\,|V(x)|\int_{-\infty}^{\infty}e^{-\operatorname{Re}\mu_j|x-s|}\,|f(s)|\,ds\,dx\\
&\quad\le\frac{\|f\|_p}{2k|\lambda|^{1-(l+1)/(2k)}}\sum_{j=1}^{k}\int_{-\infty}^{\infty}|g(x)|\,|V(x)|\bigg(\int_{-\infty}^{\infty}e^{-q\operatorname{Re}\mu_j|x-s|}ds\bigg)^{1/q}dx\\
&\quad=\frac{\|f\|_p}{2k|\lambda|^{1-(l+1)/(2k)}}\sum_{j=1}^{k}\bigg(\frac{2}{q\operatorname{Re}\mu_j}\bigg)^{1/q}\int_{-\infty}^{\infty}|g(x)|\,|V(x)|dx\\
&\quad=\frac{c(p)}{|\lambda|^{1-(l+1)/(2k)}}\frac{1}{k}\sum_{j=1}^{k}\bigg(\frac{1}{\operatorname{Re}\mu_j}\bigg)^{1/q}\|V\|_p\|g\|_q\|f\|_p
\end{aligned}$$

where $c(p)\le 1$ is a constant only depending on p. Therefore $D(B)\supseteq D(A)$ and

$$\begin{aligned}
\|BR(\lambda,A_{2k})\| &\le \frac{c(p)\|V\|_p}{|\lambda|^{1-(l+1)/(2k)}}\frac{1}{k}\sum_{j=1}^{k}\bigg(\frac{1}{\operatorname{Re}\mu_j}\bigg)^{1/q}\\
&\le \frac{c(p)\|V\|_p}{|\lambda|^{1-(l+1)/(2k)}\min\{(\operatorname{Re}\mu_j)^{1/q}:j=1,\dots,k\}}
\end{aligned}$$

for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>0$. As above we see that

$$\min\{(\operatorname{Re}\mu_j)^{1/q}:j=1,\dots,k\}=|\lambda|^{1/(2kq)}(\cos\psi_k)^{1/q}.$$

So we obtain

$$\begin{aligned}
|\lambda|^{1-(l+1)/(2k)}\min\{(\operatorname{Re}\mu_j)^{1/q}:j=1,\dots,k\}&=|\lambda|^{1-(l+1)/(2k)+1/(2kq)}(\cos\psi_k)^{1/q}\\
&=|\lambda|^{1-(lp+1)/(2kp)}(\cos\psi_k)^{1/q}\\
&=(\operatorname{Re}\lambda)^{1-(lp+1)/(2kp)}\frac{(\cos\psi_k)^{1/q}}{(\cos\varphi)^{1-(lp+1)/(2kp)}}\\
&=(\operatorname{Re}\lambda)^{1-(lp+1)/(2kp)}\bigg(\frac{\cos\psi_k}{\cos\varphi}\bigg)^{1/q}(\cos\varphi)^{1/q-1+(lp+1)/(2kp)}.
\end{aligned}$$

If we assume that $l\le\frac{1}{p}(2k-1)$, we obtain $\frac{1}{q}-1+\frac{lp+1}{2kp}=\frac{lp+1}{2kp}-\frac{1}{p}\le\frac{2k-1+1}{2kp}-\frac{1}{p}=0$. So there is a positive constant $c_k>0$ that only depends on k such that

$$|\lambda|^{1-1/(2k)}\min\{(\operatorname{Re}\mu_j)^{1/q}:j=1,\dots,k\}\ge c_k^{-1}(\operatorname{Re}\lambda)^{1-(lp+1)/(2kp)}.$$

Hence for all $\lambda\in\mathbb{C}$ with $\operatorname{Re}\lambda>0$,

$$\|BR(\lambda,A_{2k})\|\le\frac{c_k\|V\|_p}{(\operatorname{Re}\lambda)^{1-(lp+1)/(2kp)}}. \tag{8}$$

If we assume (i), the estimate (8) yields that there is $\lambda_0>0$ such that $\|BR(\lambda,A_{2k})\|\le M<1$ for all $\operatorname{Re}\lambda\ge\lambda_0$.

If (ii) holds, V can be written as $V_p + V_\infty$ where $V_p \in L^p(\mathbb{R})$ and $V_\infty \in L^\infty(\mathbb{R})$. Let $B_p f := V_p \cdot f$ with maximal domain $D(B_p) = D(B)$. The operator B_∞ defined by $B_\infty f := V_\infty \cdot f$ is a bounded on $L^p(\mathbb{R})$ and $B = B_p + B_\infty$. Using (8) to estimate $\|B_p R(\lambda, A_{2k})\|$ and (7) for $\|B_\infty R(\lambda, A_{2k})\|$, we again obtain that there is $\lambda_0 > 0$ such that $\|BR(\lambda, A_{2k})\| \leq M < 1$ for all $\operatorname{Re}\lambda \geq \lambda_0$.

Since $(A_{2k}, D(A_{2k}))$ generates an α-times integrated semigroup for $\alpha > \left|\frac{1}{2} - \frac{1}{p}\right|$, the assumptions of Theorem 3.3 are satisfied in both cases. Hence the operator $(A_{2k} + B, D(A_{2k}))$ generates a β-times integrated semigroup for $\beta > \left|\frac{1}{2} - \frac{1}{p}\right| + \max\left\{\frac{1}{p}, 1 - \frac{1}{p}\right\} = \sigma_p$. □

References

[1] Mostafa Adimy, Hassane Bouzahir, and Khalil Ezzinbi, *Existence for a class of partial functional differential equations with infinite delay*, Nonlinear Anal., Theory Methods Appl. **46A** (2001), no. 1, 91–112.

[2] Wolfgang Arendt, *Resolvent positive operators*, Proc. London Math. Soc. (3) **54** (1987), 321–349.

[3] ______, *Vector-valued Laplace transforms and Cauchy problems*, Israel J. Math. **59** (1987), 327–352.

[4] Wolfgang Arendt, Charles J. K. Batty, Matthias Hieber, and Frank Neubrander, *Vector-valued Laplace transforms and Cauchy problems*, Birkhäuser Basel Boston Berlin, 2001.

[5] Jean Bourgain, *A Hausdorff-Young inequality for B-convex Banach spaces*, Pacific J. Math. **101** (1982), no. 2, 255–262.

[6] ______, *Vector-valued Hausdorff-Young inequalities and applications*, Geometric aspects of functional analysis (1986/87), Lecture Notes in Math., 1317, Springer, 1988, pp. 239–249.

[7] John B. Conway, *Functions of one complex variable*, Springer Verlag, 1978.

[8] Nelson Dunford and Jacob T. Schwartz, *Linear operators, Part I: General theory*, Interscience Publishers, Inc., New york, 1957.

[9] Matthias Hieber, *Integrated semigroups and differential operators on L^p*, Dissertation, Tübingen, 1989.

[10] ______, *Integrated semigroups and differential operators on L^p spaces*, Math. Ann. **291** (1991), 1–16.

[11] E. Hille and R. S. Phillips, *Functional analysis and semigroups*, Am. Math. Soc., 1957.

[12] Lars Hörmander, *Estimates for translation invariant operators in L^p-spaces*, Acta Math. **104** (1960), 93–139.

[13] Cornelia Kaiser, *Integrated semigroups and linear differential equations with delay*, submitted.

[14] Cornelia Kaiser and Lutz Weis, *A perturbation theorem for C_0-semigroups on Hilbert spaces*, to appear in Semigroup Forum.

[15] Hermann Kellermann and Matthias Hieber, *Integrated semigroups*, J. Funct. Anal. **84** (1989), 160–180.

[16] S. Kwapien, *Isomorphic characterisations of inner product spaces by orthogonal series with vector valued coefficients*, Studia Math. **44** (1972), 583–595.

[17] Frank Neubrander, *Integrated semigroups and their application to complete second order Cauchy problems*, Semigroup Forum **38** (1989), 233–251.

[18] A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Applied Mathematical Sciences, 44, Springer-Verlag, New York, 1983.

[19] Jaak Peetre, *Sur la transformation de Fourier des fonctions à valeurs vectorielles*, Rend. Sem. Mat. Univ. Padova **42** (1969), 15–26.

Mathematisches Institut I, Universität Karlsruhe, Englerstrasse 2, 76128 Karlsruhe, Germany

E-mail address: cornelia.kaiser@math.uni-karlsruhe.de

Mathematisches Institut I, Universität Karlsruhe, Englerstrasse 2, 76128 Karlsruhe, Germany

E-mail address: lutz.weis@math.uni-karlsruhe.de